G3. RCL Circuits

I. INTRODUCTION

An RCL circuit is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel.The behaviour of an RLC circuit is generally described by a second-order differential equation. RLC circuits are used to make frequency filters, or impedance transformers.

Equations that appear in certain fields of physics, or science in general, are often identical, and therefore, a given phenomenon tends to have analogous phenomena in different fields. The experiment you will conduct is the study of oscillating electric RCL circuits, can also simulate mechanical oscillations.

II. THEORY

Consider the following circuit (Fig. 1):

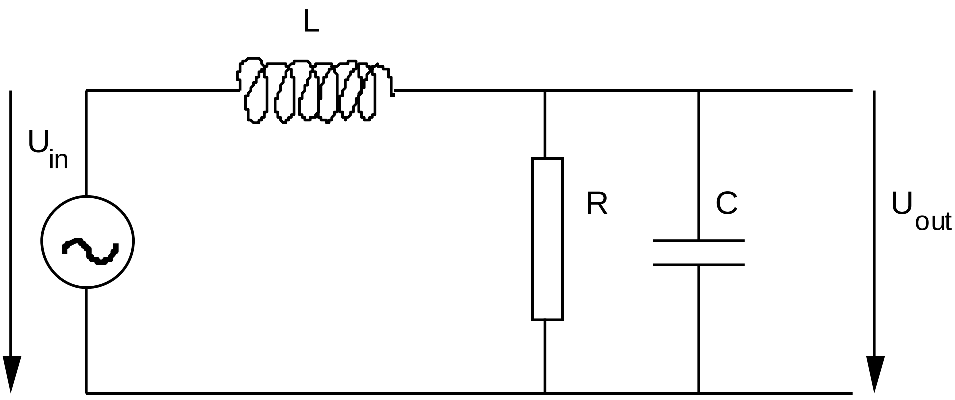


Fig. 1: Electric RCL circuit

Recall the voltage-current relationships of the different electric elements:

For the resistance R: $U_R = I_R R$

For the capacitor C : $\int I_C dt = CU_C$

For the inductance L : $U_C = -L\frac{dI_l}{dt}$

(U_x and I_x represent the voltage difference and the current across an *x* element)

Using the Kirchhoff laws, and defining $U_{out} = U(t) = U$ yields:

$$C \cdot U(t) = \int I_c \cdot dt \qquad \text{et} \qquad U(t) = R \cdot I_R$$

Since: $I_C + I_R = I_L$

Then: $$C\frac{dU}{dt}+\frac{U}{R}=\int\frac{U-U_{\text{int}}}{L}dt$$

Differentiating with respect to time yields: $$\ddot{U}+\frac{\dot{U}}{RC}+\frac{U}{LC}=\frac{U_{in}}{LC} \quad (1)$$

Equation (1) is similar to the equation defining a damped harmonic oscillator

Model harmonic oscillator.

Let's consider the following model of a harmonic oscillator (fig. 2): a point mass m moves along an axis Ox (single degree of liberty). It is subject to:

1) A spring force : $\vec{F}_r=-k\cdot\vec{x}$ that tries to bring the mass back to its equilibrium position.

2) A viscous friction :

$\vec{F}_f=-\eta\cdot\vec{v}$

3) An external time dependent perturbation force $P(\text{t})$

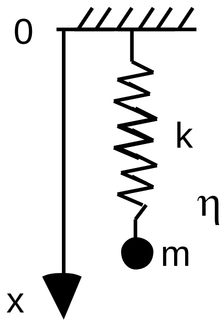


Fig. 2: Damped oscillator

The mass' differential equation of motion is given by (to be shown by reader):

$$\ddot{x}+2\lambda\cdot\dot{x}+\omega_0^2\cdot x=p\cdot\sin(\Omega\cdot t) \quad (2)$$

where: $\lambda=\frac{\eta}{2m}$ $\qquad \omega_0^2=\frac{k}{m}$ $\qquad \frac{P(\text{t})}{m}=p\sin\Omega\cdot t$

1) Free oscillations

The perturbation isn't applied, i.e. $P(\text{t})=0$, which leads to: $\ddot{x}+2\lambda\cdot\dot{x}+\omega_0^2\cdot x=0$ (3)

As initial condition, we choose to let go of the mass with no initial velocity ($\dot{x}(0)=0$) from the position x_0 with respect to its initial condition. Three distinguishable cases can arise, leading to different solutions (fig. 3):

a) $\lambda^2<\omega_0^2$ ***weak damping***:

$$x=x_0\cdot e^{-\lambda t}\cdot\cos(\omega t-\varphi) \quad (4)$$

with: $\omega^2=\omega_0^2-\lambda^2$ and $\tan\varphi=-\frac{\lambda}{\omega}$

Define: $T=\frac{2\pi}{\omega}$ =pseudo-period and $\Delta=\lambda\cdot T$ = logarithmic decay

In particular: $\lambda=0$ (no friction), this is the standard *harmonic oscillator.*

$$x=x_0\cos(\omega_0 t-\varphi_0)$$

($\omega_0 = \sqrt{\frac{k}{m}}$ is the angular frequency of the oscillator, φ_0 = random phase shift)

b) $\lambda^2 = \omega_0^2$ ***critical damping***

$$x(t) = x_0 e^{-\lambda t}(1+\lambda t) \tag{5}$$

c) $\lambda^2 > \omega_0^2$ ***strong damping****:* non periodic motion

$$x = x_0 e^{-\lambda t}(C_1 e^{\omega t} + C_2 e^{-\omega t}) \tag{6}$$

With: $\omega^2 = \lambda^2 - \omega_0^2$ $\quad C_1 = \frac{\omega - \lambda}{2\omega}$ $\quad C_2 = \frac{\omega + \lambda}{2\omega}$

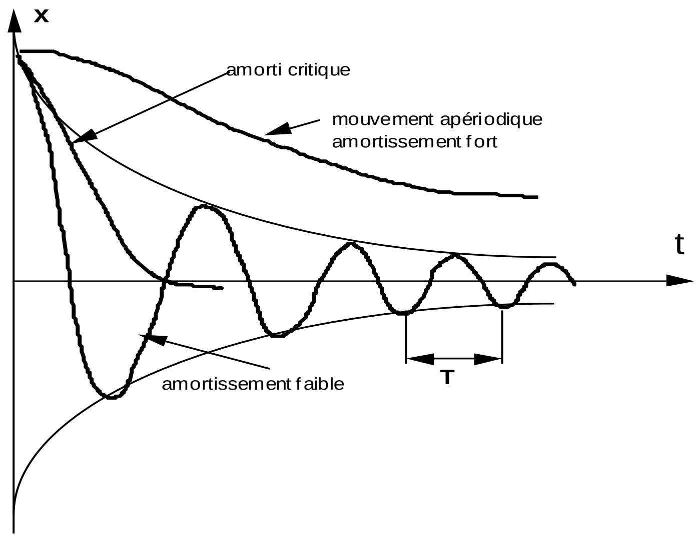


Fig. 3: Free oscillations in three different cases: weak, critical and strong damping.

2).Forced oscillations

Suppose the system is initially at rest ($x(0) = \dot{x}(0) = 0$), and you apply a sinusoidal perturbation ($p = p\sin(\Omega t)$). To simplify the expression, the phase shift is chosen to be zero (by redefining the time $t = 0$)

The solution to equation (2) is:

$x(t) = A(\Omega)\sin(\Omega t - \psi) + Ce^{-\lambda t}\cos(\omega t - \varphi)$	if $\lambda^2 < \omega_0^2$	weak damping
$x(t) = A(\Omega)\sin(\Omega t - \psi) + e^{-\lambda t}(C_1 + C_2 t)$	if $\lambda^2 = \omega_0^2$	critical damping
$x(t) = A(\Omega)\sin(\Omega t - \psi) + e^{-\lambda t}(C_1 e^{\omega t} + C_2\,\mathrm{e}^{-\omega t})$	if $\lambda^2 > \omega_0^2$	strong damping

permanent oscillations transitory oscillations with

$$\psi = \arctan\left(\frac{2 \cdot \lambda \cdot \Omega}{\omega_0^2 - \Omega^2}\right) \qquad \text{et} \qquad A(\Omega) = \frac{p}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4 \cdot \lambda^2 \cdot \Omega^2}} \tag{7}$$

The constants (C and φ or C_1 and C_2) are determined by the initial conditions in both cases.

The amplitude of the stationary forced oscillations $A(\Omega)$ as well as the phase shift $\psi(\Omega)$ depend on the value of Ω (Fig. 4).

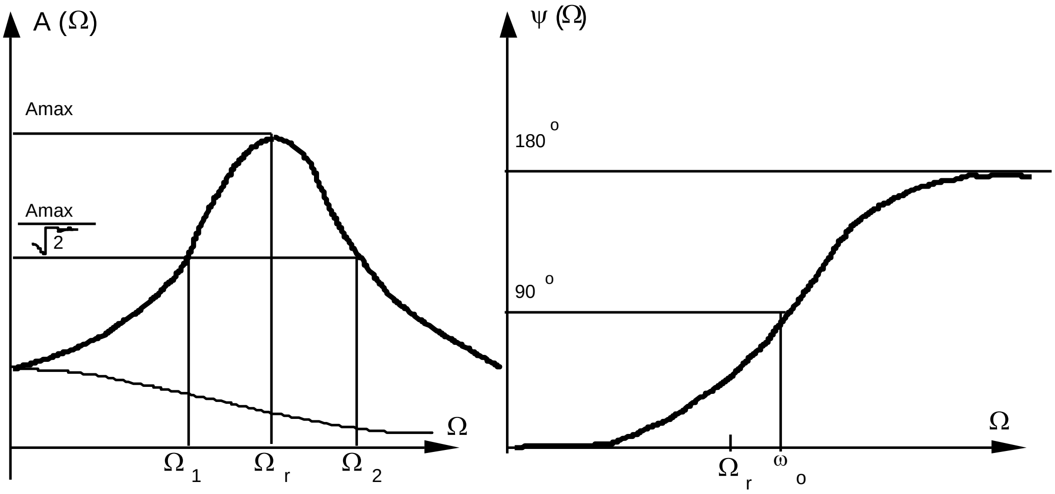


Fig. 4: Amplitude and phase shift of oscillations in a forced regime.

a) weak damping:

A typical resonance curve $A(\Omega)$ is characterized by:

Resonant angular frequency $\Omega_r = \sqrt{\omega_0^2 - 2\lambda^2}$ and peak width $\Delta\Omega = \Omega_2 - \Omega_1 = \dfrac{2\lambda\omega}{\Omega_r}$

Ω_1 and Ω_2 correspond to the values of the amplitude $\dfrac{A_{\max}}{\sqrt{2}}$, also given by an attenuation of 3dB with respect to the maximal value.

The Q-factor of the resonance is given by: $$Q = \frac{\Omega_r}{\Delta\Omega} = \frac{\Omega_r^2}{2\lambda\omega}$$

It is possible to calculate λ from following expression: $\lambda = \Omega_r \sqrt{\dfrac{1}{2}\left(\sqrt{\dfrac{G_{\max}^2}{G_{\max}^2 - 1}} - 1\right)}$

where $G_{\max} = \dfrac{u_{out}}{u_{in}}$ is the gain at the resonance.

Comment: The frequencies ω_0, ω and Ω_r are such that: $\omega_0^2 - \omega^2 = \omega^2 - \Omega_r^2 = \lambda^2$

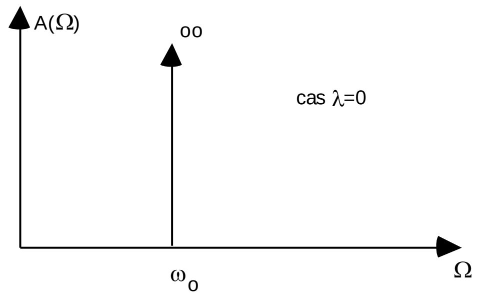


Fig. 5: Amplitude of the undamped forced oscillations

In the absence of friction $(\lambda = 0)$, the resonance curve $A(\Omega)$ is made of a singularity at $\Omega = \omega_0$ only (Fig. 5)

Transition period.

The second term of the equations in (7) represent *own damped oscillations* of angular frequency ω. As t increases, this term quickly loses importance with respect to the first term, which represents stationary forced oscillations of angular frequency Ω and phase shift ψ. As Ω and ω are close, the system oscillates at the *beat frequency* $\omega_B = |\omega - \Omega|$

b) strong damping: $\lambda^2 > \frac{\omega_0^2}{2}$

The curve is maximal in $\Omega = 0$: there are no forced oscillations, but a fixed displacement equal to $\frac{p}{\omega_0}$.

The solutions of the mechanical damped oscillator apply directly to the oscillating RCL circuit's differential equation (1). These oscillators have the following analogies:

"electrical"		*"mechanical"*
$\omega_0 = \frac{1}{\sqrt{L \cdot C}}$	⟷	$\sqrt{\frac{k}{m}}$
$2\lambda = \frac{1}{R \cdot C}$	⟷	$\frac{\eta}{m}$
U	⟷	x

Therefore, by studying an RCL circuit, we can observe and simulate a mechanical oscillator. In particular, for $U_P = 0$, if we give a current impulse to the system at $t = 0$, we will have results for "free oscillations".

For $\frac{U_P}{L \cdot C} = p \sin \Omega t$, we will be observing results for "forced oscillations".

III. REQUESTED EXPERIMENTS

1) Setup the RLC circuit from figure 1 with:

L = variable inductance up to 1 H

C = variable capacitor up to 0.22 $\mu\mathrm{F}$

R = variable resistance up to 1 $\mathrm{M\Omega}$

Use the NI ELVIS platform as a function generator with frequency measurement to power the circuit. Use the NI ELVIS oscilloscope to view and record numeric values.
A multimeter provides the internal resistance value of the inductor.

2) <u>Demonstrate non forced oscillations</u>

Choose a resonant frequency of approximately 7 kHz ($L \approx 5\cdot 10^{-2}$ H, $C \approx 10^{-2}$ μF) with $R \approx 900$ kΩ. Turn the function generator on “square wave”, with a low frequency of 100 Hz. Use the FGEN BNC output of the generator. The CH0 channel measures the generator signal and the CH1 channel measures the output signal of the circuit.

Explain what is observed on the oscilloscope, and verify. Reduce the resistance progressively. Demonstrate weak, critical and strong damping. Discuss qualitatively.
By fitting the obtained curve for weak damping, determine the damping coefficient λ. Compare this value to the theoretical one. Calculate the damping coefficient using the correction with the internal resistance:

$$\lambda' = \lambda + \frac{R_{in}}{2L}$$

Repeat for three different low damping circuits (varying R).

3) <u>Demonstrate forced oscillations</u>

Use the NI ELVIS platform as a function generator for “sine waves”. Connect CH0 to measure Uin and CH1 to measure Uout (Fig. 6).

For the same values of R, C and L as in the case for free oscillations (3 cases of weak damping) do the following:

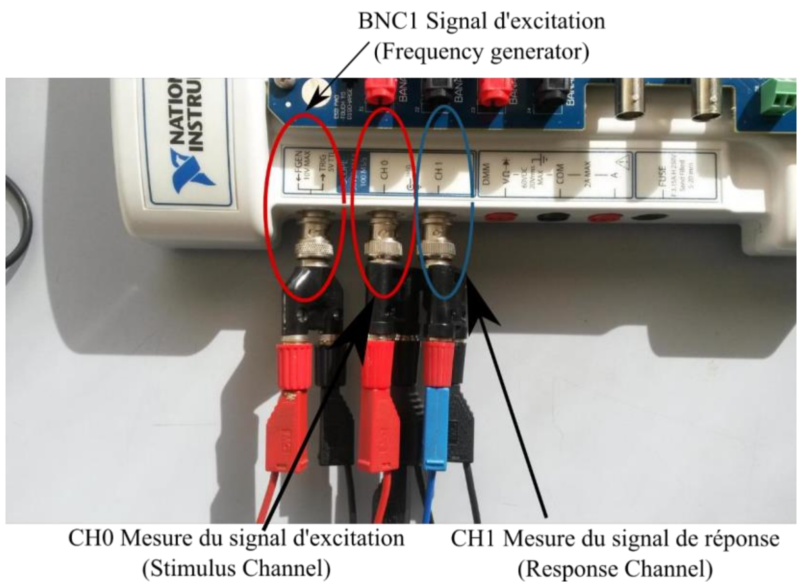
- Calculate the value of the resonant frequency.
- Measure and plot $G(\Omega)$ and $\psi(\Omega)$ using the LabVIEW program "Bode Analyzer" on the computer (use a sufficient number of steps per decade).
- Determine Ω_r, Q and λ

Compare the experimental values for λ with those determined for free oscillations in the three cases of weak damping. Compare with the corrected values of the damping coefficient. Discuss and explain the differences.

Demonstrate what happens if you exchange the position of the inductance L and capacitor C in the circuit?

Diagramme de Bode avec NI Elvis

Câblage



Programme « Bode Analyzer »

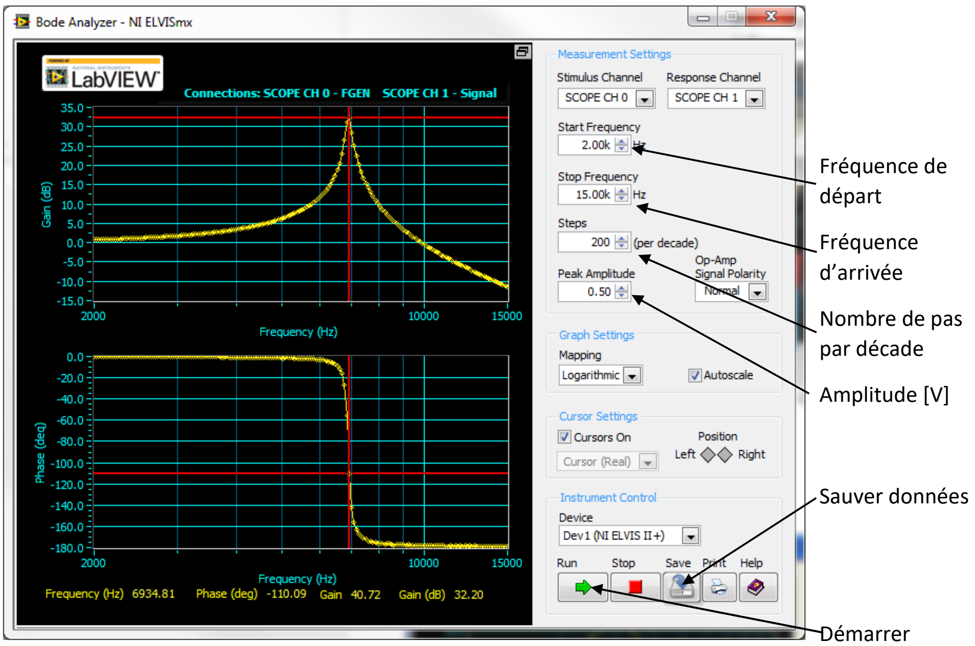


Fig. 6: Image of the setup with NI ELVIS