Quantum Physics IV, Solutions 1

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Exercise 1 : Double slit experiment

The kinetic energy of the electrons coming out of the lamp will be roughly given by

$$k_B T \sim 0.86 \left(\frac{T}{10^4 K} \right) \text{eV}. \tag{1}$$

So that the emitted electrons are non-relativistic for any possible thermionic lamp on Earth ($m_e = 0.511 \cdot 10^6$eV):

$$\begin{aligned} E &= m_e v^2/2 \\ p &= m_e v. \end{aligned}$$

The DeBroglie wavelength of such electrons is

$$\lambda = 2\pi \frac{\hbar}{p}.$$

If $a \gg \lambda$, the waves simply pass through, and we obtain two peaks on the screen. To have interference we have to consider :

$$\boxed{a \ll \lambda} \tag{2}$$

In a system with point like slits d would play no role. We assume to work in such a setup, corrections to the plane-wave approximation will enter through the ratio b/d.

If (2) is satisfied the two slits are a source for spherical waves

$$\psi_i(x,t) = \frac{A_i}{|\mathbf{x} - \mathbf{x}_i|} \exp i \left(\frac{p\,|\mathbf{x} - \mathbf{x}_i|}{\hbar} - \frac{E\,t}{\hbar} + \phi_i \right) \tag{3}$$

where $(\mathbf{x}, t)$ is a generic point and $\mathbf{x}_i$ is the position of the point like sources (slits). On a generic point on the screen the two wave amplitudes are summed

$$\Psi(x,t) = \psi_1(x,t) + \psi_2(x,t). \tag{4}$$

Some simplifications are in order. First, both the amplitudes $A_{1,2}$ and the phases $\phi_{1,2}$ are equal, since the two waves coming out of the two slits come from the same plane wave behind the first screen. Second, the time of observation is the same, so that also the E dependent terms in the amplitudes are equal and irrelevant. To maximize the interference we will assume a large L limit

$$\boxed{L \gg b, |\mathbf{x}|} \tag{5}$$

so that $|\mathbf{x} - \mathbf{x}_1| \sim |\mathbf{x} - \mathbf{x}_2|$. So up to a normalization factor

$$\Psi(x) \propto \exp i \left(\frac{p\,|\mathbf{x} - \mathbf{x}_1|}{\hbar} \right) + \exp i \left(\frac{p\,|\mathbf{x} - \mathbf{x}_2|}{\hbar} \right) \tag{6}$$

and

$$|\mathbf{x} - \mathbf{x}_{1,2}| = \sqrt{(b/2 \mp x)^2 + L^2} \sim L \left(1 + \tfrac{1}{2} \frac{(b/2 \mp x)^2}{L^2} \right) \tag{7}$$

The interference pattern is thus given, as a function of x (the coordinate on the screen), and up to a normalization factor independent of x by

$$|\Psi|^2(x) = |\psi_1(x,t) + \psi_2(x,t)|^2 \propto \cos\left(\pi \frac{bx}{\lambda L} \right)^2. \tag{8}$$

The positions of the maxima are

$$x^{(1)}_{\max} = n\frac{\lambda L}{b}. \tag{9}$$

Too see at least one maximum, $x^{(1)}$, must be within the range of the approximation we made, that is $x^{(1)} \ll L$ so that in terms of the other parameters

$$\boxed{L \gg b, x}. \tag{10}$$

Exercise 2 : Gaussian integral 1

For all integrals, $\alpha > 0$.

- One usual trick is to compute the square of the integral :

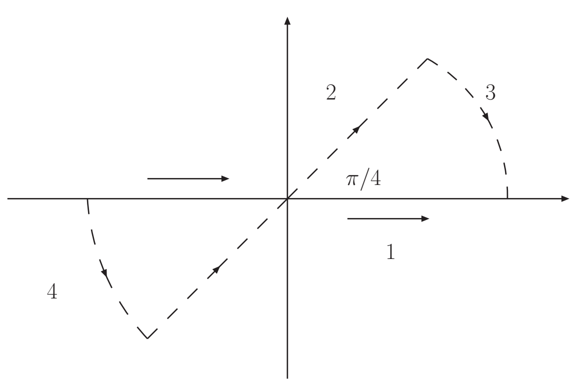
$$\begin{aligned} G^2_{\text{Re}} &= \left(\int_{-\infty}^{+\infty} \mathrm{d}x\ e^{-\alpha x^2}\right)\left(\int_{-\infty}^{+\infty} \mathrm{d}y\ e^{-\alpha y^2}\right) \\ &= \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\ e^{-\alpha(x^2+y^2)} \\ &= 2\pi \int_0^\infty \mathrm{d}r\ re^{-\alpha r^2} \\ &= -\frac{\pi}{\alpha}\left|e^{-\alpha r^2}\right|_0^\infty \\ &= \frac{\pi}{\alpha} \end{aligned}$$

$$\boxed{G_{\text{Re}} = \sqrt{\frac{\pi}{\alpha}}} \tag{11}$$

I fill in the following some mathematical detail left open during the class about the evaluation of G^+_{Im}. The intuitive procedure was to tilt the countour of integration 45 degrees counterclockwise. This can indeed be justified using Cauchy theorem.

Consider the contour of integration indicated below with R fixed and big

$$\int_C = \int_{\theta=0,-R\to R} + \int_{C_R\, 0\to\pi/4} + \int_{\theta=\pi/4, R\to -R} + \int_{C_R\, \pi\to 5/4\pi} = (a)+(b)+(c)+(d). \tag{12}$$



For finite R

$$\int_C dz\ \exp i\alpha z^2 = 0 \tag{13}$$

by Cauchy theorem being $\exp i\alpha z^2$ an entire function. Assuming for the moment $\lim_{R\to\infty}\ (b)+(d)=0$,

$$\lim_{R\to\infty}\int_{\theta=0,-R\to R} = \int_{-\infty}^{\infty} dx\ \exp\ i\alpha x^2 = -\lim_{R\to\infty}\int_{\theta=\pi/4,R\to -R} = e^{i\frac{\pi}{4}}\int_{-\infty}^{\infty}\exp -\alpha x^2 = e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{\alpha}} \tag{14}$$

that is what you already proved during the class. To show $\lim_{R\to\infty}\ (b)=0$, we parametrize the arc countour as $z(\theta)=R\exp i\theta$

$$|(\mathrm{b})| = \left|\int_{C_R\,0\to\pi/4}\right| = \left|\int_0^{\pi/4} d\theta\, iR\,\exp\left(i\alpha R^2 e^{2i\theta}\right)\right| = \tag{15}$$

$$= R\left|\int_0^{\pi/4} d\theta \exp\left(-\alpha R^2\sin 2\theta\right)\right| \leq R\int_0^{\pi/4} d\theta\left|\exp\left(-\alpha R^2\sin 2\theta\right)\right| \tag{16}$$

Since $\sin 2\theta>0$ for $0\leq\theta\leq\pi/4$, $R\to\infty$ imply $|(b)|\to 0$ as we desired. the same reasonings works for (d).

What does go wrong the integral?

$$\int_{-\infty}^{\infty}\exp\ \alpha x^2,\qquad \alpha>0. \tag{17}$$

You may think to evaluate it tilting the contour of integration by 90 degrees counter-clockwise. This time we must ask for

$$\lim_{R\to\infty}\left|\int_{C_R\,0\to\pi/2}\right| + \left|\int_{C_R\,\pi\to 3/2\pi}\right| = 0. \tag{18}$$

As it can be readily checked in the same way I explained before, this time both integrals diverge in the large R limit.

The evaluation of G^-_{Im} proceeds in the same way. The countour this time must be tilted by 45 degrees but in the clockwise direction. The result is

$$G^-_{\mathrm{Im}} = \int_{-\infty}^{\infty} dx\ \exp -i\alpha x^2 = e^{-i\frac{\pi}{4}}\sqrt{\frac{\pi}{\alpha}} \tag{19}$$

- The integral is vanishing for odd n, since the integrand would be an odd function of x. For even values of n

$$\int_{-\infty}^{\infty} x^{2k}e^{-\alpha x^2} = (-1)^k\int_{-\infty}^{\infty} x^{2k}\frac{d^k}{d\lambda^k}e^{-\lambda x^2}\Big|_{\lambda=\alpha} = (-1)^k\frac{d^k}{d\lambda^k}\sqrt{\frac{\pi}{\lambda}}\Big|_{\lambda=\alpha} = \tag{20}$$

$$= \sqrt{\pi}\alpha^{-k-1/2}\frac{(2k-1)!!}{2^k} \tag{21}$$

- For the real integral, the naive guess would be that the error is of order $e^{-\alpha L^2}$. But there is actually a further suppression. The error δ is defined by

$$\delta_{\mathrm{Re}} = 2\int_L^{\infty} dx\, e^{-\alpha x^2} = -\frac{2}{\sqrt{\alpha}}\int_{\sqrt{\alpha}L}^{\infty} dx\,\frac{1}{2x}\frac{d}{dx}\left(e^{-x^2}\right) = \tag{22}$$

$$= -\frac{2}{\sqrt{\alpha}}\left[-\frac{e^{-\alpha L^2}}{2\sqrt{\alpha}L} + \int_{\sqrt{\alpha}L}^{\infty} dx\,\frac{1}{2x^2}e^{-x^2}\right] = \frac{e^{-\alpha L^2}}{\alpha L} + \mathcal{O}(\frac{e^{-\alpha L^2}}{\alpha^2 L^3}). \tag{23}$$

For the imaginary gaussian integral the exponential damping of the error turns into a phase and only the $1/\alpha L$ suppression remains

$$|\delta_{\mathrm{Im}}| = \frac{1}{\alpha L} + \mathcal{O}(\frac{1}{\alpha^2 L^3}), \tag{24}$$

so that the convergence in terms of L is much slower.

Exercise 3 : Gaussian integral 2

- A can be diagonalized with help of an orthogonal matrix. This change of variables has a determinant $= 1$; thus we end up with a product of n integrals of $e^{i\lambda_i \tilde{x}_i^2}$, where λ_i are the eigenvalues with eigenvector $\tilde{x}_i$. The n-dimensional integral factors into a product of n integrals that we already calculated. Recall that the sign of the eigenvalues λ_i determined the overall phase of the result through the contour rotation:

$$\boxed{G_A(0) = \frac{e^{i(n_+ - n_-)\frac{\pi}{4}}}{\sqrt{\left|\det\left(\frac{A}{\pi}\right)\right|}}} \tag{25}$$

The factor $e^{i(n_+ - n_-)\frac{\pi}{4}}$ keeps track of these sign counting the difference of the number of the positive eigenvalues (n_+) and that of the negative ones (n_-).

- The exponent can be rewritten as

$$x^T Ax + J^T x = (x + JA^{-1}/2)^T A(x + A^{-1}J/2) - J^T A^{-1} J/4 \equiv x'^T Ax' - J^T A^{-1} J/4 \tag{26}$$

Since the transformation $x \to x'$ is just a shift, the measure of integration is invariant and the final result is simply

$$G_A(J) = \exp\left(-\frac{i}{4} J_i (A^{-1})_{ij} J_j\right) G_A(0) \tag{27}$$