I. OBJECTIVE OF THE EXPERIMENT

We will measure the shear modulus $G$ of different materials, using static and dynamic methods, and discuss the pros and cons of each method.

II. INTRODUCTION: ELASTICITY

When a solid body is put under external stress, it deforms. If the deformation is reversible, we say it is elastic. We generally distinguish three different types of elastic deformation: uniaxial deformation, shearing and compression.

Uniaxial deformation

When applying a tensile force $F_x$ in the direction $x$, the sample (metal, polymer, ceramic, ...) stretches by $\Delta x$. We define the strain as:

$$\varepsilon_x = \frac{\Delta x}{x}$$

where $\Delta x$ is the variation in length of a specimen of length $x$ (fig 1.).

By defining the tensile stress in the $x$ direction by:

$$\sigma_x = \frac{F_x}{S}$$

where $S$ is the area of the sample, we can express Hooke's law:

$$\sigma_x = E \cdot \varepsilon_x$$

where $E$ is the elasticity modulus, also called Young modulus.

The stretching of the specimen in the direction of the tensile strength generally comes with a lateral contraction, in such a way that the strain in the orthogonal direction are: $\varepsilon_y = \frac{\Delta y}{y}$ et $\varepsilon_z = \frac{\Delta z}{z}$.

By symmetry, $\varepsilon_y = \varepsilon_z$. 
The **Poisson coefficient** \( \nu \) is defined as:

\[
\nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}
\]

Since for small strains, the volume deformation \( \Delta V \) is given by:

\[
\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_x \cdot (1 - 2\nu)
\]

we see that \( \nu \) must be between 0 and 0.5. In metals \( \nu \approx 0.35 \).

**Shearing (fig 2.)**

An example of shearing is represented in figure 2, where the deformation is:

\[
\alpha = \frac{\delta x}{z_0}
\]

And the shear stress:

\[
\tau = \frac{F}{S_0}
\]

Equilibrium conditions on a parallelepiped force the tangential stress on all four faces to be equal.

The **shear modulus** \( G \) is defined by:

\[
\tau = G \cdot \alpha
\]

**Uniform hydrostatic compression**

\[
P = -k \cdot \frac{\Delta V}{V}
\]

where \( P \) is the hydrostatic pressure and the **compressibility modulus** \( k \).

The four constants \( E, G, k \) and the Poisson coefficient are linked by

\[
E = \frac{9kG}{G + 3k} = 3k(1 - 2\nu) = 2G(1 + \nu)
\]

Therefore, amongst the four elastic constants, only two are independent.
III. TORSION OF CYLINDRICAL RODS

III.1. THEORY

Let's consider a cylindrical sample of radius $r$, length $l$, and axis $OO'$, that receives a torque $M_z$ parallel to the $OO'$ axis (fig. 3).

Under the effect of the torque, each line (perpendicular to the $OO'$ axis) rotates by an angle $\theta$ without deformation, which generates a shear strain of the volume elements at a distance $\rho$ of the axis.

The strain can be expressed by the angle $\alpha$:

$$\alpha = \frac{\rho \theta}{l}.$$

The shear strain is due to the shear stresses $\tau$ equal to (at a distance $\rho$ from the center of the cylinder):

$$\tau = G\alpha = G\frac{\rho \theta}{l}.$$

Fig. 3 : Torsion of a cylindrical rod

and the stress $\tau$ is connected to the torque $M_z$ according to:

$$M_z = \int_0^{\rho} \rho G \alpha dS = \int_0^{\rho} \rho G \alpha 2\pi \rho d\rho = \frac{2\pi G \theta}{l} \int_0^{\rho} \rho^3 d\rho.$$

We can then get an expression of the torque as a function of the torsion angle $\theta$:

$$M(\theta) = \frac{\pi Gr^4}{2l} \theta = \frac{\pi Gd^4}{32l} \theta$$

with $d = 2r =$ diameter of cylinder of length $l$. 

III.2. STATIC METHOD

The cylindrical rods to be studied (Fig. 4), attached to a pulley (D), are placed between a fixed (F) and mobile (M) stand with a rigid, graduated base (B). The rod can turn freely in (F), is attached in (M) a distance \( l \) away from (D) thanks to a tightening screw (V). The torque is generated using a weight \( F = mg \) hanging from a graduated disk (D). The torsion angle is measured by redirecting a light beam using a mirror (m) attached to the disk (D).

![Fig. 4: Frame for torsion tests.](image)

The torque of the bar is therefore:

\[
M(\theta) = \frac{\pi G d^4}{32 l} \theta = mg \frac{D}{2}
\]

Thus:

\[
G = \frac{16 l mg D}{\pi d^4 \theta}
\]

III.3. DYNAMIC METHOD

If we hang a solid of inertial moment \( I \) at the end of a string, allowed to move freely on its other end, we can create a torsion pendulum (fig. 5), whose equation of motion is that of a harmonic oscillator (see mechanics lecture):

\[
I \ddot{\theta} + M(\theta) = 0
\]

i.e. by replacing \( M(\theta) \) by its expression:

\[
\ddot{\theta} + \frac{\pi G d^4}{32 I l} \theta = 0
\]

The solution is

\[
\theta = \theta_0 \cos(\omega t + \varphi)
\]
The system oscillates at an angular frequency $\omega$ given by:

$$\omega = \sqrt{\frac{\pi G d^4}{32 I \cdot l}}$$

The measure of the period $T = \frac{2\pi}{\omega}$ gives us the shear modulus $G$

$$G = \frac{128\pi I \cdot l}{d^3 T^2}$$

(For a rectangular section $(b \times c)$, with $b = 4c$:

$$G = \frac{140.5I \cdot l}{bc^3 T^2}$$

**Damped oscillator**

In practice, we observe the oscillations of the pendulum oscillating freely decrease over time. This decrease of the amplitude is partly due to friction with air, but also to the damping property of the material. In fact, the decrease in amplitude depends on the material.

The equation of motion is modified, by adding a viscous friction factor that represents the damping of the material.

The equation of motion becomes:

$$I \cdot \ddot{\theta} + C \cdot \dot{\theta} + M(\theta) = 0$$

and the solution:

$$\theta = \theta_0 \cdot \exp(-\lambda \cdot t) \cdot \cos(\omega t + \phi)$$

with $\lambda = \frac{C}{2I}$ the damping coefficient, that represents the damping capacity of the material.

We can determine $\lambda$ using the logarithmic decay of the oscillations (show this result):

$$\lambda = \frac{1}{nT} \ln \frac{\theta(t)}{\theta(t + nT)}$$

where $\theta(t)$ and $\theta(t + nT)$ are the amplitudes at the instant $t$, and $n$ periods after the instant $t$ (instant $t + nT$)
IV. SUGGESTED EXPERIMENTS

Measure the shear modulus $G$ of at least two different materials, and compare the obtained values

a) Static method (Fig. 6)

- Measure the diameter $D$ of the disk (D) used for calculating the torque
- Measure the average diameter $d$ of the rod by measuring it in 10 different places and averaging
- Mount the rod on the extremities (F) and (M). Place the mirror (m) attached to the disk (D) in order to obtain a light mark on the ruler.
- In order to take a measurement, measure the height of the light on the ruler as well as the distance between the mirror and the ruler, and use basic trigonometry to calculate $\theta$. Don’t forget that for a rotation of an angle $\alpha$ of the mirror, the light beam is redirected by an angle $2\alpha$.
- Measure the torsion at a constant length $l$ by varying the weight applied ($P = mg$). Plot $\theta = \theta(P)$
- Measure the torsion for a constant weight, by varying the length $l$. Plot $\theta = \theta(l)$ - From these measures, calculate the shear modulus $G$ by using the slope of the graphs $\theta = \theta(P)$ and $\theta = \theta(l)$.
b) Dynamic Method (Fig.7)

- Measure the dimensions and the mass of the inertial disk, and calculate its inertial moment.
\[ I = \int r^2 \cdot dm = \frac{M \cdot r^2}{2} \]

- Measure the length of the sample, and its diameter.

- Place the inertial disk in order for the reflection of the light beam to be centered on the captor. The voltmeter should be close to zero, to avoid saturation during the measurement.

- Set the pendulum in a pure torsion motion. This can be achieved by starting the motion with the rod at the bottom of the setup. The oscillations are saved on a computer using the program `Plotter_Y1Y2_t`. Make sure to put the cover an before measuring for a better signal, and avoid touching the table during a measurement.

- Measure the length of 10 oscillations and deduce the period \( T \) of the oscillations.

- Measure the logarithmic decay to get an order of magnitude of the damping of the material.

- Repeat the experiment 2 to 3 times per sample.

Fig. 7: Setup for dynamic method
### Elasticities of metals and alloys

<table>
<thead>
<tr>
<th>Material</th>
<th>$E$</th>
<th>$G$</th>
<th>$v$</th>
<th>$K'$</th>
</tr>
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<tbody>
<tr>
<td>to 20 °C</td>
<td>GPa</td>
<td>GPa</td>
<td></td>
<td>GPa</td>
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<tr>
<td>Aluminium</td>
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<td>0.345</td>
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<td>12.0</td>
<td>0.330</td>
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<td>115.4</td>
<td>0.210</td>
<td>160.1</td>
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<td>48.3</td>
<td>0.343</td>
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<td>27.0</td>
<td>0.44</td>
<td>217.0</td>
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<tr>
<td>Iron (soft)</td>
<td>211.4</td>
<td>81.6</td>
<td>0.293</td>
<td>169.8</td>
</tr>
<tr>
<td>Iron (cast)</td>
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<td>60.0</td>
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<td>109.5</td>
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<td>Lead</td>
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<tr>
<td>Nickel (unmag., soft)</td>
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<tr>
<td>&quot; hard)</td>
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<td>0.306</td>
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<td>Brass (70 Zn, 30 Cu)</td>
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<td>Constantan</td>
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<td>156.4</td>
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<td>Hidurax Special††</td>
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<td>54.4</td>
<td>0.333</td>
<td>144.1</td>
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<tr>
<td>Invar (36 Ni, 63.8 Fe, 0.2 Cu)</td>
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<td>57.2</td>
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<td>99.4</td>
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<td>Nickel Silver§</td>
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<td>81.1</td>
<td>0.293</td>
<td>168.7</td>
</tr>
<tr>
<td>&quot; (3% C hardened)</td>
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<td>77.8</td>
<td>0.296</td>
<td>165.0</td>
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<td>82.2</td>
<td>0.287</td>
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<tr>
<td>&quot; Tool (hardened)‖</td>
<td>203.2</td>
<td>78.5</td>
<td>0.295</td>
<td>165.2</td>
</tr>
<tr>
<td>&quot; Stainless§§</td>
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<td>83.9</td>
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<td>Tungsten Carbide†</td>
<td>534.4</td>
<td>219.0</td>
<td>0.22</td>
<td>319.0</td>
</tr>
</tbody>
</table>

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Fig. 8: Orders of magnitude of elastic moduli of several metals