Nonlinear Analysis 196 (2020) 111787

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On the equation $A\,\nabla u + (\nabla u)^t\,A = G$

S. Bandyopadhyay [a], B. Dacorogna [b,*], D. Strütt [b]

[a] Department of Mathematics and Statistics, IISER Kolkata, Mohanpur 741246, India
[b] Section de Mathématiques, EPFL, 1015 Lausanne, Switzerland

ARTICLE INFO

Article history:
Received 16 October 2019
Accepted 25 January 2020
Communicated by Vicentiu D. Radulescu

Keywords:
Symmetrized gradient
Poincare lemma
Dirichlet problem

ABSTRACT

Let $\Omega \subset \mathbb{R}^n$ be an open set, $A \in \mathbb{R}^{n\times n}$ and $G : \Omega \to \mathbb{R}^{n\times n}$ be given. We look for a solution $u : \Omega \to \mathbb{R}^n$ of the equation

$$A\,\nabla u + (\nabla u)^t\,A = G.$$

We also study the associated Dirichlet problem.

1. Introduction

Given $\Omega \subset \mathbb{R}^n$ an open set, $G : \Omega \to \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$, we wish to find $u : \Omega \to \mathbb{R}^n$ verifying

$$A\,\nabla u + (\nabla u)^t\,A = G \quad \text{in } \Omega$$

sometimes coupled with

– either Dirichlet data, i.e. $u = u_0$ on $\partial\Omega$ (replacing G by $G - A\,\nabla u_0 - (\nabla u_0)^t\,A$ we can and will always assume that $u_0 = 0$)

– or Cauchy data

$$u(x_0) = c_0 \quad \text{and} \quad \nabla u(x_0) = C_0,$$

where $(x_0, c_0, C_0) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ are given and satisfy the natural condition

$$A\,C_0 + C_0^t\,A = A\,\nabla u(x_0) + (\nabla u(x_0))^t\,A = G(x_0).$$

Observe immediately that the Cauchy problem follows at once from the unconstrained problem. Indeed first find a solution v of the unconstrained problem $A\,\nabla v + (\nabla v)^t\,A = G$ and then recover a solution of the Cauchy problem by setting

$$u(x) = v(x) + (C_0 - \nabla v(x_0))(x - x_0) + c_0 - v(x_0).$$

* Corresponding author.
E-mail addresses: saugata.bandyopadhyay@iiserkol.ac.in (S. Bandyopadhyay), bernard.dacorogna@epfl.ch (B. Dacorogna), david.strutt@epfl.ch (D. Strütt).

https://doi.org/10.1016/j.na.2020.111787
0362-546X/

This equation has been much studied when $A = I$ (or more generally A is symmetric and invertible) or when A is skew-symmetric and invertible (in this case the dimension n must be even). Setting $v = A\,u$ the equation, when A is symmetric, becomes

$$\nabla v + (\nabla v)^t = G$$

(i.e. the *symmetrized gradient* is prescribed, see, for example, [1–3,11]), while, when A is skew-symmetric, the equation is

$$\nabla v - (\nabla v)^t = G$$

or, by abuse of notations, $\operatorname{curl} v = G$; in other words the problem is then reduced to the classical *Poincaré lemma* for 1-forms (see, for example, [4]). The first case is more elementary, notably from the point of view of regularity, which is straightforward, and uniqueness (the solution being unique up to an affine map whose gradient is a constant skew-symmetric matrix). The case of Poincaré lemma is more involved, the existence and regularity being a consequence of Hodge decomposition and of elliptic regularity; the solution being unique, in this case, up to a gradient of a scalar function. Both problems arise in various areas of physics; for example in elasticity when A is symmetric while when A is skew symmetric in problems involving the divergence or the curl operators.

The main features of the present article, with respect to the classical studies, are the following. We do not assume, in general, that the matrix A is invertible and we also consider the case where A is neither symmetric, nor skew-symmetric. We also study problems for general domains, not necessarily simply connected, as well as Dirichlet problems in general domains. Although we deal here only with C^r and $C^{r,\alpha}$ spaces, our analysis carries over in a straightforward way to the Sobolev setting.

We end this introduction with an observation that will not be explicitly used in the following developments, but may shed a different light on our analysis. We recall (see [6]) that the operator $\mathcal{L}$ defined by

$$\mathcal{L}u = A\,\nabla u + (\nabla u)^t\,A$$

is *elliptic* if, for every $\xi \in \mathbb{R}^n \setminus \{0\}$, the system of algebraic equations

$$\mathcal{A}\,(\lambda, \xi) = (A\,\lambda) \otimes \xi + \xi \otimes \left(A^t\,\lambda\right) = 0$$

has $\lambda = 0 \in \mathbb{R}^n$ as the only solution. It is then easy to see that $\mathcal{L}$ is elliptic if and only if the symmetric part of A, denoted A_s , is invertible. In particular if A is skew-symmetric, then $\mathcal{L}$ *is not* elliptic.

2. Preliminaries

2.1. General notations

We will use the following notations in this article (see [4] for more details.).

(i) Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be the set of $(n \times n)$-matrices with real entries.

– For every $i, j = 1, \ldots, n$, A^{ij} denotes the (i, j)th element of A. Furthermore, we write $A^{i,*}$ and $A^{*,j}$ to denote the ith row and jth column of A respectively.

– We write $\mathbb{R}^{n\times n}_s$ and $\mathbb{R}^{n\times n}_a$ for the set of symmetric and skew-symmetric $(n \times n)$-matrices, respectively. Accordingly A_s and A_a denote, respectively, the symmetric and skew-symmetric parts of A, namely

$$A_s = \frac{1}{2}\left(A + A^t\right) \quad \text{and} \quad A_a = \frac{1}{2}\left(A - A^t\right).$$

(ii) For $a, b \in \mathbb{R}^n$, we denote the tensor product of a and b by $a \otimes b \in \mathbb{R}^{n\times n}$. Note that, for every $A \in \mathbb{R}^{n\times n}$ and $a, b, c \in \mathbb{R}^n$, the following relations are easy to verify

$$(b \otimes a) = (a \otimes b)^t\,, \quad (a \otimes b)\,c = \langle b; c\rangle\,a, \quad A\,(b \otimes c) = (A\,b) \otimes c, \quad (b \otimes c)\,A = b \otimes \left(A^t c\right).$$

Moreover, we define the wedge product of a and b, denoted by $a \wedge b$, as

$$a \wedge b = a \otimes b - b \otimes a.$$

(iii) In the sequel $\{e_1, \ldots, e_n\}$ will denote the standard Euclidean basis.

(iv) Let $m \in \mathbb{N}$ and $n = 2m$. We let J_m (or simply J when there is no scope of confusion) be the standard symplectic matrix, namely J_m is a skew-symmetric $(n \times n)$-matrix defined as

$$J_m = \operatorname{diag}\underbrace{\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)}_{m-\text{times}}.$$

(v) We write partial differentiation with respect to x_i, $i = 1, \ldots, n$, of a function f as

$$\partial_i f = \frac{\partial f}{\partial x_i}.$$

(vi) Let $\Omega \subset \mathbb{R}^n$ be open and $\omega : \Omega \to \mathbb{R}^n$ be a differentiable vector field. We define

$$\operatorname{div} \omega = \sum_{k=1}^{n} \partial_k \omega^k \quad \text{and} \quad \operatorname{curl} \omega = \nabla \omega - (\nabla \omega)^t = \left(\partial_j \omega^i - \partial_i \omega^j\right)^{1 \le i,j \le n}.$$

Furthermore, on identifying ω with a differential 1-form, $\operatorname{curl} \omega$ can be identified, up to a change of sign, with the exterior derivative $d\omega$, where

$$d\omega = \sum_{1 \le i < j \le n} \left(\partial_i \omega^j - \partial_j \omega^i\right) dx^i \wedge dx^j \in \Lambda^2(\mathbb{R}^n) \sim \mathbb{R}^{\frac{n(n-1)}{2}}.$$

(vii) Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. For every $x \in \partial\Omega$, $\nu(x)$ denotes the outward unit normal at x and, when there is no ambiguity, we write only ν.

(viii) Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. We define the sets of *harmonic forms* with vanishing tangential and normal parts, respectively as

$$\mathcal{H}_T(\Omega; \mathbb{R}^n) = \left\{\omega \in C^\infty\left(\overline{\Omega}; \mathbb{R}^n\right) : \operatorname{curl} \omega = 0,\ \operatorname{div} \omega = 0,\ \nu \wedge \omega = 0 \text{ on } \partial\Omega\right\}$$

$$\mathcal{H}_N(\Omega; \mathbb{R}^n) = \left\{\omega \in C^\infty\left(\overline{\Omega}; \mathbb{R}^n\right) : \operatorname{curl} \omega = 0,\ \operatorname{div} \omega = 0,\ \langle \nu; \omega \rangle = 0 \text{ on } \partial\Omega\right\}.$$

Note that, if Ω is simply connected $\mathcal{H}_T(\Omega; \mathbb{R}^n) = \mathcal{H}_N(\Omega; \mathbb{R}^n) = \{0\}$.

2.2. On the equation $\nabla u = F$

Theorem 1. *Let $r, N \ge 0$ and $n \ge 2$ be integers. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and smooth. Let $F = \left(F^{ij}\right)^{1 \le i,j \le n} \in C^r\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$ and $\{a_1, \ldots, a_N\} \subset \mathbb{R}^n$ be linearly independent. Then there exists $u \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying, in Ω,*

$$\nabla u = F \quad \text{and} \quad \langle u; a_1 \rangle = \cdots = \langle u; a_N \rangle = 0 \tag{1}$$

if and only if

(i) *in Ω and for every $i, j, k = 1, \ldots, n$,*

$$\operatorname{curl} F^{i,*} = 0 \quad \text{i.e.} \quad \partial_j F^{ik} - \partial_k F^{ij} = 0$$

(ii) *for every $i = 1, \ldots, n$ and for every $\chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$*

$$\int_\Omega \left\langle F^{i,*}; \chi \right\rangle = 0 \quad \text{i.e.} \quad \sum_{j=1}^{n} \int_\Omega F^{ij} \chi^j = 0$$

(iii) *for every* $k = 1, \ldots, N$ *and* $x \in \Omega$

$$(F(x))^t a_k = 0.$$

Moreover, the solution of (1) *is unique, up to a constant* $c \in \mathbb{R}^n$ *verifying*

$$\langle c; a_1 \rangle = \cdots = \langle c; a_N \rangle = 0.$$

Proof. If $N = 0$, the result is standard (see, for example, Theorem 8.3 in [4] applied component by component). When $N \geq 1$, the theorem follows at once from the combination of the case $N = 0$ and (iii). ■

We next turn to the Dirichlet problem.

Theorem 2. *Let* $r, N \geq 0$ *and* $n \geq 2$ *be integers. Let* $\Omega \subset \mathbb{R}^n$ *be open, bounded, connected and smooth. Let* $F = \left(F^{ij}\right)^{1 \leq i,j \leq n} \in C^r\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$, $g \in C^{r+1}\left(\partial\Omega; \mathbb{R}^n\right)$ *and* $\{a_1, \ldots, a_N\} \subset \mathbb{R}^n$ *be linearly independent. There exists* $u \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ *satisfying*

$$\begin{cases} \nabla u = F & \text{in } \Omega \\ \langle u; a_1 \rangle = \cdots = \langle u; a_N \rangle = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2}$$

if and only if

(i) *in* Ω *and for every* $i, j, k = 1, \ldots, n$,

$$\operatorname{curl} F^{i,*} = 0 \text{ in } \Omega \quad \text{i.e.} \quad \partial_j F^{ik} - \partial_k F^{ij} = 0$$

(ii) *on* $\partial\Omega$ *and for every* $i, j, k = 1, \ldots, n$,

$$(F - \nabla g)^{ik} \nu^j - (F - \nabla g)^{ij} \nu^k = 0$$

(iii) *for every* $i = 1, \ldots, n$ *and for every* $\chi \in \mathcal{H}_T(\Omega; \mathbb{R}^n)$,

$$\int_\Omega \langle F^{i,*}; \chi \rangle = \int_{\partial\Omega} g^i \langle \nu; \chi \rangle \quad \text{i.e.} \quad \sum_{j=1}^n \int_\Omega F^{ij} \chi^j = \int_{\partial\Omega} g^i \langle \nu; \chi \rangle$$

(iv) *on* $\partial\Omega$

$$\langle g; a_1 \rangle = \cdots = \langle g; a_N \rangle = 0$$

(v) *for every* $k = 1, \ldots, N$ *and* $x \in \Omega$,

$$(F(x))^t a_k = 0.$$

Moreover, the solution of (2) *is unique.*

Proof. If $N = 0$, the theorem follows from standard results (see, for example, Theorem 8.16 in [4] applied component by component); while if $N \geq 1$, the result easily follows from the case $N = 0$ and from (v). ■

3. The symmetric case

3.1. Preliminaries and notations

The following form will play a crucial role in the subsequent discussion.

Notation 3. *Let* $\Omega \subset \mathbb{R}^n$ *be open, bounded, connected and smooth,* $r \geq 1$ *be an integer and* $G \in C^r\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$. *For every* $i, j, k = 1, \ldots, n$, *we define*

$$M_k^{ij} = \partial_j G^{ik} - \partial_i G^{jk}.$$

Furthermore, for every $i, j, k = 1, \ldots, n$, *let* $\mathbf{M}^{ij} \in C^{r-1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ *and* $\mathbf{M}_k \in C^{r-1}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ *be defined as*

$$\left(\mathbf{M}^{ij}\right)^k = \left(\mathbf{M}_k\right)^{ij} = M_k^{ij}.$$

Remark 4. Note that $\mathbf{M}_k$ is skew-symmetric. Observe also that

$$\operatorname{curl} \mathbf{M}^{ij} = 0 \quad \text{for every } i, j = 1, \ldots, n,$$

means that, for every $i, j, k, l = 1, \ldots, n$,

$$\partial_l M_k^{ij} - \partial_k M_l^{ij} = \partial_{jl} G^{ik} - \partial_{il} G^{jk} + \partial_{ik} G^{jl} - \partial_{jk} G^{il} = 0.$$

The following definition will be useful to handle domains that are not simply connected.

Definition 5 (*Dirichlet and Neumann Potentials*)**.** Let r be an integer, $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and $f \in C^{r,\alpha}(\overline{\Omega})$. We write $\mathcal{D}(f)$, $\mathcal{N}(f) \in C^{r+2,\alpha}(\overline{\Omega})$ to denote the *Dirichlet* and *Neumann potentials* of f respectively. In other words,

$$\begin{cases} \Delta\left[\mathcal{D}(f)\right] = f & \text{in } \Omega \\ \mathcal{D}(f) = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta\left[\mathcal{N}(f)\right] = f - \frac{1}{\operatorname{meas}\Omega}\int_\Omega f & \text{in } \Omega \\ \partial_\nu\left[\mathcal{N}(f)\right] = 0 & \text{on } \partial\Omega \\ \int_\Omega \left[\mathcal{N}(f)\right] = 0 & \end{cases}$$

where $\partial_\nu[\cdot]$ denotes the normal derivative. One defines Dirichlet and Neumann potentials of a vector field componentwise.

The following selection principle will help us construct solutions in the singular case.

Lemma 6. *Let* r *be an integer,* $\Omega \subset \mathbb{R}^n$ *be open and* $A \in \mathbb{R}^{n\times n}$. *Let* $v \in C^r\left(\overline{\Omega}; \mathbb{R}^n\right)$ *satisfy* $v(\Omega) \subset \operatorname{im} A$. *Then there exists* $u \in C^r\left(\overline{\Omega}; \mathbb{R}^n\right)$ *satisfying*

$$v = A\, u \quad \text{in } \overline{\Omega}.$$

Furthermore,

(i) $u = 0$ *on* $\partial\Omega$, *if* $v = 0$ *on* $\partial\Omega$.

(ii) *For* $(x_0, c_0, C_0) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, *if* $v(x_0) = A\, c_0$ *and* $\nabla v(x_0) = A\, C_0$, *then*

$$u(x_0) = c_0 \quad \text{and} \quad \nabla u(x_0) = C_0.$$

Proof. We assume that $A \neq 0$. We first find invertible $P, Q \in \mathbb{R}^{n\times n}$ such that $A = P D_m Q$, where $m = \operatorname{rank} A$ and $D_m \in \mathbb{R}^{n\times n}$ is given by

$$D_m = \operatorname{diag}(\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{n-m}).$$

It is then easy to see that it is enough to prove the lemma for $A = D_m$. We define $u = \left(u^1, \ldots, u^n\right) \in C^r\left(\overline{\Omega}; \mathbb{R}^n\right)$ by

$$u^j = \begin{cases} v^j & \text{if } 1 \leqslant j \leqslant m \\ 0 & \text{if } m < j \leqslant n. \end{cases}$$

Clearly $v = A\, u$ and u satisfies the extra statements (i) and (ii). ■

We will also need the following algebraic lemma.

Lemma 7. *Let $A \in \mathbb{R}^{n \times n}$ be symmetric and let $\nu \in \mathbb{R}^n$ with $|\nu| = 1$. The two following conditions are then equivalent*

$$\text{(i)} \qquad A = (A\,\nu) \otimes \nu + \nu \otimes (A\,\nu) - \langle A\,\nu; \nu \rangle\, \nu \otimes \nu$$

$$\text{(ii)} \qquad \left(A^{ik}\nu^j - A^{jk}\nu^i\right)\nu^l = \left(A^{il}\nu^j - A^{jl}\nu^i\right)\nu^k, \quad \textit{for every } i, j, k, l = 1, \ldots, n.$$

Proof. *Step 1:* (i) $\Rightarrow$ (ii). By hypothesis we have, for every $k = 1, \ldots, n$,

$$A\,e_k = \nu^k\,(A\,\nu) + \langle A\,\nu; e_k \rangle\, \nu - \nu^k\, \langle A\,\nu; \nu \rangle\, \nu.$$

Therefore

$$A\,e_k \wedge \nu = (A\,\nu \wedge \nu)\,\nu^k, \quad \text{for every } k = 1, \ldots, n,$$

which implies that

$$(A\,e_k \wedge \nu)\,\nu^l = (A\,\nu \wedge \nu)\,\nu^k \nu^l = (A\,e_l \wedge \nu)\,\nu^k, \quad \text{for every } k, l = 1, \ldots, n,$$

from where the lemma follows.

Step 2: (ii) $\Rightarrow$ (i). Conversely assume that, for every $i, j, k, l = 1, \ldots, n$,

$$\left(A^{ik}\nu^j - A^{jk}\nu^i\right)\nu^l = \left(A^{il}\nu^j - A^{jl}\nu^i\right)\nu^k.$$

Then, for every $j, k = 1, \ldots, n$,

$$\sum_{i,l=1}^{n} \left(A^{ik}\nu^j - A^{jk}\nu^i\right)\nu^l\nu^l\nu^i = \sum_{i,l=1}^{n} \left(A^{il}\nu^j - A^{jl}\nu^i\right)\nu^k\nu^l\nu^i.$$

Since A is symmetric, we infer that, for every $j, k = 1, \ldots, n$,

$$(A\,\nu)^k\,\nu^j - A^{jk} = \langle A\,\nu; \nu \rangle\, \nu^j \nu^k - (A\,\nu)^j\,\nu^k$$

which is exactly our claim. ■

3.2. The unconstrained problem

Theorem 8. *Let $n, r \geqslant 2$ be integers, $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and smooth. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $G \in C^r\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$.*

Part 1 (existence). *The following statements are equivalent.*

(i) *There exists $u \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying*

$$A\,\nabla u + (\nabla u)^t\,A = G \quad \textit{in } \Omega. \tag{3}$$

(ii) *The following properties hold.*

– G is symmetric,

$$\langle G(x)\,a; a' \rangle = 0 \quad \textit{for every } x \in \Omega \textit{ and for every } a, a' \in \ker A \tag{4}$$

– in Ω and for every $i, j, k, l = 1, \ldots, n$

$$\operatorname{curl} \mathbf{M}^{ij} = 0 \quad \textit{i.e.} \quad \partial_{jl}G^{ik} - \partial_{il}G^{jk} + \partial_{ik}G^{jl} - \partial_{jk}G^{il} = 0 \tag{5}$$

– *in* Ω *and for every* $a \in \ker A$ *and* $i, k = 1, \ldots, n$

$$[\partial_k G - \mathbf{M}_k]\, a = 0 \quad \text{i.e.} \quad \sum_{j=1}^{n} \left[\partial_k G^{ij} - \partial_j G^{ik} + \partial_i G^{jk}\right] a^j = 0 \tag{6}$$

– *for every* $\chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$

$$\int_\Omega \langle \mathbf{M}^{ij}; \chi \rangle = \sum_{k=1}^{n} \int_\Omega \left(\partial_j G^{ik} - \partial_i G^{jk}\right) \chi^k = 0, \quad \text{for every } i, j = 1, \ldots, n \tag{7}$$

$$\sum_{j=1}^{n} \int_\Omega \left[\langle \mathbf{M}^{ij}; \nabla\left[\mathcal{N}\left(\chi^j\right)\right]\rangle - G^{ij}\chi^j\right] = 0, \quad \text{for every } i = 1, \ldots, n. \tag{8}$$

Part 2 (uniqueness). *If* A *is invertible, then the solution is unique up to an affine map of the form*

$$u(x) = C\,x + c \quad \text{with } A\,C + C^t A = 0.$$

Remark 9. *(i)* Eq. (3) may not have unique solution (up to an affine map) if A is not invertible. To see this, choose $A = \operatorname{diag}(1, 0)$ and $G = 0$, so that the equation becomes

$$A\,\nabla u + (\nabla u)^t A = \begin{pmatrix} 2\,\partial_1 u^1 & \partial_2 u^1 \\ \partial_2 u^1 & 0 \end{pmatrix} = G = 0.$$

This implies that u^1 is constant. However u^2 is free; thus the non-uniqueness.

(ii) When A is invertible, see Proposition 2.8 in [11] for related results.

As an immediate corollary, we have the following standard result, see [1,3] and references therein.

Corollary 10. *Let* $n, r \geqslant 2$ *be integers,* $\Omega \subset \mathbb{R}^n$ *be open, bounded, simply connected and smooth. Let* $G \in C^r\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. *There exists* $u \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ *satisfying*

$$\nabla u + (\nabla u)^t = G \quad \text{in } \Omega \tag{9}$$

if and only if G *is symmetric and*

$$\operatorname{curl} \mathbf{M}^{ij} = 0 \quad \text{in } \Omega \text{ and for every } i, j = 1, \ldots, n.$$

Moreover, the solution of (9) *is unique up to an affine map of the form*

$$u(x) = C\,x + c \quad \text{with} \quad C + C^t = 0.$$

Proof of Theorem 8. *Step 1 (Necessity).* Let $u \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfy (3). Then $v = A\,u$ verifies

$$\left[\nabla v + (\nabla v)^t = G\right] \quad \text{and} \quad \left[\langle v; a \rangle = 0 \text{ for every } a \in \ker A\right]. \tag{10}$$

Clearly G must be symmetric and (4) holds. We then prove the other statements.

Step 1.1 (proof of (5)). Note that, for every $i, j, k = 1, \ldots, n$ and in Ω,

$$M_k^{ij} = \partial_j G^{ik} - \partial_i G^{jk} = \partial_j\left(\partial_k v^i + \partial_i v^k\right) - \partial_i\left(\partial_k v^j + \partial_j v^k\right) = \partial_k\left(\partial_j v^i - \partial_i v^j\right). \tag{11}$$

It immediately follows that, for every $i, j, k, l = 1, \ldots, n$ and in Ω,

$$\partial_l M_k^{ij} = \partial_k M_l^{ij} \quad \text{i.e.} \quad \operatorname{curl} \mathbf{M}^{ij} = 0.$$

Step 1.2 (proof of (6)). Using the second statement in (10), we infer that, for every $a \in \ker A$, $(\nabla v)^t a = 0$ and, since $\nabla v + (\nabla v)^t = G$, we obtain

$$0 = (\nabla v)^t a = (G - \nabla v)\, a \quad \text{in } \Omega. \tag{12}$$

Hence, for every $k = 1, \ldots, n$ and for every $a \in \ker A$,

$$(\partial_k G)\, a = (\partial_k (\nabla v))\, a \quad \text{in } \Omega. \tag{13}$$

From (11) we find, for every $i, k = 1, \ldots, n$ and every $a \in \ker A$,

$$\langle \mathbf{M}_k a; e_i \rangle = \sum_{j=1}^{n} M_k^{ij} a^j = \sum_{j=1}^{n} \left[\partial_k \left(\partial_j v^i - \partial_i v^j\right)\right] a^j \quad \text{in } \Omega.$$

Therefore, it follows from (12) and (13) that, for every $k = 1, \ldots, n$ and for every $a \in \ker A$,

$$\mathbf{M}_k a = \partial_k \left[(\nabla v)\, a\right] - \partial_k \left[(\nabla v)^t a\right] = (\partial_k (\nabla v))\, a = (\partial_k G)\, a \quad \text{in } \Omega.$$

Step 1.3 (proof of (7)). It follows from (11) that, for every $i, j = 1, \ldots, n$ and $\chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$,

$$\begin{aligned}\int_\Omega \langle \mathbf{M}^{ij}; \chi \rangle &= \sum_{k=1}^{n} \int_\Omega \partial_k \left(\partial_j v^i - \partial_i v^j\right) \chi^k \\ &= -\int_\Omega \left(\partial_j v^i - \partial_i v^j\right) \operatorname{div} \chi + \int_{\partial\Omega} \left(\partial_j v^i - \partial_i v^j\right) \langle \chi; \nu \rangle = 0.\end{aligned}$$

Step 1.4 (proof of (8)). We first observe that if $\chi = \left(\chi^j\right)^{1 \le j \le n} \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$, then

$$\int_\Omega \chi^j = \int_\Omega \langle \chi; \nabla (x_j) \rangle = 0 \quad \text{for every } j = 1, \ldots, n.$$

This implies that

$$\Delta \left[\mathcal{N}\left(\chi^j\right)\right] = \chi^j \quad \text{for every } j = 1, \ldots, n.$$

We now show (8). Note that, using (11) and after integration by parts, we obtain, for every $i = 1, \ldots, n$ and $\chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$,

$$\begin{aligned}\sum_{j=1}^{n} \int_\Omega \left\langle \mathbf{M}^{ij}; \nabla \left[\mathcal{N}\left(\chi^j\right)\right] \right\rangle &= \sum_{j,k=1}^{n} \int_\Omega \partial_k \left(\partial_j v^i - \partial_i v^j\right) \partial_k \left[\mathcal{N}\left(\chi^j\right)\right] \\ &= -\sum_{j=1}^{n} \int_\Omega \left(\partial_j v^i - \partial_i v^j\right) \Delta \left[\mathcal{N}\left(\chi^j\right)\right] \\ &\quad + \sum_{j,k=1}^{n} \int_{\partial\Omega} \left(\partial_j v^i - \partial_i v^j\right) \partial_k \left[\mathcal{N}\left(\chi^j\right)\right] \nu^k\end{aligned}$$

and thus

$$\sum_{j=1}^{n} \int_\Omega \left\langle \mathbf{M}^{ij}; \nabla \left[\mathcal{N}\left(\chi^j\right)\right] \right\rangle = -\sum_{j=1}^{n} \int_\Omega \left(\partial_j v^i - \partial_i v^j\right) \chi^j. \tag{14}$$

Since, for every $i = 1, \ldots, n$ and $\chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$,

$$\sum_{j=1}^{n} \int_\Omega \left(\partial_j v^i - \partial_i v^j\right) \chi^j + \sum_{j=1}^{n} \int_\Omega G^{ij} \chi^j = 2 \sum_{j=1}^{n} \int_\Omega \partial_j v^i \chi^j = -2 \int_\Omega (\operatorname{div} \chi)\, v^i + 2 \int_{\partial\Omega} v^i \langle \chi; \nu \rangle = 0$$

we have that (8) follows from (14).

Step 2 (Sufficiency). Let $\{a_1, \ldots, a_N\} \subset \mathbb{R}^n$ be linearly independent such that

$$\ker A = \operatorname{span}\{a_1, \ldots, a_N\}.$$

Step 2.1. Let $x_0 \in \Omega$. Note that (4) (i.e. $\langle G(x_0)\, a; a' \rangle = 0$) coupled with Proposition 33 guarantees that there exists $C_0 \in \mathbb{R}^{n \times n}$ verifying

$$A\, C_0 + C_0^t\, A = G(x_0).$$

Step 2.2. For every $i, j = 1, \ldots, n$, we use the vector valued version (see, for example, Theorem 8.3 in [4]) of Theorem 1 (with $N = 0$) to find $\Phi \in C^r\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$ solving

$$\nabla \Phi^{ij} = \mathbf{M}^{ij} \quad \text{in } \Omega; \tag{15}$$

this is possible in view of (5) and (7). Note that, since $M_k^{ij} = -M_k^{ji}$, we can choose Φ to be skew-symmetric and thus we have that

$$\Phi^t = -\Phi \quad \text{and} \quad \partial_k \Phi = \mathbf{M}_k \quad \text{in } \Omega \text{ and for every } k = 1, \ldots, n. \tag{16}$$

We claim that

$$[G(x) - \Phi(x)]\, a = [G(x_0) - \Phi(x_0)]\, a, \quad \text{for every } x \in \Omega,\ a \in \ker A. \tag{17}$$

To see this, let $a \in \ker A$ be fixed. It follows from (6) and (16) that, for every $k = 1, \ldots, n$,

$$[\partial_k (G - \Phi)]\, a = [(\partial_k G) - \mathbf{M}_k]\, a = 0, \quad \text{in } \Omega$$

and hence we obtain (17).

Step 2.3. Invoking again Theorem 1, we find $v \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying, in Ω,

$$\begin{cases} \nabla v = \frac{1}{2}(G + \Phi) - \frac{1}{2}(G(x_0) + \Phi(x_0)) + A\, C_0 \\ \langle v; a_1 \rangle = \cdots = \langle v; a_N \rangle = 0. \end{cases} \tag{18}$$

Let us now verify that the hypotheses of Theorem 1 indeed hold.

(i) By (16), for every $i, j, k = 1, \ldots, n$, we have

$$\partial_k\left(G^{ij} + \Phi^{ij}\right) = \partial_k G^{ij} + \partial_j G^{ik} - \partial_i G^{jk} = \partial_j G^{ik} + \partial_k G^{ij} - \partial_i G^{kj} = \partial_j\left(G^{ik} + \Phi^{ik}\right).$$

(ii) Appealing to (8), (15), $\int_\Omega \chi^j = 0$ and after integration by parts, we get that, for every $i = 1, \ldots, n$ and $\chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$,

$$\sum_{j=1}^n \int_\Omega G^{ij}\, \chi^j = \sum_{j=1}^n \int_\Omega \left\langle \mathbf{M}^{ij}; \nabla\left[\mathcal{N}\left(\chi^j\right)\right]\right\rangle = \sum_{j=1}^n \int_\Omega \left\langle \nabla \Phi^{ij}; \nabla\left[\mathcal{N}\left(\chi^j\right)\right]\right\rangle = -\sum_{j=1}^n \int_\Omega \Phi^{ij} \chi^j$$

i.e.

$$\int_\Omega (G + \Phi)\, \chi = 0, \quad \text{for every } \chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n).$$

Since $\int_\Omega \chi = 0$ for every $\chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$, we infer, invoking the previous identity, that

$$\int_\Omega \left(\frac{1}{2}(G + \Phi) - \frac{1}{2}(G(x_0) + \Phi(x_0)) + A\, C_0\right) \chi = 0, \quad \text{for every } \chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$$

as wished.

(iii) We finally have to show that for every $k = 1, \ldots, N$ and $x \in \Omega$

$$(F(x))^t\, a_k = 0 \quad \text{where} \quad F = \frac{1}{2}(G(x) + \Phi(x)) - \frac{1}{2}(G(x_0) + \Phi(x_0)) + A\, C_0.$$

Using (16), (17) and the fact that G is symmetric, we obtain, for every $k = 1, \ldots, N$ and $x \in \Omega$,

$$(F(x))^t a_k = \left[\frac{1}{2}(G(x) + \Phi(x)) - \frac{1}{2}(G(x_0) + \Phi(x_0)) + A\, C_0\right]^t a_k = C_0^t A\, a_k = 0.$$

Step 2.4. In view of Step 2.1 and (18), we have just shown that, in Ω,

$$\nabla v + (\nabla v)^t = G \quad \text{and} \quad \langle v; a_1\rangle = \cdots = \langle v; a_N\rangle = 0.$$

Step 2.5. Finally, we appeal to Lemma 6 to find $u \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that

$$v = A\, u \quad \text{in } \Omega.$$

Then, u satisfies (3). When A is invertible, the uniqueness of solution follows from Theorem 27. This finishes the proof. ■

3.3. The Dirichlet problem

Theorem 11. *Let $n, r \geqslant 2$ be integers, $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and smooth, $A \in \mathbb{R}^{n\times n}$ be symmetric and $G \in C^r\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. The following statements are equivalent.*

(i) *There exists $u \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying*

$$\begin{cases} A\,\nabla u + (\nabla u)^t A = G & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{19}$$

(ii) *The following properties hold.*

– *G is symmetric*

– *in Ω and for every $i, j, k, l = 1, \ldots, n$*

$$\operatorname{curl} \mathbf{M}^{ij} = 0 \quad \text{i.e.} \quad \partial_{jl} G^{ik} - \partial_{il} G^{jk} + \partial_{ik} G^{jl} - \partial_{jk} G^{il} = 0 \tag{20}$$

– *in Ω and for every $a \in \ker A$ and $i, k = 1, \ldots, n$*

$$[\partial_k G - \mathbf{M}_k]\, a = 0 \quad \text{i.e.} \quad \sum_{j=1}^{n} \left[\partial_k G^{ij} - \partial_j G^{ik} + \partial_i G^{jk}\right] a^j = 0 \tag{21}$$

– *on $\partial\Omega$ and for every $i, j = 1, \ldots, n$*

$$G = (G\,\nu) \otimes \nu + \nu \otimes (G\,\nu) - \langle G\,\nu; \nu\rangle\, \nu \otimes \nu \quad \text{i.e.} \quad G^{ik}\nu^j\nu^l - G^{jk}\nu^i\nu^l + G^{jl}\nu^i\nu^k - G^{il}\nu^j\nu^k = 0 \tag{22}$$

$$\nabla\left((G\,\nu \wedge \nu)^{ij}\right) \wedge \nu = \mathbf{M}^{ij} \wedge \nu \tag{23}$$

– *for every $\chi \in \mathcal{H}_T(\Omega; \mathbb{R}^n)$ and for every $i, j = 1, \ldots, n$,*

$$\int_\Omega \langle \mathbf{M}^{ij}; \chi\rangle = \int_{\partial\Omega} (G\,\nu \wedge \nu)^{ij} \langle \chi; \nu\rangle \tag{24}$$

$$\sum_{j=1}^{n} \int_\Omega \left[\langle \mathbf{M}^{ij}; \nabla\left[\mathcal{D}\left(\chi^j\right)\right]\rangle - G^{ij}\chi^j\right] = \sum_{j=1}^{n} \int_{\partial\Omega} (G\,\nu \wedge \nu)^{ij}\, \partial_\nu\left[\mathcal{D}\left(\chi^j\right)\right]. \tag{25}$$

Furthermore, the solution of (19) is unique when A is invertible.

Remark 12. *(i)* Obviously we can handle a general boundary datum u_0 on $\partial\Omega$. It suffices to apply the theorem replacing G by $G - A\nabla u_0 - (\nabla u_0)^t A$.

(ii) It is easy to see that if $\operatorname{supp} G \subset\subset \Omega$, then our construction gives that $\operatorname{supp} u \subset\subset \Omega$.

(iii) When Ω is simply connected, $\mathcal{H}_T(\Omega;\mathbb{R}^n) = \{0\}$. Therefore (24) and (25) are trivially satisfied in this case.

(iv) An easy computation shows that (24) is equivalent to

$$\int_\Omega (G\nabla\chi - \nabla\chi\, G) = \int_{\partial\Omega} G\nu \wedge \chi - \int_{\partial\Omega} (G\nu\wedge\nu)\langle\chi;\nu\rangle, \quad \text{for every } \chi \in \mathcal{H}_T(\Omega;\mathbb{R}^n).$$

Corollary 13. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, simply connected and smooth, $r \geqslant 2$ be an integer and $G \in C^r\left(\overline{\Omega};\mathbb{R}^{n\times n}\right)$. The following statements are equivalent.*

(i) *There exists $u \in C^{r+1}\left(\overline{\Omega};\mathbb{R}^n\right)$ satisfying*

$$\begin{cases} \nabla u + (\nabla u)^t = G & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{26}$$

(ii) *The following properties hold.*

– *G is symmetric*

– *in Ω and for every $i,j,k,l = 1,\ldots,n$*

$$\partial_{jl}G^{ik} - \partial_{il}G^{jk} + \partial_{ik}G^{jl} - \partial_{jk}G^{il} = 0$$

– *on $\partial\Omega$ and for every $i,j,k,l = 1,\ldots,n$,*

$$G^{ik}\nu^j\nu^l - G^{jk}\nu^i\nu^l + G^{jl}\nu^i\nu^k - G^{il}\nu^j\nu^k = 0$$

$$\nabla\left((G\nu\wedge\nu)^{ij}\right)\wedge\nu = \mathbf{M}^{ij}\wedge\nu.$$

Moreover, the solution of (26) is unique.

Proof of Theorem 11. Let $\{a_1,\ldots,a_N\} \subset \mathbb{R}^n$ be linearly independent such that

$$\ker A = \operatorname{span}\{a_1,\ldots,a_N\}.$$

Step 1 (Necessity). Let $u \in C^{r+1}\left(\overline{\Omega};\mathbb{R}^n\right)$ satisfy (19). Then, $v = A\,u$ verifies

$$\begin{cases} \nabla v + (\nabla v)^t = G & \text{in } \Omega \\ \langle v;a_1\rangle = \cdots = \langle v;a_N\rangle = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

The fact that G is symmetric is obvious. Both (20) and (21) are established in Steps 1.1 and 1.2 of Theorem 8.

Step 1.1 (proof of (22)). Note that $\nabla u = \alpha\otimes\nu$ on $\partial\Omega$ for some $\alpha \in C^r(\partial\Omega;\mathbb{R}^n)$ and hence

$$G = A(\alpha\otimes\nu) + (\nu\otimes\alpha)A = (A\alpha)\otimes\nu + \nu\otimes(A\alpha) \quad \text{on } \partial\Omega. \tag{28}$$

Therefore,

$$G\nu = A\alpha + \langle A\alpha;\nu\rangle\,\nu \quad \text{on } \partial\Omega, \tag{29}$$

which implies that $\langle A\alpha;\nu\rangle = \langle G\nu;\nu\rangle/2$. It then follows from (29) that

$$A\alpha = G\nu - \frac{1}{2}\langle G\nu;\nu\rangle\,\nu \quad \text{on } \partial\Omega.$$

Thus, using (28), we have indeed proved (22) .

Step 1.2 (proof of (23)). Using that $\nabla v = A\,\nabla u = (A\,\alpha) \otimes \nu$ on $\partial\Omega$ and (29) we note that

$$\operatorname{curl} v = \nabla v - (\nabla v)^t = (A\,\alpha) \otimes \nu - \nu \otimes (A\,\alpha) = (A\,\alpha) \wedge \nu = G\,\nu \wedge \nu \quad \text{on } \partial\Omega. \tag{30}$$

Therefore, for every $i, j, k, l = 1, \dots, n$ and on $\partial\Omega$,

$$\begin{aligned}\left[\nabla\left((G\,\nu \wedge \nu)^{ij}\right) \wedge \nu\right]^{kl} &= \left[\nabla\left((\operatorname{curl} v)^{ij}\right) \wedge \nu\right]^{kl} = \partial_k (\operatorname{curl} v)^{ij} \nu^l - \partial_l (\operatorname{curl} v)^{ij} \nu^k \\ &= \left[\partial_k \left(\partial_j v^i - \partial_i v^j\right)\right] \nu^l - \left[\partial_l \left(\partial_j v^i - \partial_i v^j\right)\right] \nu^k \\ &= \left[\partial_{jk} v^i - \partial_{ik} v^j\right] \nu^l - \left[\partial_{jl} v^i - \partial_{il} v^j\right] \nu^k\end{aligned}$$

and thus

$$\begin{aligned}&\left[\nabla\left((G\,\nu \wedge \nu)^{ij}\right) \wedge \nu\right]^{kl} \\ &= \left[\partial_j \left(\partial_k v^i + \partial_i v^k\right) - \partial_i \left(\partial_k v^j + \partial_j v^k\right)\right] \nu^l - \left[\partial_j \left(\partial_l v^i + \partial_i v^l\right) - \partial_i \left(\partial_l v^j + \partial_j v^l\right)\right] \nu^k \\ &= \left(\partial_j G^{ik} - \partial_i G^{jk}\right) \nu^l - \left(\partial_j G^{il} - \partial_i G^{jl}\right) \nu^k = M_k^{ij} \nu^l - M_l^{ij} \nu^k = \left(\mathbf{M}^{ij} \wedge \nu\right)^{kl}.\end{aligned}$$

Step 1.3 (proof of (24)). It follows from (11) and (30) that, for every $i, j = 1, \dots, n$ and $\chi \in \mathcal{H}_T(\Omega; \mathbb{R}^n)$,

$$\begin{aligned}\int_\Omega \langle \mathbf{M}^{ij}; \chi \rangle &= \sum_{k=1}^n \int_\Omega \left(\partial_j G^{ik} - \partial_i G^{jk}\right) \chi^k = \sum_{k=1}^n \int_\Omega \partial_k \left(\partial_j v^i - \partial_i v^j\right) \chi^k \\ &= -\int_\Omega \left(\partial_j v^i - \partial_i v^j\right) \operatorname{div} \chi + \int_{\partial\Omega} \left(\partial_j v^i - \partial_i v^j\right) \langle \chi; \nu \rangle = \int_{\partial\Omega} (G\,\nu \wedge \nu)^{ij} \langle \chi; \nu \rangle.\end{aligned}$$

Step 1.4 (proof of (25)). For every $i = 1, \dots, n$ and $\chi \in \mathcal{H}_T(\Omega; \mathbb{R}^n)$, using (11), we get

$$\begin{aligned}&\sum_{j=1}^n \int_\Omega \left\langle \mathbf{M}^{ij}; \nabla\left[\mathcal{D}\left(\chi^j\right)\right]\right\rangle \\ &= \sum_{j,k=1}^n \int_\Omega \left(\partial_j G^{ik} - \partial_i G^{jk}\right) \partial_k \left[\mathcal{D}\left(\chi^j\right)\right] = \sum_{j,k=1}^n \int_\Omega \partial_k \left(\partial_j v^i - \partial_i v^j\right) \partial_k \left[\mathcal{D}\left(\chi^j\right)\right] \\ &= -\sum_{j=1}^n \int_\Omega \left(\partial_j v^i - \partial_i v^j\right) \Delta\left[\mathcal{D}\left(\chi^j\right)\right] + \sum_{j,k=1}^n \int_{\partial\Omega} \left(\partial_j v^i - \partial_i v^j\right) \partial_k \left[\mathcal{D}\left(\chi^j\right)\right] \nu^k\end{aligned}$$

and thus, appealing to (30),

$$\sum_{j=1}^n \int_\Omega \left\langle \mathbf{M}^{ij}; \nabla\left[\mathcal{D}\left(\chi^j\right)\right]\right\rangle = -\sum_{j=1}^n \int_\Omega \left(\partial_j v^i - \partial_i v^j\right) \chi^j + \sum_{j=1}^n \int_{\partial\Omega} (G\,\nu \wedge \nu)^{ij} \partial_\nu \left[\mathcal{D}\left(\chi^j\right)\right]. \tag{31}$$

Note that, for every $i = 1, \dots, n$ and $\chi \in \mathcal{H}_T(\Omega; \mathbb{R}^n)$,

$$\sum_{j=1}^n \int_\Omega \left(\partial_j v^i - \partial_i v^j\right) \chi^j + \sum_{j=1}^n \int_\Omega G^{ij} \chi^j = 2 \sum_{j=1}^n \int_\Omega \partial_j v^i \chi^j = -2 \int_\Omega (\operatorname{div} \chi) v^i = 0.$$

Therefore, the claim follows from (31).

Step 2 (Sufficiency). We now prove the sufficiency of the conditions.

Step 2.1. For every $i, j = 1, \dots, n$, we use Theorem 2 (with $N = 0$) to find $\Phi = \left(\Phi^{ij}\right)^{1 \leq i,j \leq n} \in C^r\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$ solving

$$\begin{cases} \nabla \Phi^{ij} = \mathbf{M}^{ij} & \text{in } \Omega \\ \Phi^{ij} = (G\,\nu \wedge \nu)^{ij} & \text{on } \partial\Omega \end{cases} \tag{32}$$

That (32) is indeed solvable, follows from (20), (23) and (24). We next give some properties of Φ.

(i) Note that

$$\Phi^t = -\Phi \quad \text{and} \quad \partial_k \Phi = \mathbf{M}_k \quad \text{in } \Omega \text{ and for every } k = 1, \ldots, n. \tag{33}$$

(ii) Furthermore, using (22) and Lemma 7, for every $i, j, k = 1, \ldots, n$ and on $\partial\Omega$, we have

$$\Phi^{ij}\nu^k = (G\,\nu \wedge \nu)^{ij}\,\nu^k = \sum_{l=1}^{n} \left[G^{il}\nu^j - G^{jl}\nu^i\right]\nu^l\nu^k = \sum_{l=1}^{n} \left[G^{ik}\nu^j - G^{jk}\nu^i\right]\left(\nu^l\right)^2$$

and hence

$$\Phi^{ij}\nu^k = G^{ik}\nu^j - G^{jk}\nu^i. \tag{34}$$

(iii) We claim that

$$\Phi a = Ga \quad \text{in } \Omega \text{ and for every } a \in \ker A. \tag{35}$$

To see this, let $a \in \ker A$ be fixed. It follows from (21) and (33) that, for every $k = 1, \ldots, n$,

$$\partial_k (G - \Phi)\, a = [\partial_k G - \mathbf{M}_k]\, a = 0 \quad \text{in } \Omega.$$

Therefore, for some $c \in \mathbb{R}^n$,

$$Ga - \Phi a = c \quad \text{in } \Omega.$$

Note that, using (22), we get on $\partial\Omega$

$$\begin{aligned}
[G - \Phi]\, a &= [(G\,\nu \otimes \nu) + (\nu \otimes G\,\nu) - \langle G\,\nu; \nu\rangle (\nu \otimes \nu) - (G\,\nu \wedge \nu)]\, a \\
&= 2\,(\nu \otimes G\,\nu)\, a - \langle G\,\nu; \nu\rangle (\nu \otimes \nu)\, a = 2\,\langle G\,\nu; a\rangle\, \nu - \langle G\,\nu; \nu\rangle \langle \nu; a\rangle\, \nu
\end{aligned}$$

which implies that $(Ga - \Phi a) \wedge \nu = 0$ on $\partial\Omega$, from where it follows that $c = 0$. This proves (35) .

Step 2.2. Invoking Theorem 2, we find $v \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying

$$\begin{cases}
\nabla v = \frac{1}{2}\,(G + \Phi) & \text{in } \Omega \\
\langle v; a_1\rangle = \cdots = \langle v; a_N\rangle = 0 & \text{in } \Omega \\
v = 0 & \text{on } \partial\Omega.
\end{cases} \tag{36}$$

This is possible for the following reasons.

(i) By (32) and (34), for every $i, j, k = 1, \ldots, n$,

$$\partial_k \left(G^{ij} + \Phi^{ij}\right) = \partial_k G^{ij} + \partial_j G^{ik} - \partial_i G^{jk} = \partial_j G^{ik} + \partial_k G^{ij} - \partial_i G^{kj} = \partial_j \left(G^{ik} + \Phi^{ik}\right) \quad \text{in } \Omega$$

$$\left(G^{ij} + \Phi^{ij}\right)\nu^k = G^{ij}\nu^k + G^{ik}\nu^j - G^{jk}\nu^i = G^{ik}\nu^j + G^{ij}\nu^k - G^{kj}\nu^i = \left(G^{ik} + \Phi^{ik}\right)\nu^j \text{ on } \partial\Omega.$$

(ii) Appealing to (25) and (32), for every $i = 1, \ldots, n$ and $\chi \in \mathcal{H}_T(\Omega; \mathbb{R}^n)$,

$$\begin{aligned}
\int_\Omega (G\,\chi)^i &= \sum_{j=1}^{n} \int_\Omega \left\langle \mathbf{M}^{ij}; \nabla\left[\mathcal{D}\left(\chi^j\right)\right]\right\rangle - \int_{\partial\Omega} \left[(G\,\nu \wedge \nu)\,\partial_\nu\,(\mathcal{D}(\chi))\right]^i \\
&= \sum_{j=1}^{n} \int_\Omega \langle \nabla\Phi^{ij}; \nabla\left[\mathcal{D}\left(\chi^j\right)\right]\rangle - \int_{\partial\Omega} \left[\Phi\,\partial_\nu\,(\mathcal{D}(\chi))\right]^i \\
&= -\sum_{j=1}^{n} \int_\Omega \Phi^{ij}\chi^j + \sum_{j=1}^{n} \int_{\partial\Omega} \Phi^{ij}\partial_\nu\left[\mathcal{D}\left(\chi^j\right)\right] - \int_{\partial\Omega} \left[\Phi\,\partial_\nu\,(\mathcal{D}(\chi))\right]^i = -\int_\Omega (\Phi\,\chi)^i
\end{aligned}$$

i.e.

$$\int_\Omega (G + \Phi)\,\chi = 0 \quad \text{for every } \chi \in \mathcal{H}_T(\Omega; \mathbb{R}^n)\,.$$

(iii) Invoking (33) and (35), for every $a \in \ker A$, then $a \in \ker\left(\left(G+\Phi\right)^t\right)$, since

$$\left(G+\Phi\right)^t a = \left[G-\Phi\right] a = 0 \quad \text{in } \Omega.$$

Step 2.3. Appealing to Lemma 6, we find $u \in C^{r+1}\left(\overline{\Omega};\mathbb{R}^n\right)$ such that

$$v = A\,u \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega.$$

Then, u satisfies (19). When A is invertible, the uniqueness follows from Theorem 27. ■

4. The skew-symmetric case

4.1. Preliminaries and notations

By abuse of notations we will, in this section, identify $\mathbb{R}^{n\times n}_a$, the set of skew-symmetric matrices, with $\Lambda^2\left(\mathbb{R}^n\right)$, the set of 2-forms, in a straightforward manner. For example if $G = \left(G^{ij}\right)^{1\leq i,j\leq n} \in C^1\left(\Omega;\mathbb{R}^{n\times n}_a\right)$, $dG = 0$ respectively $\delta G = 0$ mean that, for every $i,j,k = 1,\ldots,n$,

$$\partial_k G^{ij} - \partial_j G^{ik} + \partial_i G^{jk} = 0 \quad \text{respectively} \quad \sum_{i=1}^n \partial_i G^{ij} = 0.$$

Similarly, if $\nu \in \mathbb{R}^n$ and $G = \left(G^{ij}\right)^{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}_a$, we write $\nu \wedge G = 0$ respectively $\nu \lrcorner G = 0$ meaning that, for every $i,j,k = 1,\ldots,n$,

$$G^{ij}\nu^k - G^{ik}\nu^j + G^{jk}\nu^i = 0 \quad \text{respectively} \quad \sum_{i=1}^n G^{ij}\nu^i = 0.$$

Finally we define, for $\Omega \subset \mathbb{R}^n$ an open, bounded, connected and smooth set,

$$\mathcal{H}_T\left(\Omega;\mathbb{R}^{n\times n}_a\right) = \left\{\omega \in C^\infty\left(\overline{\Omega};\mathbb{R}^{n\times n}_a\right) : d\omega = 0,\ \delta\omega = 0,\ \nu\wedge\omega = 0 \text{ on } \partial\Omega\right\}$$
$$\mathcal{H}_N\left(\Omega;\mathbb{R}^{n\times n}_a\right) = \left\{\omega \in C^\infty\left(\overline{\Omega};\mathbb{R}^{n\times n}_a\right) : d\omega = 0,\ \delta\omega = 0,\ \nu\lrcorner\omega = 0 \text{ on } \partial\Omega\right\}.$$

When $n = 2$, then $\mathcal{H}_N\left(\Omega;\mathbb{R}^{2\times 2}_a\right) = \{0\}$, while

$$\mathcal{H}_T\left(\Omega;\mathbb{R}^{2\times 2}_a\right) = \left\{\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} : \text{where } c \in \mathbb{R}\right\}. \tag{37}$$

If $n \geq 3$ and Ω is contractible, then $\mathcal{H}_T\left(\Omega;\mathbb{R}^{n\times n}_a\right) = \mathcal{H}_N\left(\Omega;\mathbb{R}^{n\times n}_a\right) = \{0\}$.

4.2. The unconstrained problem when A is invertible

Theorem 14. *Let $n \geqslant 2$, $r \geqslant 1$ be integers, $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and smooth. Let $A \in \mathbb{R}^{n\times n}$ be skew-symmetric and invertible and $G \in C^{r,\alpha}\left(\overline{\Omega};\mathbb{R}^{n\times n}\right)$.*

Part 1 (existence). *The following conditions are equivalent.*

(i) *There exists $u \in C^{r+1,\alpha}\left(\Omega;\mathbb{R}^n\right)$ satisfying*

$$A\,\nabla u + \left(\nabla u\right)^t A = G \text{ in } \Omega. \tag{38}$$

(ii) *G is skew-symmetric, $dG = 0$ in Ω and*

$$\int_\Omega \langle G;\chi\rangle = 0, \quad \text{for every } \chi \in \mathcal{H}_N\left(\Omega;\mathbb{R}^{n\times n}_a\right). \tag{39}$$

Part 2 (uniqueness). *Solutions are unique up to a gradient, i.e. given two solutions $u,v \in C^{r+1,\alpha}\left(\Omega;\mathbb{R}^n\right)$, there exists $\varphi \in C^{r+2,\alpha}\left(\Omega\right)$ such that*

$$v = u + A^{-1}\nabla\varphi.$$

Remark 15. *(i)* Note that, since $A \in \mathbb{R}_a^{n\times n}$ is assumed to be invertible, then n must be even.

(ii) When $n = 2$ or $n \geq 3$ and Ω is contractible, (39) is trivially satisfied. Therefore, if we assume that Ω is contractible and $A \in \mathbb{R}_a^{n\times n}$ is invertible, then the necessary conditions reduce to $G \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}_a^{n\times n}\right)$ and $dG = 0$ (which is trivially satisfied if $n = 2$).

(iii) Note that, as opposed to Theorem 8 where regularity is mentioned in the C^r-class, the right space to study regularity issues in the skew-symmetric case is the Hölder spaces. This is illustrated through the following example. Let $\Omega = (0,1)^2$ and $h \in C^0(\Omega)$ be chosen for which there is no $u \in C^1\left(\Omega; \mathbb{R}^2\right)$ satisfying $\operatorname{div} u = h$ in Ω (see, for example, [5] and the references therein). Define $G \in C^0\left(\Omega; \mathbb{R}_a^{2\times 2}\right)$ as $G^{12} = h$. Then, $dG = 0$ in Ω. It is easy to check that there exists no $u \in C^1\left(\Omega; \mathbb{R}^2\right)$ satisfying (38), with $A = J_2$.

Proof of Theorem 14. Setting $v = A\,u$, our problem is equivalent to

$$\nabla v - (\nabla v)^t = G \quad \Leftrightarrow \quad \operatorname{curl} v = G \tag{40}$$

and we are back to the classical result, see, for example, Theorem 8.3 in [4]. ■

4.3. The unconstrained problem when $\operatorname{rank} A \leq n-1$

We give two theorems. The first one is valid for any A with $\operatorname{rank} A \leq n-1$ and the second one when $\operatorname{rank} A = n-1$, but with different hypotheses and proofs.

Theorem 16. *Let $n \geqslant 2$, $r \geqslant 1$ be integers, $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^n$ be an open ball. Let $A \in \mathbb{R}^{n\times n}$ be skew-symmetric and $G \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. The following conditions are equivalent.*

(i) *There exists $u \in C^{r,\alpha}(\Omega; \mathbb{R}^n)$ satisfying*

$$A\,\nabla u + (\nabla u)^t A = G \quad \text{in } \Omega. \tag{41}$$

(ii) *G is skew-symmetric, $dG = 0$ in Ω and*

$$\langle G(x)\,a; a'\rangle = 0 \quad \text{for every } x \in \Omega \text{ and for every } a, a' \in \ker A. \tag{42}$$

Remark 17. *(i)* The solutions of $A\,\nabla u + (\nabla u)^t A = 0$ are, as in Theorem 14, but with a special structure. For example, if

$$A = \operatorname{diag}\left(\underbrace{J_2, \ldots, J_2}_{m-times}, \underbrace{0, \ldots, 0}_{(n-2m)-times}\right),$$

then any solution is of the form

$$u^i(x) = \partial_i \varphi \quad \text{where } \varphi = \varphi(x_1, \ldots, x_{2m})$$

if $1 \leq i \leq 2m$; while, for $2m+1 \leq i \leq n$, u^i is free.

(ii) There is an interesting difference between the regularity result when A is invertible or not. Indeed if A is invertible and $G \in C^{r,\alpha}$, there exists a $u \in C^{r+1,\alpha}$; while if A is not invertible the solution $u \in C^{r,\alpha}$ and not in a better space, in general, as shown in Example 18.

Example 18. Let $\Omega = (-2,2)^3$ and define $G \in C^{1,1}\left(\mathbb{R}^3; \mathbb{R}_a^{3\times 3}\right)$ by

$$G^{12}(x) = 2\left(x^3\right)^2\left(x^1 - x^2\right)\left|x^1x^2x^3\right| \quad \text{and} \quad G^{13}(x) = G^{23}(x) = 3\left(x^1x^2x^3\right)\left|x^1x^2x^3\right|$$

for every $x \in \mathbb{R}^3$. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Any solution of (41) is then of the form

$$u^1(x) = -\left(x^1x^2x^3\right)\left|x^1x^2x^3\right|x^3 + g\left(x^1, x^2\right) \quad \text{and} \quad u^2(x) = \left(x^1x^2x^3\right)\left|x^1x^2x^3\right|x^3 + h\left(x^1, x^2\right)$$

for some $g, h : (-2, 2)^2 \to \mathbb{R}$. It is easy to see, by contradiction, that $u^1, u^2 \notin C^2(\Omega)$ for any g, h.

Proof of Theorem 16. It is obvious that (ii) are necessary conditions, so we discuss only their sufficiency.

Step 1. Let A be such that $\operatorname{rank} A = 2m < n$. We first prove that we can assume, without loss of generality, that

$$\ker A = \operatorname{span}\{e_{2m+1}, \dots, e_n\}$$

where $\{e_1, \dots, e_n\}$ denotes the standard Euclidean basis. Indeed we can find (as in Proposition 34) $P \in O(n)$ (i.e. $P^tP = I$) and $\Lambda \in \mathbb{R}^{n\times n}$ such that

$$P^t A P = \Lambda = \operatorname{diag}\left(\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix}, 0, \dots, 0\right) \quad \text{with} \quad \lambda_i \neq 0,\ 1 \le i \le m.$$

We then define, for $y \in P^t\Omega$, v and F through

$$v(y) = P^t u(Py) \quad \text{and} \quad F(y) = P^t G(Py) P$$

and insert this into $A\nabla u + (\nabla u)^t A = G$, to get, writing $y = P^t x \in P^t\Omega$,

$$\Lambda \nabla v(y) + (\nabla v(y))^t \Lambda = F(y).$$

We then have $\ker\Lambda = \operatorname{span}\{e_{2m+1}, \dots, e_n\}$, F is skew-symmetric, $dF = 0$ in $P^t\Omega$ (writing $\varphi(y) = Py$, has transformed the equation $F(y) = P^t G(Py) P$ into $\varphi^*(G) = F$ and thus $dF = 0$) and

$$\langle F(y)a; a'\rangle = 0 \quad \text{for every } y \in P^t\Omega \text{ and for every } a, a' \in \ker\Lambda.$$

Moreover $P^t\Omega$ is a ball. This concludes the proof of Step 1.

Step 2. With the help of Step 1, setting $v = Au$, the problem becomes

$$\begin{cases} \operatorname{curl} v = G \\ \langle e_{2m+1}; v\rangle = \cdots = \langle e_n; v\rangle = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_j v^i - \partial_i v^j = G^{ij} & 1 \le i < j \le 2m \\ \partial_j v^i = G^{ij} & 1 \le i \le 2m,\ 2m+1 \le j \le n \\ v^j = 0 & 2m+1 \le j \le n. \end{cases} \tag{43}$$

The necessary conditions read as

$$\partial_{2m} G^{ij} - \partial_j G^{i(2m)} + \partial_i G^{j(2m)} = 0 \quad \text{and} \quad G^{ij} = 0 \text{ for every } 2m+1 \le i < j \le n. \tag{44}$$

We now prove that under conditions (44), the problem (43) admits a solution. We proceed by induction on n (when $n = 2$ the result is trivial) and write

$$x = (x_1, \dots, x_{n-1}, x_n) = (\widehat{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Any solution of (43) is necessarily of the form (using the second set of equations with $j = n$)

$$v^i(x) = \alpha^i(\widehat{x}) + \int_0^{x_n} G^{in}(\widehat{x}, t)\,dt \quad 1 \le i \le 2m.$$

A direct computation transforms (43) into, for $\alpha = \alpha(\widehat{x})$,

$$\begin{cases} \partial_j \alpha^i - \partial_i \alpha^j = G^{ij}(\widehat{x}, 0) & 1 \le i < j \le 2m \\ \partial_j \alpha^i = G^{ij}(\widehat{x}, 0) & 1 \le i \le 2m,\ 2m+1 \le j \le n-1 \\ \alpha^j = 0 & 2m+1 \le j \le n-1. \end{cases}$$

The hypothesis of induction leads to the result. ■

We now discuss the case where rank $A = n-1$. We provide a different proof (inspired by Proposition 8.20 in [4]) from the one above, which allows for more general domains Ω.

Theorem 19. *Let $n \geqslant 3$, $r \geqslant 1$ be integers, $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and smooth. Let $A \in \mathbb{R}^{n \times n}$ be skew-symmetric with* rank $A = n - 1$ *(and thus n is odd) and $G \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. The following statements are equivalent*

(i) *there exists $u \in C^{r,\alpha}(\Omega; \mathbb{R}^n)$ satisfying*

$$A\,\nabla u + (\nabla u)^t A = G \quad \text{in } \Omega. \tag{45}$$

(ii) *G is skew-symmetric, $dG = 0$ in Ω and*

$$\int_\Omega \langle G; \chi\rangle = 0, \quad \text{for every } \chi \in \mathcal{H}_N\left(\Omega; \mathbb{R}^{n\times n}_a\right).$$

Remark 20. We observe that, in the present context, the condition

$$\langle G(x)\, a; a'\rangle = 0 \quad \text{for every } x \in \Omega \text{ and for every } a, a' \in \ker A$$

comes for free, since the dimension of ker A is 1 and G is skew-symmetric.

Proof. *(i)* The necessity of the conditions follows from Theorem 8.3 of [4].

(ii) Let us prove the sufficient part. Let $\ker A = \operatorname{span}\{a\}$, for some $a \in \mathbb{R}^n \setminus \{0\}$. Using Theorem 8.3 of [4], we find $w \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying

$$\nabla w - (\nabla w)^t = G \quad \text{in } \Omega.$$

We then use Theorem B.5 of [7], see [8] as well, to find $\varphi \in C^{r+1,\alpha}(\overline{\Omega})$ such that

$$\langle a; \nabla\varphi\rangle = -\langle a; w\rangle \quad \text{in } \Omega.$$

Then, $v = w + \nabla\varphi \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ satisfies, in Ω,

$$\nabla v - (\nabla v)^t = G \quad \text{and} \quad \langle a; v\rangle = 0.$$

Invoking Lemma 6, we find $u \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ such that $v = A\,u$, which satisfies $A\,\nabla u + (\nabla u)^t A = G$. This finishes the proof. ■

4.4. The Dirichlet problem

We next study the Dirichlet problem when A is invertible (the result is then classical, see, for example, Theorem 8.16 of [4] and setting $v = A\,u$.).

Theorem 21 (*Dirichlet Problem*)*.Let $n \geqslant 2$, $r \geqslant 1$ be integers, $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and smooth. Let $A \in \mathbb{R}^{n\times n}$ be skew-symmetric and $G \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. The following statements hold.*

Necessary conditions: *if there exists $u \in C^{r+1,\alpha}\left(\Omega; \mathbb{R}^n\right)$ satisfying*

$$\begin{cases} A\,\nabla u + (\nabla u)^t A = G & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{46}$$

then G is skew-symmetric, $dG = 0$ in Ω, $G \wedge \nu = 0$ on $\partial\Omega$ and

$$\int_\Omega \langle G; \chi \rangle = 0 \quad \text{for every } \chi \in \mathcal{H}_T\left(\Omega; \mathbb{R}^{n\times n}_a\right). \tag{47}$$

Sufficient conditions: *if A is invertible, then the necessary conditions are also sufficient and uniqueness is up to a gradient, i.e. given two solutions $u, v \in C^{r+1,\alpha}\left(\Omega; \mathbb{R}^n\right)$, there exists $\varphi \in C^{r+2,\alpha}\left(\Omega\right)$, with $\nabla\varphi = 0$ and $\nabla^2\varphi = 0$ on $\partial\Omega$, such that $v = u + A^{-1}\nabla\varphi$.*

Remark 22. *(i)* If $n = 2$, then $dG = 0$ in Ω and $G \wedge \nu = 0$ on $\partial\Omega$ are automatically satisfied, while (47) becomes, in view of (37),

$$\int_\Omega G^{12} = 0.$$

(ii) If $n \geq 3$ and Ω is contractible, then $\mathcal{H}_T\left(\Omega; \mathbb{R}^{n\times n}_a\right) = \{0\}$ and hence (47) is trivially true.

(iii) One can show that if $\operatorname{supp} G \subset\subset \Omega$, then our construction gives that $\operatorname{supp} u \subset\subset \Omega$, invoking [10] instead of Theorem 8.16 of [4].

4.5. The Dirichlet problem when rank $A = n - 1$

We finally turn to the Dirichlet problem when rank $A = n-1$ (and therefore the dimension n is odd). The equation to be solved is $A\,\nabla u + (\nabla u)^t A = G$ and upon setting $v = Au$, we have transformed the problem into $\nabla v - (\nabla v)^t = G$. Since rank $[A] = n - 1 = 2m$, we equivalently have $\langle a; v\rangle = 0$ for a nonzero vector a. Therefore an equivalent way to rewrite the problem is

$$\begin{cases} \operatorname{curl} v = G \quad \text{and} \quad \langle a; v\rangle = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

For the sake of simplicity, we take $a = e_n$ (but, up to a change of v, we can always get back to this case, as in the proof of Theorem 16). In the sequel we write for $x \in \mathbb{R}^n$

$$x = (x_1, \ldots, x_{n-1}, x_n) = (\widehat{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$$

while $\{e_1, \ldots, e_n\}$ denotes the standard Euclidean basis. We first introduce the type of sets Ω that we will consider.

Definition 23. (i) We say that Ω is x_n-simple if

$$\Omega = \{x = (\widehat{x}, x_n) \in \mathbb{R}^n : \alpha_-(\widehat{x}) < x_n < \alpha_+(\widehat{x}) \text{ with } \widehat{x} \in O\}$$

where $O \subset \mathbb{R}^{n-1}$ is a bounded open set with Lipschitz boundary and $\alpha_\pm \in C^{0,1}(O) \cap C^0\left(\overline{O}\right)$ with $\alpha_- < \alpha_+$ in O.

(ii) The boundary of such a set is given by

$$\partial\Omega = \overline{\Gamma}_- \cup \overline{\Gamma}_+ \cup \Gamma_c$$

where

$$\Gamma_- = \{x = (\widehat{x}, x_n) \in \mathbb{R}^n : x_n = \alpha_-(\widehat{x}) \text{ with } \widehat{x} \in O\}$$

$$\Gamma_+ = \{x = (\widehat{x}, x_n) \in \mathbb{R}^n : x_n = \alpha_+(\widehat{x}) \text{ with } \widehat{x} \in O\}$$

$$\Gamma_c = \{x = (\widehat{x}, x_n) \in \mathbb{R}^n : \alpha_-(\widehat{x}) \le x_n \le \alpha_+(\widehat{x}) \text{ with } \widehat{x} \in \partial O\}.$$

The respective exterior unit normals are $\nu_- = \frac{(\nabla\alpha_-, -1)}{\sqrt{1+|\nabla\alpha_-|^2}}$, $\nu_+ = \frac{(-\nabla\alpha_+, 1)}{\sqrt{1+|\nabla\alpha_+|^2}}$ and $\nu_c = (\widehat{\nu}, 0)$ where $\widehat{\nu}$ is the exterior unit normal to ∂O.

Remark 24. *(i)* Note that Γ_c is closed. It might be that, on some part of the boundary, $\alpha_-(\widehat{x}) = \alpha_+(\widehat{x})$ (as in the examples below).

(ii) The following examples are x_n-simple.

– A convex set (and, in fact, simple in every direction); in particular if $\Omega = \left\{x \in \mathbb{R}^n : |x|_p < 1\right\}$ with $1 \le p < \infty$ and

$$|x|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} = \left(|\widehat{x}|_p^p + |x_n|^p\right)^{\frac{1}{p}}.$$

– A generalized cylinder (including a cube) of the form

$$\Omega = O \times (-1, 1) \quad \text{where} \quad O \subset \mathbb{R}^{n-1} \text{ is bounded and open.}$$

Theorem 25. *Let $\Omega \subset \mathbb{R}^n$ be x_n-simple. Let $r \ge 1$ be an integer and $G \in C^r\left(\overline{\Omega}; \mathbb{R}_a^{n\times n}\right)$ satisfy*

$$dG = 0 \text{ in } \Omega \text{ (i.e. } \partial_k G^{ij} - \partial_j G^{ik} + \partial_i G^{jk} = 0) \quad \text{and} \quad \nu \wedge G = 0 \text{ on } \partial\Omega$$

and, for every, $1 \le i \le n-1$,

$$\int_{\alpha_-(\widehat{x})}^{\alpha_+(\widehat{x})} G^{in}(\widehat{x}, t)\, dt = 0 \text{ if } \widehat{x} \in O \quad \text{and} \quad G^{in}(x) = 0 \text{ on } \Gamma_c.$$

Let

$$v^i(x) = v^i(\widehat{x}, x_n) = \int_{\alpha_-(\widehat{x})}^{x_n} G^{in}(\widehat{x}, t)\, dt \quad 1 \le i \le n-1.$$

If, in addition to the above hypotheses, $v^i \in C^r\left(\overline{\Omega}\right)$, then it satisfies

$$\begin{cases} \partial_j v^i - \partial_i v^j = G^{ij} \Leftrightarrow \operatorname{curl} v = G & \text{in } \Omega \\ v^n = 0 \Leftrightarrow \langle e_n; v\rangle = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{48}$$

Remark 26. *(i)* Note that the solution v of the problem is unique.

(ii) The conditions on G are also necessary.

(iii) The construction ensures that if $\operatorname{supp} G \subset\subset \Omega$, then also $\operatorname{supp} v \subset\subset \Omega$.

(iv) If $\alpha_\pm \in C^r\left(\overline{O}\right)$, respectively $C^r(O) \cap C^0\left(\overline{O}\right)$, then the v^i defined in the theorem is automatically in $C^r\left(\overline{\Omega}\right)$, respectively $C^r(\Omega) \cap C^0\left(\overline{\Omega}\right)$.

Proof. We recall that we have to find v such that $v^n = 0$ and

$$\begin{cases} \partial_j v^i - \partial_i v^j = G^{ij},\ 1 \le i < j \le n-1 & \text{in } \Omega \\ \partial_n v^i = G^{in},\ 1 \le i \le n-1 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

A straightforward integration (of the equation $\partial_n v^i = G^{in}$) leads that the only candidate is

$$v^i(\widehat{x}, x_n) = \int_{\alpha_-(\widehat{x})}^{x_n} G^{in}(\widehat{x}, t)\, dt, \quad \text{for } 1 \le i \le n-1, .$$

(i) Clearly $v = 0$ on $\Gamma_\pm$, since $\int_{\alpha_-(\widehat{x})}^{\alpha_+(\widehat{x})} G^{in}(\widehat{x}, t)\, dt = 0$. By continuity we have $v = 0$ on $\overline{\Gamma}_\pm$.

(ii) We also have to prove that $v = 0$ on Γ_c. Let $x \in \Gamma_c$. If $x \in \Gamma_c \cap \left(\overline{\Gamma}_+ \cup \overline{\Gamma}_-\right)$, this has already been established in (i). So let us assume that

$$\widehat{x} \in \partial O \quad \text{and} \quad \alpha_-(\widehat{x}) < \alpha_+(\widehat{x}).$$

Since, by continuity, we have $\partial_n v^i = G^{in}$ and $G^{in} = 0$ on Γ_c, we deduce that $v^i(x) = v^i(\widehat{x}, \alpha_-(\widehat{x})) = 0$; as wished.

(iii) We still have to show that $\partial_j v^i - \partial_i v^j = G^{ij}$, for every $1 \le i < j \le n-1$. Let us first observe that (since $v^i, G^{in} \in C^r(\overline{\Omega})$) we have, for every $1 \le i, j \le n-1$,

$$\partial_j v^i(\widehat{x}, x_n) = \int_{\alpha_-(\widehat{x})}^{x_n} \partial_j G^{in}(\widehat{x}, t)\, dt - \partial_j \alpha_-\, G^{in}(\widehat{x}, \alpha_-(\widehat{x}))$$

and thus the function $g^{ij}(\widehat{x}) = \partial_j \alpha_-\, G^{in}(\widehat{x}, \alpha_-(\widehat{x}))$ is $C^{r-1}(\overline{O})$. We next note that

$$\partial_j v^i - \partial_i v^j = \int_{\alpha_-(\widehat{x})}^{x_n} \left(\partial_j G^{in} - \partial_i G^{jn}\right)(\widehat{x}, t)\, dt - \partial_j \alpha_-\, G^{in}(\widehat{x}, \alpha_-(\widehat{x})) + \partial_i \alpha_-\, G^{jn}(\widehat{x}, \alpha_-(\widehat{x})).$$

Since $dG = 0$, we find, for every $1 \le i < j \le n-1$,

$$\begin{aligned}\partial_j v^i - \partial_i v^j &= \int_{\alpha_-(\widehat{x})}^{x_n} \partial_n G^{ij}(\widehat{x}, t)\, dt - \partial_j \alpha_-\, G^{in}(\widehat{x}, \alpha_-(\widehat{x})) + \partial_i \alpha_-\, G^{jn}(\widehat{x}, \alpha_-(\widehat{x})) \\ &= G^{ij}(\widehat{x}, x_n) - G^{ij}(\widehat{x}, \alpha_-(\widehat{x})) - \partial_j \alpha_-\, G^{in}(\widehat{x}, \alpha_-(\widehat{x})) + \partial_i \alpha_-\, G^{jn}(\widehat{x}, \alpha_-(\widehat{x})).\end{aligned}$$

This leads immediately to the result since $\nu \wedge G = 0$ on $\partial\Omega$, which in the present context reads as

$$-G^{ij}(\widehat{x}, \alpha_-(\widehat{x})) - \partial_j \alpha_-\, G^{in}(\widehat{x}, \alpha_-(\widehat{x})) + \partial_i \alpha_-\, G^{jn}(\widehat{x}, \alpha_-(\widehat{x})) = 0, \; 1 \le i < j \le n-1.$$

Indeed at almost all points $(\widehat{x}, \alpha_-(\widehat{x})) \in \Gamma_- \subset \partial\Omega$, we have that

$$\nu_-^i = \frac{\partial_i \alpha_-}{\sqrt{1 + |\nabla \alpha_-|^2}} \; 1 \le i \le n-1 \quad \text{and} \quad \nu_-^n = \frac{-1}{\sqrt{1 + |\nabla \alpha_-|^2}}.$$

This ends the proof. ■

5. The general case when the symmetric part is invertible

Recall that for $A \in \mathbb{R}^{n \times n}$, we denote by A_s (respectively A_a) its symmetric (respectively skew-symmetric) part. Note that finding a solution of $A\, \nabla u + (\nabla u)^t A = G$ is equivalent to solving simultaneously

$$A_s\, \nabla u + (\nabla u)^t A_s = G_s \quad \text{and} \quad A_a\, \nabla u + (\nabla u)^t A_a = G_a.$$

5.1. The kernel

Theorem 27. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and $A \in \mathbb{R}^{n\times n}$ with A_s invertible. Then, $u \in C^1(\Omega;\mathbb{R}^n)$ satisfy*

$$A\,\nabla u + (\nabla u)^t\,A = 0 \quad \text{in } \Omega, \tag{49}$$

if and only if, for some $C \in \mathbb{R}^{n\times n}$ satisfying $A\,C + C^t\,A = 0$ and $c \in \mathbb{R}^n$,

$$u(x) = C\,x + c, \text{ for every } x \in \Omega.$$

Proof. The result is standard when A is symmetric (see, for example, [3]). The sufficient part is obvious, so we discuss now the necessity of the condition. Let us suppose that $u \in C^1(\Omega;\mathbb{R}^n)$ satisfies (49). Then $A_s\,\nabla u + (\nabla u)^t\,A_s = 0$. Setting $v = A_s u$, we have $\nabla v + (\nabla v)^t = 0$. Therefore, for every $i, j, k = 1, \ldots, n$ and for every $\varphi \in C_0^2(\Omega)$,

$$\begin{aligned}\int_\Omega \partial_i v^j \partial_k \varphi &= -\int_\Omega v^j \partial_{ik}\varphi = \int_\Omega \partial_k v^j \partial_i \varphi = -\int_\Omega \partial_j v^k \partial_i \varphi = \int_\Omega v^k \partial_{ji}\varphi = -\int_\Omega \partial_i v^k \partial_j \varphi \\ &= \int_\Omega \partial_k v^i \partial_j \varphi = -\int_\Omega v^i \partial_{kj}\varphi = \int_\Omega \partial_j v^i \partial_k \varphi = -\int_\Omega \partial_i v^j \partial_k \varphi.\end{aligned}$$

This implies that, for every $i, j, k = 1, \ldots, n$,

$$\int_\Omega \partial_i v^j \partial_k \varphi = 0, \quad \text{for every } \varphi \in C_0^2(\Omega).$$

Therefore, it follows from the fundamental lemma of the calculus of variations that, for some $P \in \mathbb{R}^{n\times n}$, $\nabla v = P$ in Ω. Since A_s is invertible, we conclude that $u = A_s^{-1} v$ is affine, i.e.

$$u(x) = C\,x + c, \quad \text{for some } C \in \mathbb{R}^{n\times n} \text{ and } c \in \mathbb{R}^n.$$

Thus, since u is a solution of (49), we deduce that $A\,C + C^t A = 0$. This proves the theorem. ■

5.2. The unconstrained problem

Theorem 28. *Let $\Omega \subset \mathbb{R}^n$ be open bounded connected and smooth, $r \geqslant 2$ be an integer, $G = \left(G^{ij}\right)^{1\le i,j\le n} \in C^r\left(\overline{\Omega};\mathbb{R}^{n\times n}\right)$ and $A \in \mathbb{R}^{n\times n}$. Then, the following statements hold.*

Necessary conditions: *if there exists $u \in C^{r+1}\left(\overline{\Omega};\mathbb{R}^n\right)$ satisfying*

$$A\,\nabla u + (\nabla u)^t\,A = G \quad \text{in } \Omega \tag{50}$$

then

– *there exist $x_0 \in \Omega$ and $C_0 \in \mathbb{R}^{n\times n}$ verifying*

$$A\,C_0 + C_0^t\,A = G(x_0)$$

– *for every $x \in \Omega$*

$$\langle G_s(x)\,a; a'\rangle = 0 \quad \text{for every } a, a' \in \ker A_s$$

– *in Ω and for every $i, j, k, l = 1, \ldots, n$,*

$$\partial_{jl} G_s^{ik} - \partial_{il} G_s^{jk} + \partial_{ik} G_s^{jl} - \partial_{jk} G_s^{il} = 0$$

– *in Ω and for every $a \in \ker A_s$ and $i, k = 1, \ldots, n$*

$$\sum_{j=1}^{n} \left[\partial_k G_s^{ij} - \partial_j G_s^{ik} + \partial_i G_s^{jk} \right] a^j = 0$$

– *for every $\chi \in \mathcal{H}_N(\Omega; \mathbb{R}^n)$ and for every $i, j = 1, \ldots, n$*

$$\int_\Omega \langle \mathbf{M}_s^{ij}; \chi \rangle = 0 \quad \text{and} \quad \sum_{j=1}^{n} \int_\Omega \left[\langle \mathbf{M}_s^{ij}; \nabla \left[\mathcal{N}(\chi^j) \right] \rangle - G_s^{ij} \chi \right] = 0$$

where $\mathbf{M}_s^{ij} = \left((M_s)_k^{ij} = \partial_j G_s^{ik} - \partial_i G_s^{jk} \right)^{1 \le k \le n}$.

Sufficient conditions: *if furthermore A_s is invertible and the following necessary condition holds, for every $k = 1, \ldots, n$,*

$$2\,\partial_k G_a = A_a A_s^{-1} \left[(M_s)_k + \partial_k G_s \right] + \left[\partial_k G_s - (M_s)_k \right] A_s^{-1} A_a \quad \text{in } \Omega, \tag{51}$$

then the above conditions are sufficient and, moreover, the solution of (50) is unique, up to an affine map of the form

$$u(x) = C\,x + c, \quad \text{for some } C \in \mathbb{R}^{n \times n} \text{ and } c \in \mathbb{R}^n$$

where $A\,C + C^t A = 0$.

Remark 29. *(i)* Note that, (51) determines G_a up to a constant once G_s is fixed. See Remark 31 for a comparison with the corresponding Dirichlet problem.

(ii) There are also several hidden necessary conditions of analytical, algebraic and topological nature on the skew-symmetric part G_a. For example, we must have $dG_a = 0$ and

$$\langle G_a(x)\,e; e' \rangle = 0 \quad \text{for every } x \in \Omega \text{ and for every } e, e' \in \ker A_a.$$

Proof of Theorem 28. The equation $A\,\nabla u + (\nabla u)^t A = G$ is equivalent to

$$A_s \nabla u + (\nabla u)^t A_s = G_s \quad \text{and} \quad A_a \nabla u + (\nabla u)^t A_a = G_a.$$

Step 1 (Necessity). The necessary conditions on the symmetric side follow from Theorem 8 applied to A_s, G_s and the equation $A_s \nabla u + (\nabla u)^t A_s = G_s$. To establish the necessity of (51), when A_s is invertible, we proceed as follows. We note that $v = A_s u$ satisfies the equation $\nabla v + (\nabla v)^t = G_s$. Therefore, for every $i, j, k = 1, \ldots, n$,

$$\begin{aligned}
((M_s)_k + \partial_k G_s)^{ij} &= \partial_j G_s^{ik} - \partial_i G_s^{jk} + \partial_k G_s^{ij} \\
&= \partial_j \left(\partial_i v^k + \partial_k v^i \right) - \partial_i \left(\partial_j v^k + \partial_k v^j \right) + \partial_k \left(\partial_i v^j + \partial_j v^i \right) \\
&= 2\,\partial_k \left(\partial_j v^i \right) = 2\,\partial_k (\nabla v)^{ij}
\end{aligned}$$

and similarly

$$((M_s)_k - \partial_k G_s)^{ij} = \partial_j G_s^{ik} - \partial_i G_s^{jk} - \partial_k G_s^{ij} = -2\,\partial_k (\nabla v)^{ji}.$$

Since $A_a \nabla u + (\nabla u)^t A_a = G_a$, we deduce that, in Ω,

$$\begin{aligned}
2\,\partial_k G_a &= 2\,\partial_k \left[A_a \nabla u + (\nabla u)^t A_a \right] = 2\,\partial_k \left[A_a A_s^{-1} \nabla v + (\nabla v)^t A_s^{-1} A_a \right] \\
&= A_a A_s^{-1} \left[(M_s)_k + \partial_k G_s \right] + \left[\partial_k G_s - (M_s)_k \right] A_s^{-1} A_a
\end{aligned}$$

which proves (51).

Step 2 (Sufficiency). We first invoke Theorem 8 to find $u \in C^{r+1}(\overline{\Omega};\mathbb{R}^n)$ satisfying

$$A_s \nabla u + (\nabla u)^t A_s = G_s \quad \text{in } \Omega.$$

As explained in the introduction, we can assume, without loss of generality, that $\nabla u(x_0) = C_0$, since $A\,C_0 + C_0^t A = G(x_0)$ (and thus $A_s C_0 + C_0^t A_s = G_s(x_0)$). It remains to show that u satisfies

$$A_a \nabla u + (\nabla u)^t A_a = G_a \quad \text{in } \Omega.$$

Setting $v = A_s u$, we need to show that

$$A_a A_s^{-1} \nabla v + (\nabla v)^t A_s^{-1} A_a = G_a \quad \text{in } \Omega.$$

Since $A\,C_0 + C_0^t A = G(x_0)$ (and thus $A_a C_0 + C_0^t A_a = G_a(x_0)$) we have

$$A_a A_s^{-1}\nabla v(x_0) + (\nabla v(x_0))^t A_s^{-1} A_a - G_a(x_0) = A_a C_0 + C_0^t A_a - G_a(x_0) = 0$$

and hence it is enough to establish that, for every $k = 1, \ldots, n$

$$\partial_k \left[A_a A_s^{-1}\nabla v + (\nabla v)^t A_s^{-1} A_a - G_a \right] = 0. \tag{52}$$

We have already seen in Step 1 that, in Ω and for every $k = 1, \ldots, n$,

$$(M_s)_k + \partial_k G_s = 2\,\partial_k (\nabla v) \quad \text{and} \quad (M_s)_k - \partial_k G_s = -2\,\partial_k (\nabla v)^t .$$

Therefore, using (51), for every $k = 1, \ldots, n$ and in Ω, we find

$$\begin{aligned} &2\,\partial_k \left[G_a - A_a A_s^{-1}\nabla v - (\nabla v)^t A_s^{-1} A_a \right] \\ &= 2\,\partial_k G_a - A_a A_s^{-1} \left[(M_s)_k + \partial_k G_s \right] + \left[(M_s)_k - \partial_k G_s \right] A_s^{-1} A_a = 0 \end{aligned}$$

and thus (52) holds.

Step 3. Uniqueness follows from Theorem 27. This proves the theorem. ■

5.3. The Dirichlet problem

Theorem 30 (*Dirichlet Problem*)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded connected and smooth, $r \geqslant 2$ be an integer, $G = \left(G^{ij}\right)^{1 \le i,j \le n} \in C^r\left(\overline{\Omega};\mathbb{R}^{n\times n}\right)$ and $A \in \mathbb{R}^{n\times n}$. Then, the following holds.*

Necessary conditions: *if there exists $u \in C^{r+1}\left(\overline{\Omega};\mathbb{R}^n\right)$ satisfying*

$$\begin{cases} A \nabla u + (\nabla u)^t A = G & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{53}$$

then G satisfies the following conditions

– *in Ω and for every $i, j, k, l = 1, \ldots, n$,*

$$\partial_{jl} G_s^{ik} - \partial_{il} G_s^{jk} + \partial_{ik} G_s^{jl} - \partial_{jk} G_s^{il} = 0$$

– *in Ω and for every $a \in \ker A_s$ and $i, k = 1, \ldots, n$*

$$\sum_{j=1}^{n} \left[\partial_k G_s^{ij} - \partial_j G_s^{ik} + \partial_i G_s^{jk} \right] a^j = 0$$

– *on $\partial\Omega$ and for every $i,j=1,\dots,n$,*

$$G_s = (G_s\,\nu)\otimes\nu + \nu\otimes(G_s\,\nu) - \langle G_s\,\nu;\nu\rangle\nu\otimes\nu$$

$$\nabla\left((G_s\,\nu\wedge\nu)^{ij}\right)\wedge\nu = \mathbf{M}_s^{ij}\wedge\nu$$

– *for every $\chi\in\mathcal{H}_T(\Omega;\mathbb{R}^n)$ and for every $i,j=1,\dots,n$,*

$$\int_\Omega \langle \mathbf{M}_s^{ij};\chi\rangle = \int_{\partial\Omega}(G_s\nu\wedge\nu)^{ij}\,\langle\chi;\nu\rangle$$

$$\sum_{j=1}^n\int_\Omega\left[\langle\mathbf{M}_s^{ij};\nabla\left[\mathcal{D}\left(\chi^j\right)\right]\rangle - G_s^{ij}\,\chi^j\right] = \sum_{j=1}^n\int_{\partial\Omega}(G_s\nu\wedge\nu)^{ij}\,\partial_\nu\left(\mathcal{D}(\chi^j)\right)$$

where $\mathbf{M}_s^{ij} = \left((M_s)_k^{ij} = \partial_j G_s^{ik} - \partial_i G_s^{jk}\right)^{1\le k\le n}$.

– *on $\partial\Omega$*

$$[G_a\wedge\nu = 0 \text{ if } n\ge 3] \quad or \quad \left[\int_\Omega G_a^{12} = 0 \text{ if } n=2\right] \tag{54}$$

Sufficient conditions: *if furthermore A_s is invertible and the following necessary condition holds, for every $k=1,\dots,n$,*

$$2\,\partial_k G_a = A_a\,A_s^{-1}\left[(M_s)_k + \partial_k G_s\right] + \left[\partial_k G_s - (M_s)_k\right]A_s^{-1}\,A_a \quad in\ \Omega \tag{55}$$

then the above conditions are sufficient and, moreover, the solution of (53) is unique.

Remark 31. *(i)* Once G_s is fixed in Theorem 30, G_a is uniquely determined by (54) and (55). Indeed, (55) determines G_a up to a constant. Then, uniqueness follows from the boundary condition on G_a i.e. (54).

(ii) In a similar way, (54) and (55) also imply that G is symmetric whenever A is.

(iii) When Ω is simply connected we have $\mathcal{H}_T(\Omega;\mathbb{R}^n)=\{0\}$.

(iv) As already said, there are several hidden necessary conditions, notably that $dG_a=0$.

Proof of Theorem 30. As already observed, $u\in C^{r+1}\left(\overline{\Omega};\mathbb{R}^n\right)$ is a solution of (53) if and only if

$$\begin{cases} A_s\,\nabla u + (\nabla u)^t\,A_s = G_s & \text{in } \Omega\\ A_a\,\nabla u + (\nabla u)^t\,A_a = G_a & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega.\end{cases}$$

Step 1 (Necessary conditions). Let $u\in C^{r+1}\left(\overline{\Omega};\mathbb{R}^n\right)$ satisfy (53). Then the necessary conditions on the symmetric side follow from Theorem 11 applied to A_s and G_s. The condition (54) follows from the corresponding one in Theorem 21 (and, when $n=2$, cf. Remark 22 (i)) applied to A_a and G_a. The necessity of (55), if A_s is invertible, has already been established in Step 1 of the proof of Theorem 28, see (51) .

Step 2 (Sufficient conditions). Using Theorem 11, we find $u\in C^{r+1}(\overline{\Omega};\mathbb{R}^n)$ satisfying

$$\begin{cases} A_s\,\nabla u + (\nabla u)^t\,A_s = G_s & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega.\end{cases}$$

It remains to show that u satisfies

$$A_a\nabla u + (\nabla u)^t\,A_a = G_a \quad \text{in } \Omega.$$

Setting $v=A_s u$ (which satisfies $\nabla v + (\nabla v)^t = G_s$), it is enough to show that

$$A_a\,A_s^{-1}\nabla v + (\nabla v)^t\,A_s^{-1}\,A_a = G_a \quad \text{in } \Omega. \tag{56}$$

But we have already proved in (52) that, for every $k = 1, \ldots, n$,

$$\partial_k \left[G_a - A_a A_s^{-1} \nabla v - (\nabla v)^t A_s^{-1} A_a \right] = 0 \quad \text{in } \Omega.$$

Therefore there exists $C \in \mathbb{R}^{n\times n}$ with $C^t = -C$, such that

$$G_a - A_a A_s^{-1} \nabla v - (\nabla v)^t A_s^{-1} A_a = C \quad \text{in } \Omega.$$

It remains to show that $C = 0$.

– In the case $n = 2$, since $\int G_a^{12} = 0$ and $v = 0$ on $\partial\Omega$, it follows from the divergence theorem that $C = 0$.

– We next discuss the case $n \geq 3$. Since $v = 0$ on $\partial\Omega$, $\nabla v = \beta \otimes \nu$ on $\partial\Omega$, for some $\beta \in C^r(\partial\Omega; \mathbb{R}^n)$. We therefore have on $\partial\Omega$

$$\begin{aligned} C &= G_a - \left[A_a A_s^{-1} \nabla v - \left(A_a A_s^{-1} \nabla v \right)^t \right] = G_a - \left[A_a A_s^{-1} \beta \otimes \nu - \left(A_a A_s^{-1} \beta \otimes \nu \right)^t \right] \\ &= G_a - \left(A_a A_s^{-1} \beta \right) \wedge \nu. \end{aligned}$$

It hence follows from (54) that $C \wedge \nu = 0$ on $\partial\Omega$, which implies that $C = 0$ and thus (56) holds.

Step 3. Uniqueness follows from Theorem 27 and the fact that $u = 0$ on $\partial\Omega$. ■

Acknowledgments

We thank S. Sil for several interesting discussions. We are also grateful to two anonymous referees for their helpful comments. Part of this work was carried out during visits of S. Bandyopadhyay to EPFL, whose hospitality and support is gratefully acknowledged. The research of S. Bandyopadhyay is supported by the MATRICS research project, India grant (File No. MTR/2017/000414) titled “On the Equation $(\nabla u)^t A \nabla u = G$ & its Linearization, & Applications to Calculus of Variations”.

Appendix A. The algebraic case

We wish to discuss the existence of $X \in \mathbb{R}^{n\times n}$ satisfying

$$A X + X^t A = G.$$

Recall that $\mathbb{R}_s^{n\times n}$ and $\mathbb{R}_a^{n\times n}$ denote respectively the set of $n\times n$ symmetric and skew-symmetric matrices with real entries and for $A \in \mathbb{R}^{n\times n}$, we let A_s (respectively A_a) be its symmetric (respectively skew-symmetric) part. We start with an elementary observation, whose proof is immediate.

Proposition 32. *Let $A, G \in \mathbb{R}^{n\times n}$.*

Part 1. *The equation $A X + X^t A = G$ is equivalent to*

$$\begin{cases} A_s X + X^t A_s = (A_s X) + (A_s X)^t = G_s \\ A_a X + X^t A_a = (A_a X) - (A_a X)^t = G_a\,. \end{cases}$$

Part 2. *If $X \in \mathbb{R}^{n\times n}$ is a solution of $A X + X^t A = G$, then*

$$\langle G e; e' \rangle = 0, \quad \forall\, e \in \ker A \text{ and } \forall\, e' \in \ker A^t$$

$$\langle G_s e; e' \rangle = 0,\ \forall\, e, e' \in \ker A_s \quad \text{and} \quad \langle G_a e; e' \rangle = 0,\ \forall\, e, e' \in \ker A_a.$$

A.1. The symmetric and skew-symmetric cases

Proposition 33. *Let $A, G \in \mathbb{R}^{n\times n}_s$. The following statements are then equivalent.*

(i) *There exists $X \in \mathbb{R}^{n\times n}$ satisfying*

$$A\,X + X^t\,A = G.$$

(ii) *$A, G \in \mathbb{R}^{n\times n}_s$ verify*

$$\langle Ga; a'\rangle = 0 \quad \text{for every } a, a' \in \ker A.$$

Proof. The implication (i) ⇒ (ii) is elementary, so we discuss only the reverse one.

Step 1: A invertible. Trivially if A is invertible, then $\ker A = \{0\}$ and the proposition follows at once; any solution of $A\,X + X^t\,A = G$ being of the form

$$X = A^{-1}\left(\frac{G}{2} + Y\right) \quad \text{with} \quad Y^t = -Y.$$

Step 2: $\operatorname{rank} A = k$ *with* $1 \le k \le n-1$. Let $\{e_1, \ldots, e_n\}$ be the standard Euclidean basis.

Special case. Assume first that

$$A = \operatorname{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \quad \text{with} \quad \lambda_i \neq 0,\ i = 1, \ldots, k.$$

In this case (ii) is equivalent to

$$\langle Ge_r; e_s\rangle = 0 \quad \text{for every } r, s = k+1, \ldots, n.$$

and thus, as G is symmetric,

$$G = \begin{pmatrix} G_1 & G_2 \\ G_2^t & 0_{(n-k)\times(n-k)} \end{pmatrix} \quad \text{where} \quad G_1 \in \mathbb{R}^{k\times k}_s,\ G_2 \in \mathbb{R}^{k\times(n-k)}.$$

We then let $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^{k\times k}_s$ which is invertible by assumption. Invoking Step 1, we find $Y \in \mathbb{R}^{k\times k}$ verifying $D\,Y + Y^t\,D = G_1$ and set

$$X = \begin{pmatrix} Y & D^{-1}G_2 \\ 0_{(n-k)\times k} & 0_{(n-k)\times(n-k)} \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

A straightforward computation gives that $A\,X + X^t\,A = G$.

General case. The classical theorem for symmetric matrices gives that there exist $P \in O(n)$ (i.e. $P^tP = I$) such that

$$P^t\,A\,P = \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \quad \text{with} \quad \lambda_i \neq 0,\ i = 1, \ldots, k.$$

Then $\ker A = \operatorname{span}\{P\,e_{k+1}, \ldots, P\,e_n\}$ and, using (ii), we get

$$\left\langle \left(P^t\,G\,P\right) e_r; e_s\right\rangle = \langle G\,(P\,e_r); P\,e_s\rangle = 0 \quad \text{for every } r, s = k+1, \ldots, n.$$

We may therefore apply the special case to $(A, G) = (\Lambda, P^t\,G\,P)$ to find $Z \in \mathbb{R}^{n\times n}$ satisfying

$$\Lambda\,Z + Z^t\,\Lambda = P^t\,G\,P.$$

Setting $X = P\,Z\,P^t$, we get the result and the proof is complete. ■

Proposition 34. *Let $A, G \in \mathbb{R}^{n\times n}_a$. The following statements are then equivalent.*

(i) *There exists $X \in \mathbb{R}^{n\times n}$ satisfying*

$$A\,X + X^t\,A = G.$$

(ii) *$A, G \in \mathbb{R}^{n\times n}_a$ verify*

$$\langle Ga; a'\rangle = 0 \quad \text{for every } a, a' \in \ker A.$$

Proof. *Step 1.* Trivially if A is invertible (this implies necessarily that the dimension n is even), then $\ker A = \{0\}$ and any solution of $A\,X + X^t\,A = G$ is of the form

$$X = A^{-1}\left(\frac{G}{2} + Y\right) \quad \text{with} \quad Y^t = Y.$$

Step 2. We next assume that rank $A = k = 2m$ with $1 \leq k \leq n-1$. Then the classical theorem for skew-symmetric matrices (see, for example, Corollary 2.5.14 in [9]) gives that there exist $P \in O\,(n)$ (i.e. $P^t P = I$) such that

$$P^t\,A\,P = \Lambda = \operatorname{diag}\left(\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix}, 0, \ldots, 0\right) \quad \text{with} \quad \lambda_i \neq 0,\ 1 \leq i \leq m.$$

From there the proof follows exactly as the one of Proposition 33. ■

A.2. The general case

The following proposition is straightforward.

Proposition 35. *Let A_s be invertible and $G \in \mathbb{R}^{n\times n}$ verifying*

$$G_a = A_a\,A_s^{-1}\,\frac{G_s}{2} + \frac{G_s}{2}\,A_s^{-1}\,A_a\,. \tag{57}$$

Then there exists $X \in \mathbb{R}^{n\times n}$ such that

$$A\,X + X^t\,A = G.$$

Proof. Since $X = A_s^{-1}\,G_s/2$ is a solution of $A_s\,X + X^t\,A_s = G_s$ and (57) holds, we see that it also solves $A_a\,X + X^t\,A_a = G_a$ and the proposition is proved. ■

The above proposition can be improved if we further assume that A_a is invertible.

Proposition 36. *Let $n = 2m$ and assume that both A_s and A_a are invertible. Then*

$$A\,X + X^t\,A = G$$

has a solution provided

$$A_s\,A_a^{-1}\,G_a + G_a\,A_a^{-1}\,A_s = G_s + A_a\,A_s^{-1}\,\frac{G_s}{2}\,A_a^{-1}\,A_s + A_s\,A_a^{-1}\,\frac{G_s}{2}\,A_s^{-1}\,A_a\,. \tag{58}$$

Moreover a solution of the equation is given by

$$X = A_a^{-1}\,\frac{G_a}{2} + A_s^{-1}\,\frac{G_s}{4} - A_a^{-1}\,\frac{G_s}{4}\,A_s^{-1}\,A_a + A_a^{-1}\,H \tag{59}$$

for any H satisfying (in particular $H = 0$)

$$H\,A_a^{-1}\,A_s - A_s\,A_a^{-1}\,H = 0 \quad \text{and} \quad H^t = H.$$

Furthermore if $A_s\,A_a^{-1}$ and $A_a\,A_s^{-1}$ are linearly dependent, then (58) is also necessary.

Example 37. Let $n = 2m, \lambda\,\mu \neq 0$ and

$$A = \lambda\,I + \mu\,J \quad \Rightarrow \quad (A_s, A_a) = (\lambda\,I, \mu\,J) \quad \Rightarrow \quad \left(A_s^{-1}, A_a^{-1}\right) = \left(\frac{1}{\lambda}\,I, -\frac{1}{\mu}\,J\right).$$

(note that $A_s A_a^{-1}$ and $A_a A_s^{-1}$ are linearly dependent) and (58) reads as

$$\lambda [J G_a + G_a J] + \mu [G_s - J G_s J] = 0$$

while a solution is given by

$$X = \left(-J \frac{G_a}{2\mu} + \frac{G_s}{4\lambda} + J \frac{G_s}{4\lambda} J \right) + J H$$

for any H satisfying (in particular $H = 0$) $J H - H J = 0 \quad \text{and} \quad H^t = H.$

Proof. *Step 1 (Sufficient conditions).* Inserting the explicit form (59) of the solution into the equation, bearing in mind (58), we immediately get the result.

Step 2 (Necessary conditions). The equation $A X + X^t A = G$ is equivalent (cf. Proposition 32) to the following system

$$\begin{cases} A_s X + X^t A_s = (A_s X) + (A_s X)^t = G_s \\ A_a X + X^t A_a = (A_a X) - (A_a X)^t = G_a . \end{cases} \tag{60}$$

We assume now that $A_s A_a^{-1}$ and $A_a A_s^{-1}$ are linearly dependent. Call

$$\Phi = A_s A_a^{-1} G_a + G_a A_a^{-1} A_s - \left(G_s + A_a A_s^{-1} \frac{G_s}{2} A_a^{-1} A_s + A_s A_a^{-1} \frac{G_s}{2} A_s^{-1} A_a \right)$$

We have to show that $\Phi = 0$. Return to (60) to get that

$$\begin{cases} A_s A_a^{-1} G_a = A_s X + A_s A_a^{-1} X^t A_a \\ G_a A_a^{-1} A_s = A_a X A_a^{-1} A_s + X^t A_s \\ -\left(A_a A_s^{-1} \frac{G_s}{2} A_a^{-1} A_s\right) = -\frac{1}{2}\left(A_a X A_a^{-1} A_s + A_a A_s^{-1} X^t A_s A_a^{-1} A_s\right) \\ -\left(A_s A_a^{-1} \frac{G_s}{2} A_s^{-1} A_a\right) = -\frac{1}{2}\left(A_s A_a^{-1} A_s X A_s^{-1} A_a + A_s A_a^{-1} X^t A_a\right) \\ -G_s = -\left(A_s X + X^t A_s\right). \end{cases}$$

Summing up the five equations, we find that

$$2\,\Phi = A_a X A_a^{-1} A_s + A_s A_a^{-1} X^t A_a - A_s A_a^{-1} A_s X A_s^{-1} A_a - A_a A_s^{-1} X^t A_s A_a^{-1} A_s.$$

Since A_s and A_a are invertible and $A_s A_a^{-1}$ and $A_a A_s^{-1}$ are linearly dependent, there exists $\alpha \neq 0$ such that $A_s A_a^{-1} = \alpha A_a A_s^{-1}$ ($\Rightarrow A_a^{-1} A_s = \alpha A_s^{-1} A_a$). Inserting this fact into the previous equation, we have indeed obtained our claim. ■

Appendix B. Ellipticity

We recall (cf. [6]) the notion of ellipticity that we use here.

Definition 38. Let $A \in \mathbb{R}^{n \times n}$. We say that the operator $\mathcal{L}$ defined by

$$\mathcal{L}u = A \nabla u + (\nabla u)^t A$$

is *elliptic* if, for every $\xi \in \mathbb{R}^n \setminus \{0\}$, the system of algebraic equations

$$\mathcal{A}(\lambda, \xi) = (A\lambda) \otimes \xi + \xi \otimes \left(A^t \lambda\right) = 0$$

has $\lambda = 0 \in \mathbb{R}^n$ as the only solution.

Proposition 39. *The operator $\mathcal{L}$ is elliptic if and only if A_s, the symmetric part of A, is invertible.*

Proof. We start by observing that $\mathcal{A}(\lambda,\xi)=0$ is equivalent to

$$(A_s\,\lambda)\otimes\xi+\xi\otimes(A_s\,\lambda)=0 \quad\text{and}\quad (A_a\,\lambda)\otimes\xi-\xi\otimes(A_a\,\lambda)=0.$$

(i) Assume first that A_s is invertible. Setting $\mu=A_s\,\lambda$, we find that the first set of equations is equivalent to $\mu\otimes\xi+\xi\otimes\mu=0$. When $\xi\neq 0$, the only solution is then $\mu=0$. Since A_s is invertible, we have the claim.

(ii) If A_s is not invertible, we can find $\lambda\neq 0$ with $\lambda\in\ker A_s$ and therefore

$$(A_s\,\lambda)\otimes\xi+\xi\otimes(A_s\,\lambda)=0 \quad\text{for every } \xi\neq 0.$$

Then two cases can happen. Either $\lambda\in\ker A_a$ and thus

$$(A_a\,\lambda)\otimes\xi-\xi\otimes(A_a\,\lambda)=0 \quad\text{for every } \xi\neq 0$$

concluding the claim. Or $\lambda\notin\ker A_a$ and hence $\xi=A_a\,\lambda\neq 0$ satisfies trivially

$$(A_a\,\lambda)\otimes\xi-\xi\otimes(A_a\,\lambda)=0.$$

Our claim is therefore established. ■

References

[1] C. Amrouche, P. Ciarlet, L. Gratie, S. Kesavan, On the characterizations of matrix fields as linearized strain tensor fields, J. Math. Pures Appl. 86 (2006) 116–132.

[2] A. Angoshtari, A. Yavari, On the compatibility equations of nonlinear and linear elasticity in the presence of boundary conditions, Z. Angew. Math. Phys. 66 (2015) 3627–3644.

[3] P. Ciarlet, Linear and nonlinear functional analysis with applications, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[4] G. Csato, B. Dacorogna, O. Kneuss, The Pullback Equation for Differential Forms, Birkhaüser, 2012.

[5] B. Dacorogna, N. Fusco, L. Tartar, On the solvability of the equation $\operatorname{div} u=f$ in L^1 and in C^0, Rend. Mat. Acc. Lincei 14 (2003) 239–245.

[6] B. Dacorogna, W. Gangbo, O. Kneuss, Symplectic factorization, Darboux theorem and ellipticity, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018) 327–356.

[7] B. Dacorogna, O. Kneuss, Neves W., Some remarks on the Lie derivative and the pullback equation for contact forms, Adv. Nonlinear Stud. 17 (2017) 269–282.

[8] J.J. Duistermaat, L. Hörmander, Fourier integral operators. II, Acta Math. 128 (1972) 183–269.

[9] R.A. Horn, C.A. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[10] S. Takahashi, On the Poincaré-Bogovski lemma on differential forms, Proc. Japan Acad. Ser. A Math. Sci. 68 (1992) 1–6.

[11] A. Yavari, Compatibility equations of nonlinear elasticity for non-simply-connected bodies, Arch. Ration. Mech. Anal. 209 (2013) 237–253.