Topological matter and its exploration with quantum gases

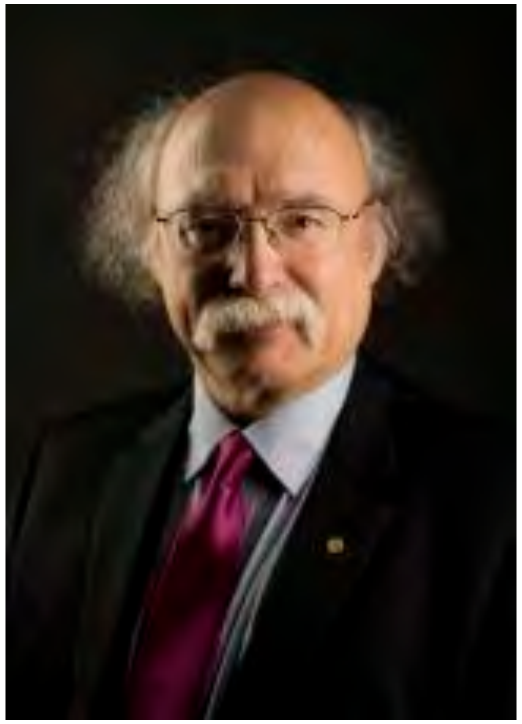
Jean Dalibard

Lectures at EPFL
November 2019

Physics and topology

J.M. Kosterlitz

D.J. Thouless



F.D. Haldane

Nobel prize 2016: "for theoretical discoveries of topological phase transitions and topological phases of matter."

Link between topology and physics

George Gamow, in *The Great Physicists, from Galileo to Einstein,* 1961

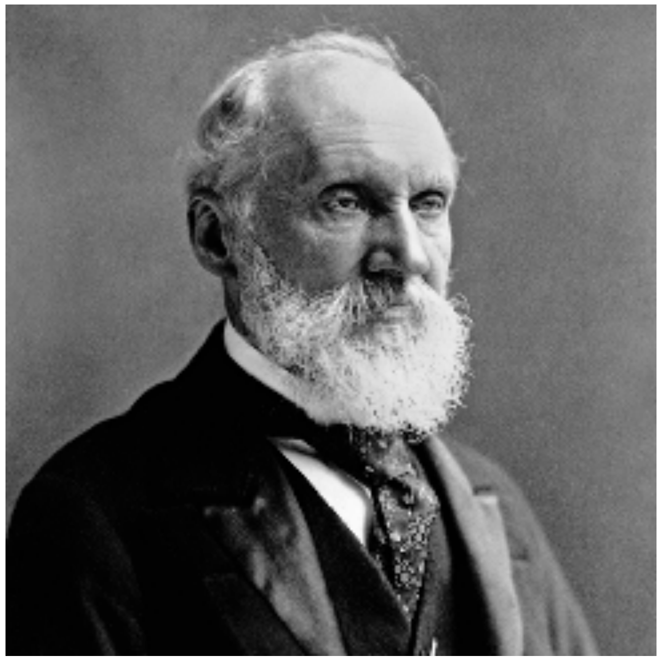
When Einstein wanted to interpret gravity as the curvature of four-dimensional, space-time continuum, he found waiting for him Riemann's theory of curved multidimensional space.

When Heisenberg looked for some unusual kind of mathematics to describe the motion of electrons inside of an atom, noncommutative algebra was ready for him.

Only the number theory and topology (analysis situs) still remain purely mathematical disciplines without any application to physics. Could it be that they will be called to help in our further understanding of the riddles of nature?

2018 : 20 new manuscrits every day on arXiv-Physics with the word « topology » in title or abstract

Gamow's statement was a bit too strong...



William Thomson
(Lord Kelvin)

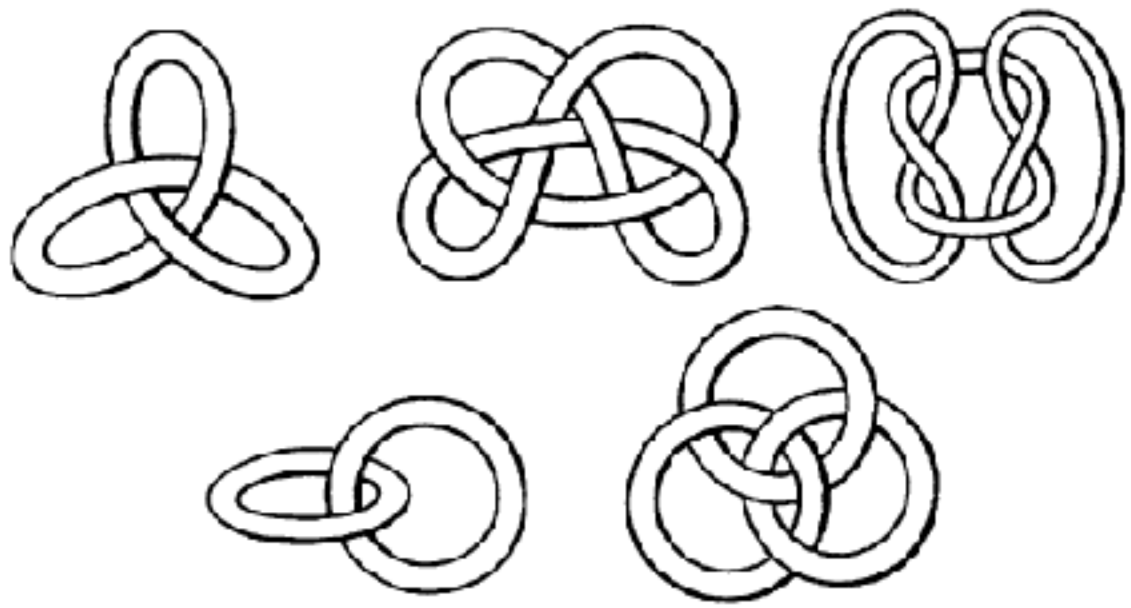
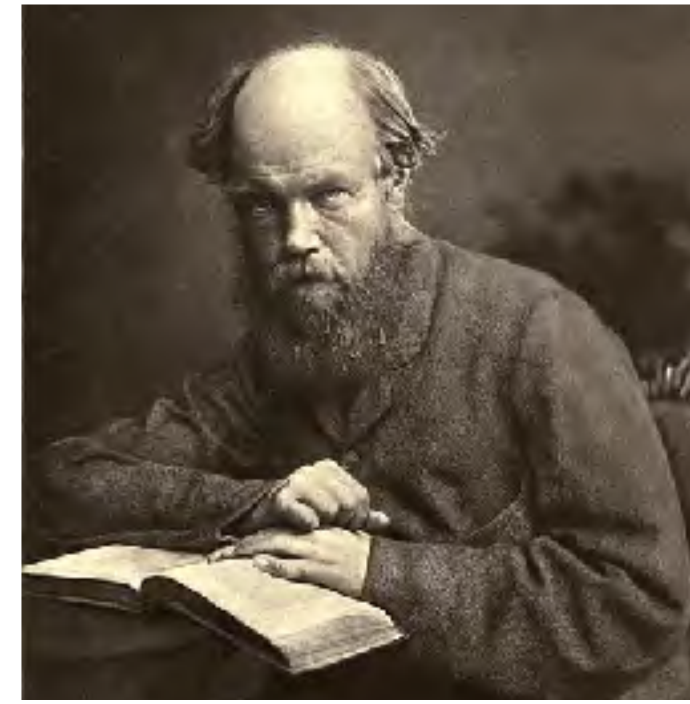


Fig. 4. Candidates for atoms? Knots and links from (Thomson 1869)



Peter Guthrie Tait

... every variety of combinations might exist. Thus a long chain of vortex rings, or three rings, each running through each other, would give each very characteristic reactions upon other such kinetic atoms. (1867)

Also Maxwell, Poincaré,...

What topology is (naively) for a physicist

Mathematical study of shapes, aiming at establishing an equivalence between objects that can be transformed into one another by a continuous deformation

Number of handles g of an object : "genus" of a closed surface in 3D space

$g = 0$

$g = 1$

$g = 2$

A change of class can only occur through a singularity:
topological protection

Link between topology and geometry

In geometry one can define at any point of a regular and orientable closed surface its curvature $\Omega(\boldsymbol{r})$. For example, one gets for a sphere $\Omega = 1/R^2$

Gauss - Bonnet theorem: $$\frac{1}{4\pi}\int_{\mathcal{S}} \Omega(\boldsymbol{r})\ \mathrm{d}^2 r = 1 - g$$

Extracted from the web site of Scientific American, by Keenan Crane (Columbia U.)

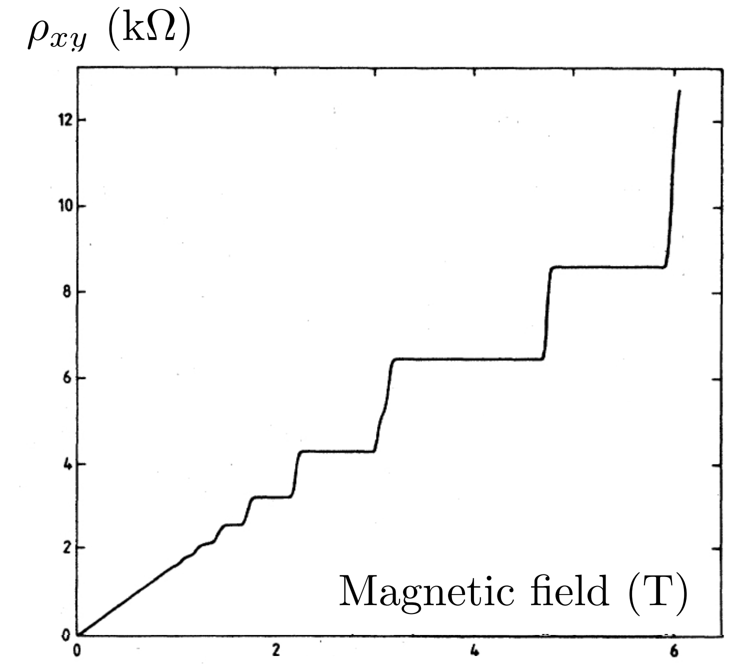
Topological protection in physics

If a physical quantity can be expressed as

$$\int_\Lambda \Omega(\lambda)\ \mathrm{d}\lambda$$

where Λ is a closed line or surface, this quantity may be :

- quantized (for instance, conductivity in Quantum Hall effect)
- topologically protected (unchanged by a perturbation or disorder)



Important both for conceptual and practical aspects

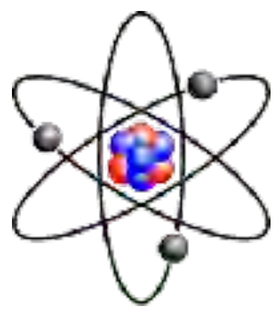
Realization of macroscopic standards that are invariant between various labs

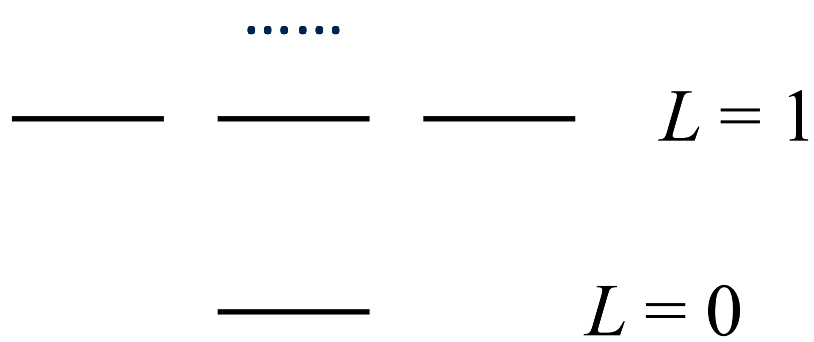
Geometrical vs. topological order

Geometrical order: appears for systems with a certain symmetry

Example : rotation in variant system

→ One can classify energy eigenstates using their angular momentum





non robust: this order may be broken by a perturbation

Topological order: defined by classes of objects that are stable by deformation

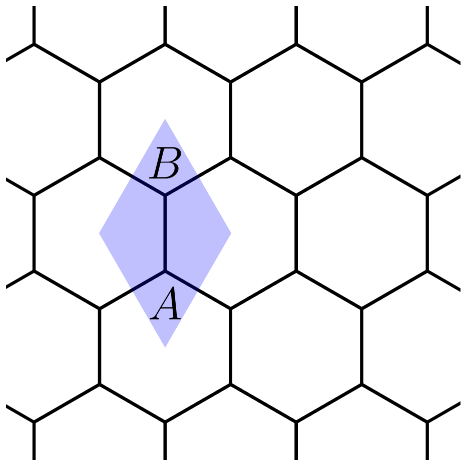
By essence, robust classification

Goal of these lectures

Introduction to topological phases of matter,
from the point of view of atom and photons gases

Simple systems:

- One ou two dimensions
- Periodic or uniform systems



Optical lattices, regular arrangement of wave guides

- Essentially no interaction between particules, except for the case of Majorana modes [and for topological phase transitions, week 4]

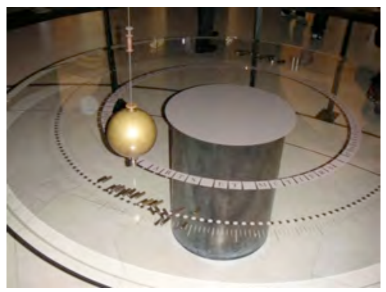
Ozawa, Price, Amo, et al.,*Topological Photonics*, Rev. Mod. Phys. **91**, 015006 (2019)
Cooper, Dalibard, Spielman, *Topological Bands for Ultracold Atoms*, Rev. Mod. Phys. **91**, 015005 (2019)

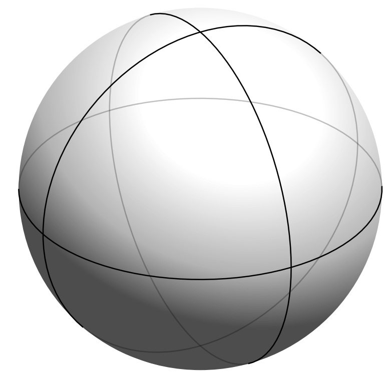
Hasan & Kane, *Topological insulators*, Rev. Mod. Phys. **82**, 3045 (2010)
Qi & Zhang, *Topological insulators and superconductors*, Rev. Mod. Phys. **83**,1057 (2011)

Goal for this first lecture

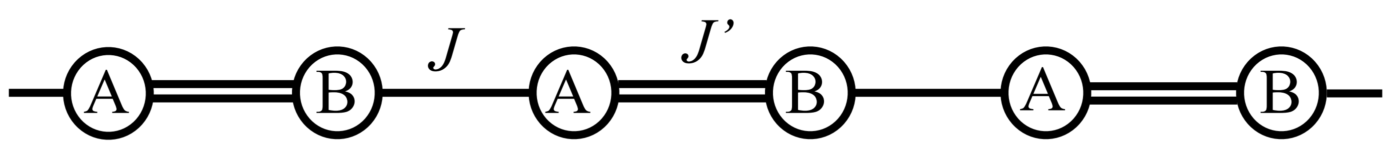
Setup of the notions that will be essential for the whole series

- Berry phase, Bloch sphere, band theory





- First steps towards topological bands (1D for the moment)



Su-Schriffer-Hegger (SSH) 1D model for polyacetylene

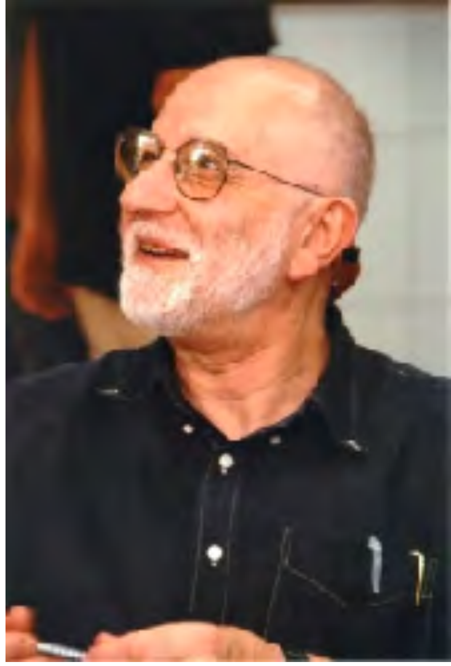
Kitaev 1D model for a superconductor

Bulk vs. edge states, role of symmetries (sublattice, particle-hole)

1.

A key ingredient:
Berry geometrical phase

Adiabatic evolution in quantum physics



Berry 1984

Pancharatnam (1956)
Mead & Truhlar (1979)

Quantum system (atom, molecule, macroscopic body) depending on an external parameter $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$

Evolution described by the Hamiltonian $\hat{H}_{\boldsymbol{\lambda}}$:

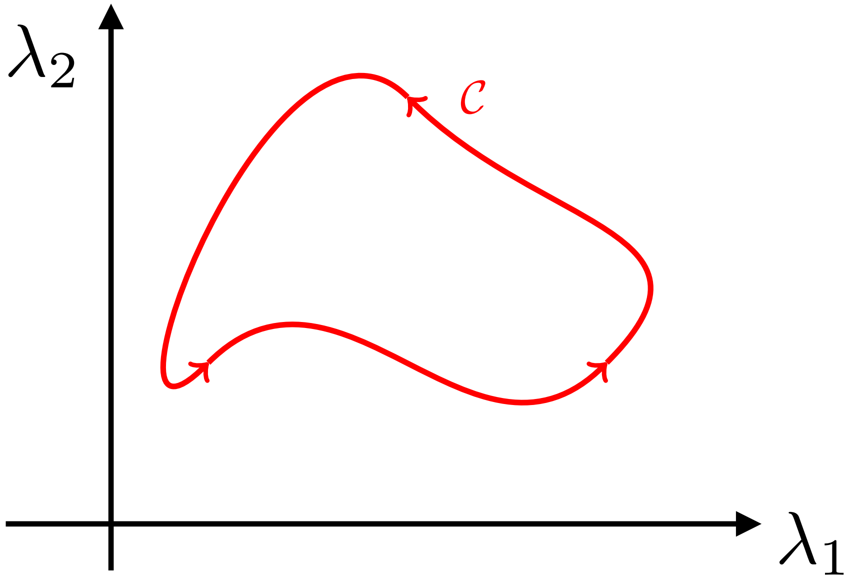
$$\mathrm{i}\hbar \frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \hat{H}_{\boldsymbol{\lambda}(t)}\ |\psi(t)\rangle$$

What happens when:

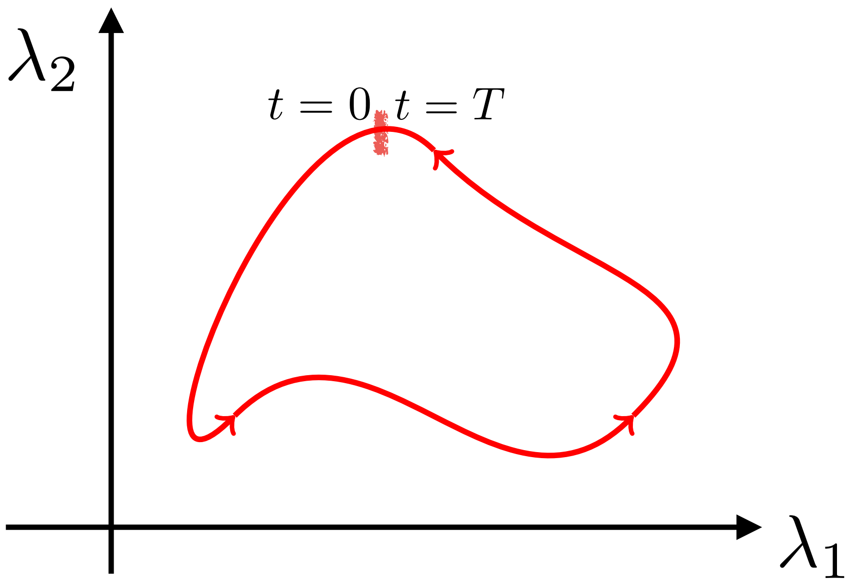
- The parameter $\boldsymbol{\lambda}$ depends slowly on time?

and

- The parameter $\boldsymbol{\lambda}$ follows a closed trajectory?



Dynamical phase and geometrical phase



Suppose that the system is initially prepared in an eigenstate of the Hamiltonian

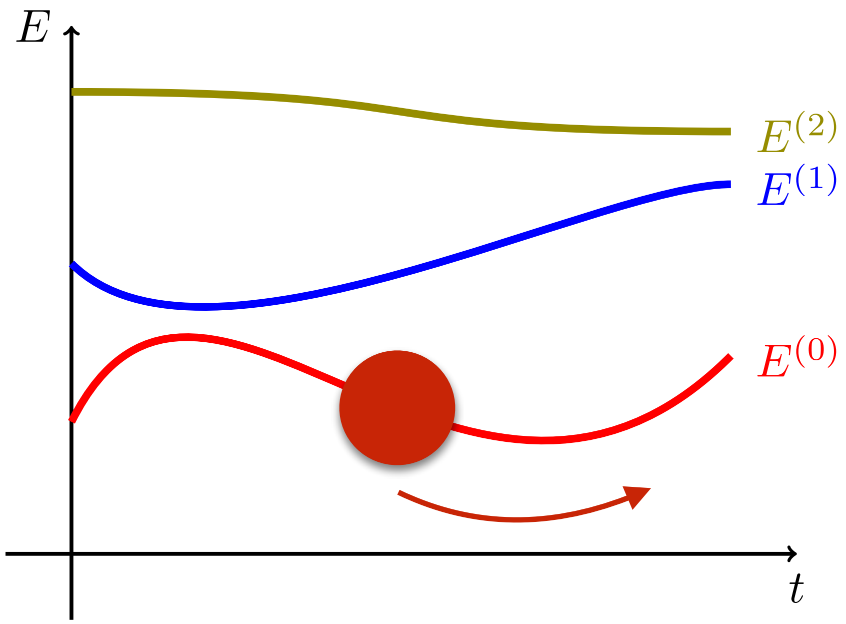
If the evolution of **λ** is slow enough, the final state reads (in the absence of degeneracies):

$$|\psi(T)\rangle \;\approx\; \mathrm{e}^{\mathrm{i}\Phi}\,|\psi(0)\rangle$$

Initial and final states coincide, up to a phase

$$\Phi = \Phi_{\mathrm{dyn}} + \Phi_{\mathrm{geom}} \qquad \begin{cases} \Phi_{\mathrm{dyn}} \propto \displaystyle\int E(t)\,\mathrm{d}t & \text{usual dynamical phase} \\ \\ \Phi_{\mathrm{geom}} = ? \end{cases}$$

Geometrical phase and Berry connection



At each time *t*, we introduce an eigenstate basis of the Hamiltonian $\hat{H}_{\boldsymbol{\lambda}}$

$$|\psi_{\boldsymbol{\lambda}}^{(n)}\rangle$$

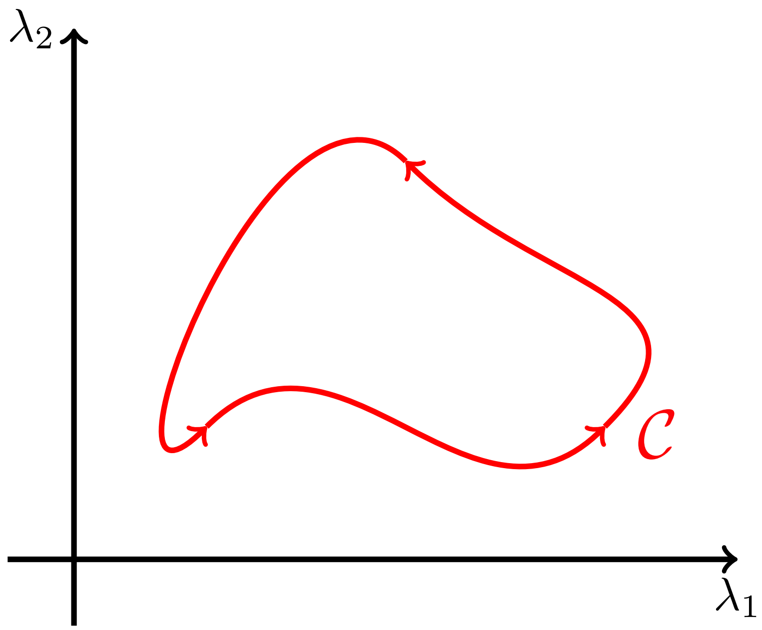
Define Berry connection:

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} = \mathrm{i}\,\langle \psi_{\boldsymbol{\lambda}}^{(n)} | \boldsymbol{\nabla} \psi_{\boldsymbol{\lambda}}^{(n)} \rangle \qquad |\boldsymbol{\nabla}\psi\rangle = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\lambda_1}|\psi\rangle \\ \frac{\mathrm{d}}{\mathrm{d}\lambda_2}|\psi\rangle \\ \vdots \end{pmatrix}$$

Real quantity since $\langle \psi | \boldsymbol{\nabla} \psi \rangle$ is purely imaginary (or zero)

The geometrical phase then reads : $\Phi_{\mathrm{geom}} = \int_{\boldsymbol{\lambda}(0)}^{\boldsymbol{\lambda}(t)} \boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(0)} \cdot \mathrm{d}\boldsymbol{\lambda}$

Depends on the path followed to go from $\boldsymbol{\lambda}(0)$ to $\boldsymbol{\lambda}(t)$, but not on the time spent to go along this path



Case of a closed contour

$\boldsymbol{\lambda}$: parameter in 1, 2 or 3 dimensions

$$\Phi_{\mathrm{geom}} = \oint_{\mathcal{C}} \boldsymbol{\mathcal{A}}^{(0)}_{\boldsymbol{\lambda}} \cdot \mathrm{d}\boldsymbol{\lambda}$$

Calculation via Stokes formula: $\boldsymbol{\Omega}^{(n)}_{\boldsymbol{\lambda}} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}^{(n)}_{\boldsymbol{\lambda}} = \mathrm{i}\, \langle \boldsymbol{\nabla}\psi^{(n)}_{\boldsymbol{\lambda}}| \times |\boldsymbol{\nabla}\psi^{(n)}_{\boldsymbol{\lambda}}\rangle$

Berry curvature

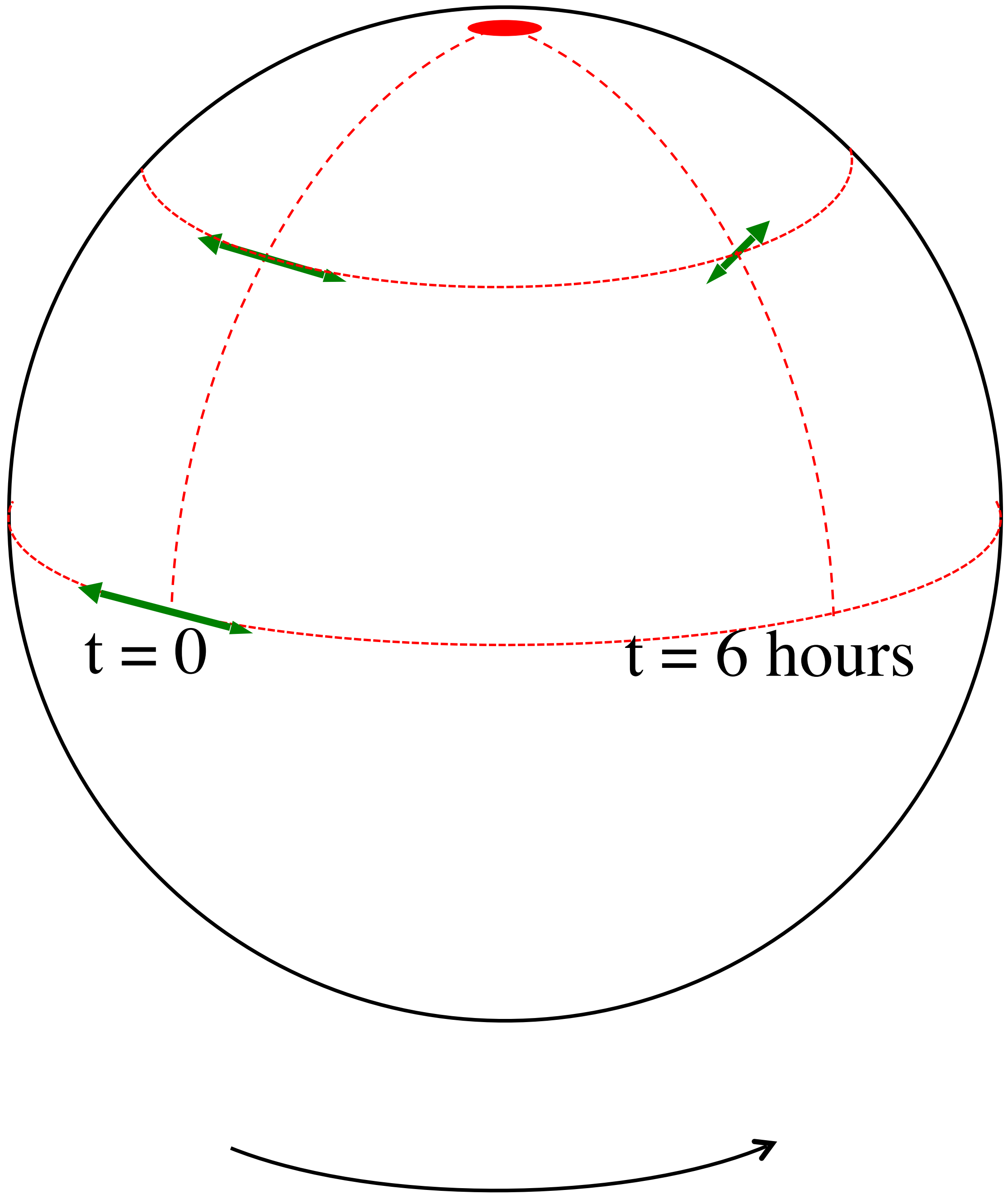
$$\longrightarrow \quad \Phi_{\mathrm{geom}} = \iint_{\Sigma} \boldsymbol{n} \cdot \boldsymbol{\Omega}^{(n)}_{\boldsymbol{\lambda}}\ \mathrm{d}^2\lambda$$

Analogy with magnetostatics:

Berry connection	$\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}} \longleftrightarrow \boldsymbol{A}(\boldsymbol{r})$	vector potential
Berry curvature	$\boldsymbol{\Omega}_{\boldsymbol{\lambda}} \longleftrightarrow \boldsymbol{B}(\boldsymbol{r})$	magnetic field

Does not seem very interesting in 1D: mere round trip?

Foucault pendulum



Wikipedia

In 24 heures, the external parameter — the hanging point of the pendulum — follows a closed loop

The two eigenmodes of the circular motion (clockwise and counterclockwise rotations) acquire different geometrical phases

Evidenced by a rotation of the oscillation plane for a linear motion

Berry curvature and gauge invariance

A gauge change in quantum mechanics reads

$$|\psi_{\boldsymbol{\lambda}}^{(n)}\rangle \quad \longrightarrow \quad |\tilde{\psi}_{\boldsymbol{\lambda}}^{(n)}\rangle = \mathrm{e}^{\mathrm{i}\chi_{\boldsymbol{\lambda}}^{(n)}}\, |\psi_{\boldsymbol{\lambda}}^{(n)}\rangle$$

Berry connection $\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} = \mathrm{i}\,\langle\psi_{\boldsymbol{\lambda}}^{(n)}|\boldsymbol{\nabla}\psi_{\boldsymbol{\lambda}}^{(n)}\rangle$ is not invariant in this transformation:

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} \quad \longrightarrow \quad \tilde{\boldsymbol{\mathcal{A}}}_{\boldsymbol{\lambda}}^{(n)} = \boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} - \boldsymbol{\nabla}\chi_{\boldsymbol{\lambda}}^{(n)}$$

Berry curvature $\boldsymbol{\Omega}_{\boldsymbol{\lambda}}^{(n)} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)}$ is unchanged:

$$\boldsymbol{\Omega}_{\boldsymbol{\lambda}}^{(n)} \quad \longrightarrow \quad \tilde{\boldsymbol{\Omega}}_{\boldsymbol{\lambda}}^{(n)} = \boldsymbol{\Omega}_{\boldsymbol{\lambda}}^{(n)}$$

cf. magnetostatics

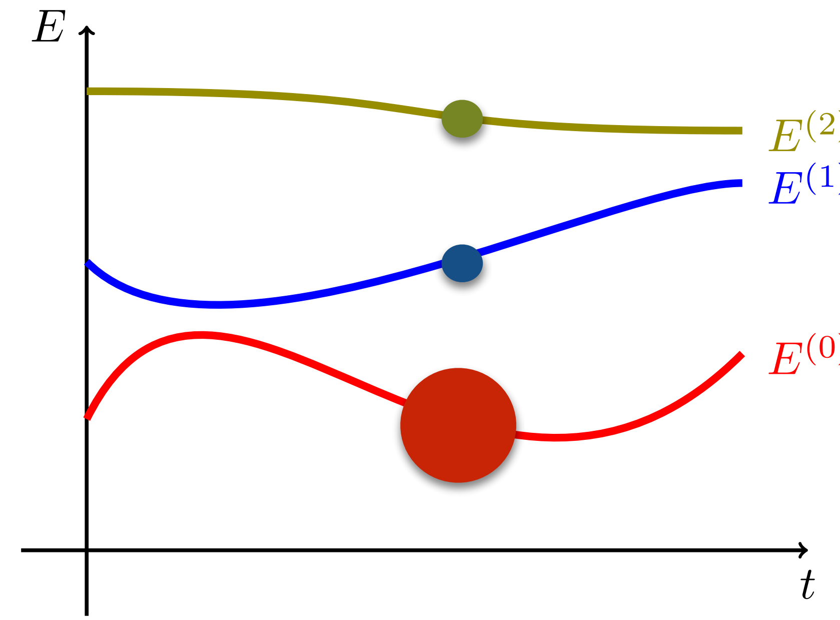
$\boldsymbol{\Omega}^{(n)}$ *is a quantity that can be physically measured*

same for the geometrical phase: $\mathrm{e}^{\mathrm{i}\tilde{\Phi}_{\mathrm{geom}}} = \mathrm{e}^{\mathrm{i}\Phi_{\mathrm{geom}}}$

Next order in the adiabatic approximation

Up to now we looked at 0th order : $|\psi(t)\rangle \approx \mathrm{e}^{\mathrm{i}\Phi}\ |\psi^{(0)}_{\boldsymbol{\lambda}}\rangle.$

The state of the system remains at any time parallel to the state we started from.



We will also need the 1st order :

$$|\psi(t)\rangle \approx \mathrm{e}^{\mathrm{i}\Phi}\ |\psi^{(0)}_{\boldsymbol{\lambda}}\rangle + \sum_{n\neq 0} c_n\ |\psi^{(n)}_{\boldsymbol{\lambda}}\rangle$$

A relatively simple calculation gives $c^{(n)}(t) \approx \mathrm{i}\hbar\ \dot{\boldsymbol{\lambda}} \cdot \dfrac{\langle \psi^{(n)}_{\boldsymbol{\lambda}} | \boldsymbol{\nabla} \psi^{(0)}_{\boldsymbol{\lambda}} \rangle}{E^{(n)}_{\boldsymbol{\lambda}} - E^{(0)}_{\boldsymbol{\lambda}}}$

valid if $|c^{(n)}| \ll 1$ at any time

Linear response of the system to the motion of the external parameter

Berry's phase for a spin 1/2

Two-level system

Hilbert space with dimension 2 : $|\psi\rangle = \alpha|+\rangle \; + \; \beta|-\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\qquad |\alpha|^2 + |\beta|^2 = 1$

Most general Hamiltonian for this type of system: Hermitian 2x2 matrix

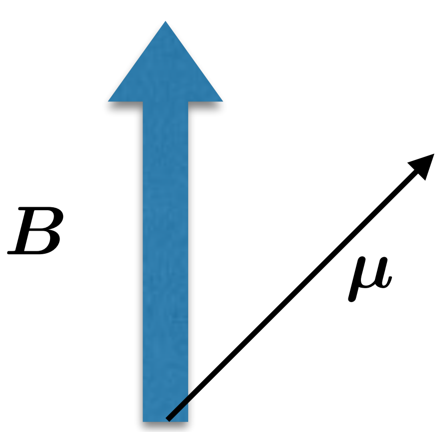
$$\hat{H} = E_0 \, \hat{1} \; - \; \boldsymbol{h} \cdot \hat{\boldsymbol{\sigma}}$$

$E_0, \; (h_x, h_y, h_z)$: real

$\hat{\sigma}_j$: Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

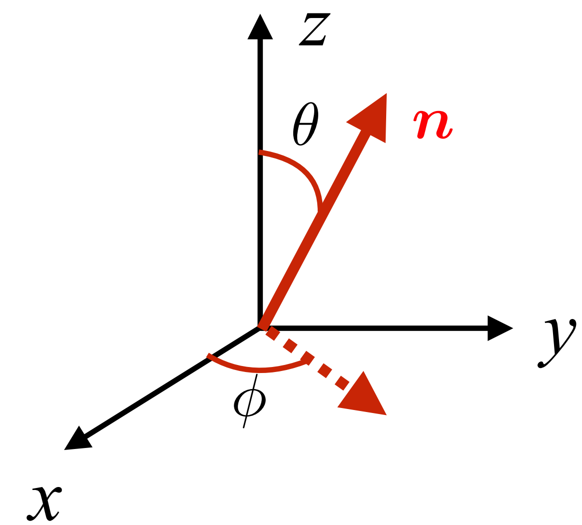
Example: spin 1/2 magnetic moment in a magnetic field



$\hat{\boldsymbol{\mu}} = \mu \hat{\boldsymbol{\sigma}}$: "magnetic moment" operator

$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B}$ $\qquad \boldsymbol{h} = \mu \, \boldsymbol{B}$

Bloch sphere



Energies and eigenstates of the Hamiltonian $\hat{H} = E_0\,\hat{1} \;-\; \boldsymbol{h}\cdot\hat{\boldsymbol{\sigma}}$?

$\boldsymbol{n} = \dfrac{\boldsymbol{h}}{|\boldsymbol{h}|}$ unit vector of spherical coordinates (θ, ϕ)

$$\boldsymbol{n} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$$

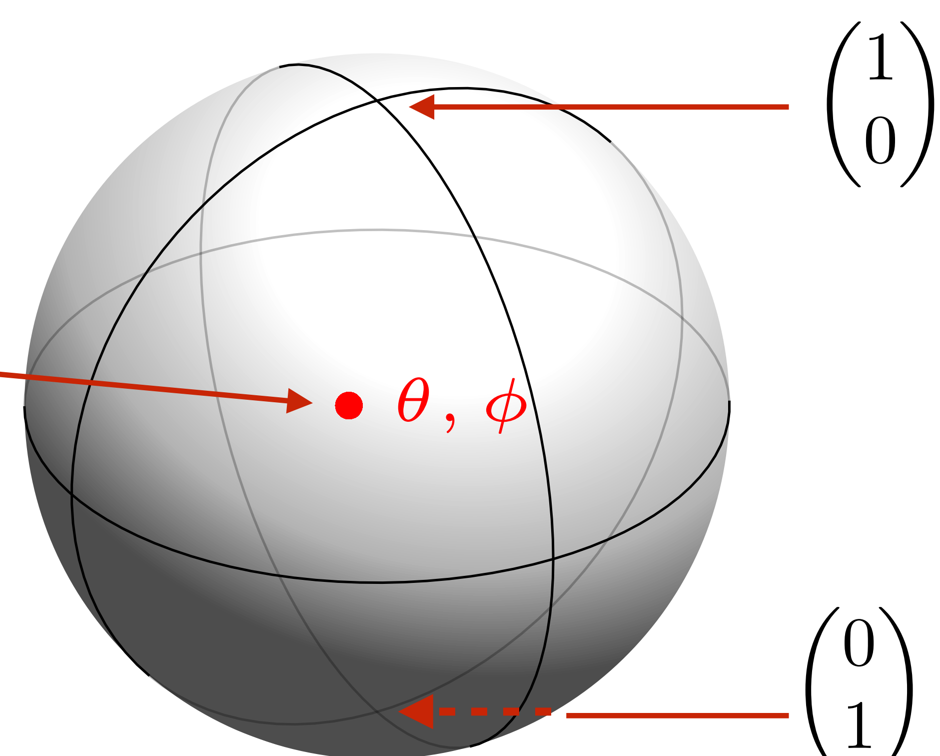
With this parametrization : $\hat{H} = E_0\,\hat{1} \;-\; |\boldsymbol{h}| \begin{pmatrix} \cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & -\cos\theta \end{pmatrix}$

Eigenenergies: $E^{(\pm)} = E_0 \pm |\boldsymbol{h}|$

Eigenstates :

$$|\psi^{(-)}\rangle = \begin{pmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{pmatrix}$$

$$|\psi^{(+)}\rangle = \begin{pmatrix} \sin(\theta/2) \\ -\mathrm{e}^{\mathrm{i}\phi}\cos(\theta/2) \end{pmatrix}$$

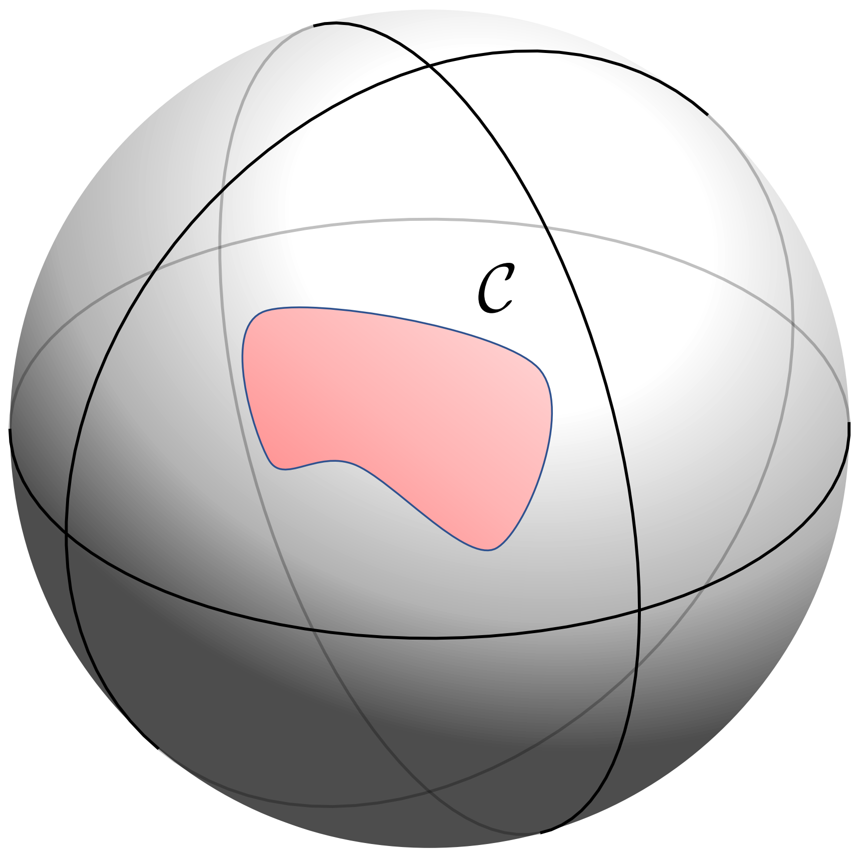


Berry curvature and geometrical phase

$$\hat{H} = E_0\,\hat{1}\;-\;\boldsymbol{h}\cdot\hat{\boldsymbol{\sigma}}$$

Case where the vector $\boldsymbol{h}$ varies in time and follows a closed contour

What is the geometrical phase acquired by the state of the system when it follows adiabatically $|\psi^{(-)}\rangle$?



We note $\boldsymbol{\alpha}$ the solid angle sustained by the trajectory $\mathcal{C}$ of the vector $\boldsymbol{h}$

$$\Phi_{\text{geom.}}(\mathcal{C}) = \pm\frac{\alpha}{2}$$

For a trajectory around the equator:

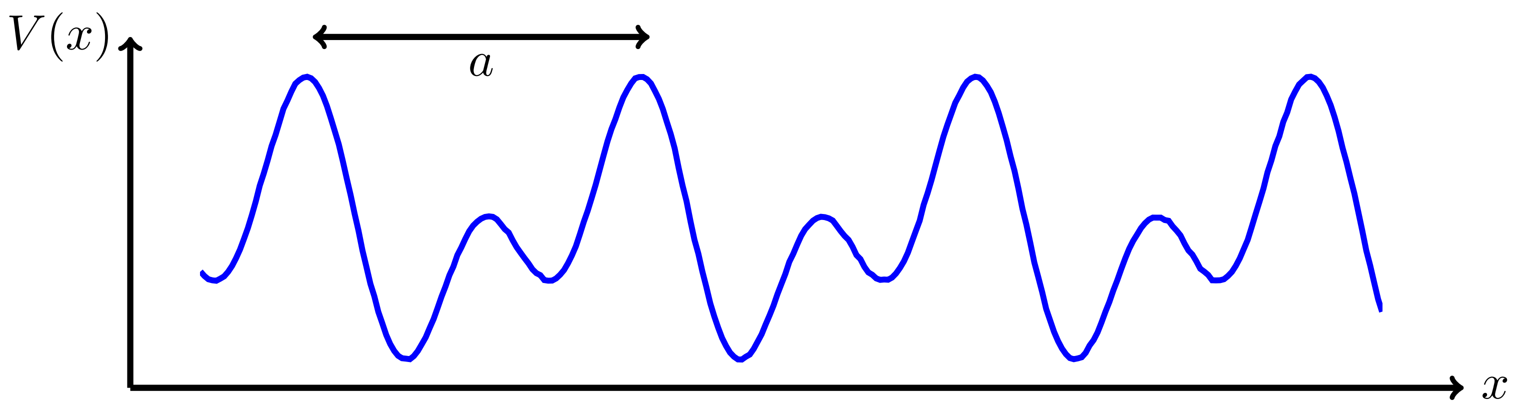
$$\alpha = 2\pi \longrightarrow \Phi_{\text{geom.}}(\mathcal{C}) = \pm\pi$$

2.

Periodic potential in Quantum Mechanics and the SSH problem

Bloch theorem

Motion of a particle of mass m in the periodic potential $V(x)$



$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

$$\hat{p} = -\mathrm{i}\hbar\,\partial_x$$

One can look for the eigenstates of $\hat{H}$ with the form:

$$\psi_q(x) \;=\; \mathrm{e}^{\mathrm{i}qx}\; u_q(x)$$

Plane wave with wavevector ***q***

Periodic function
$u_q(x+a) = u_q(x)$

The "periodic" Hamiltonian $\hat{H}_{\boldsymbol{q}}$

Insert Bloch's form $\psi_q(x) \;=\; \mathrm{e}^{\mathrm{i}qx}\, u_q(x)$ in the eigenvalue equation for $\hat{H}$

$$\hat{H}\,\psi_q(x) = E_q\,\psi_q(x) \longrightarrow \hat{H}_q\,u_q(x) = E_q\,u_q(x)$$

$\hat{H}_{\boldsymbol{q}}$: Hamiltonian for the periodic part $u_q(x)$

$$\hat{H} = \frac{\hbar^2}{2m}\left(-\mathrm{i}\partial_x\right)^2 + V(x) \longrightarrow \hat{H}_q = \frac{\hbar^2}{2m}\left(-\mathrm{i}\partial_x + q\right)^2 + V(x)$$

Example of a Hamiltonian that depends on a parameter, here q

The notion of a geometrical phase will emerge in a natural way
if one is able to control and modify the wavevector q in an experiment

Bloch oscillations

Brillouin zone

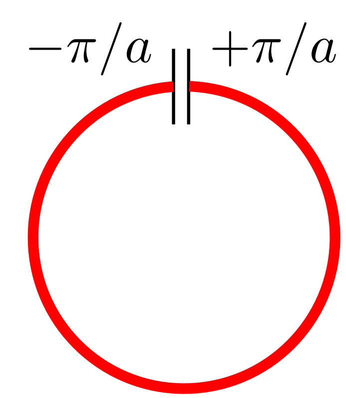
The wavevector q can take any value but there are not all independent

Consider q and $q + \dfrac{2\pi}{a}$

$$\psi_q(x) = \mathrm{e}^{\mathrm{i}qx}\, u_q(x)$$

$$\begin{aligned}\psi_{q+2\pi/a}(x) &= \mathrm{e}^{\mathrm{i}(q+2\pi/a)x}\, u_{q+2\pi/a}(x) \\ &= \mathrm{e}^{\mathrm{i}qx} \left(\underbrace{\mathrm{e}^{\mathrm{i}2\pi x/a}\, u_{q+2\pi/a}(x)}_{\text{Periodic function}} \right)\end{aligned}$$

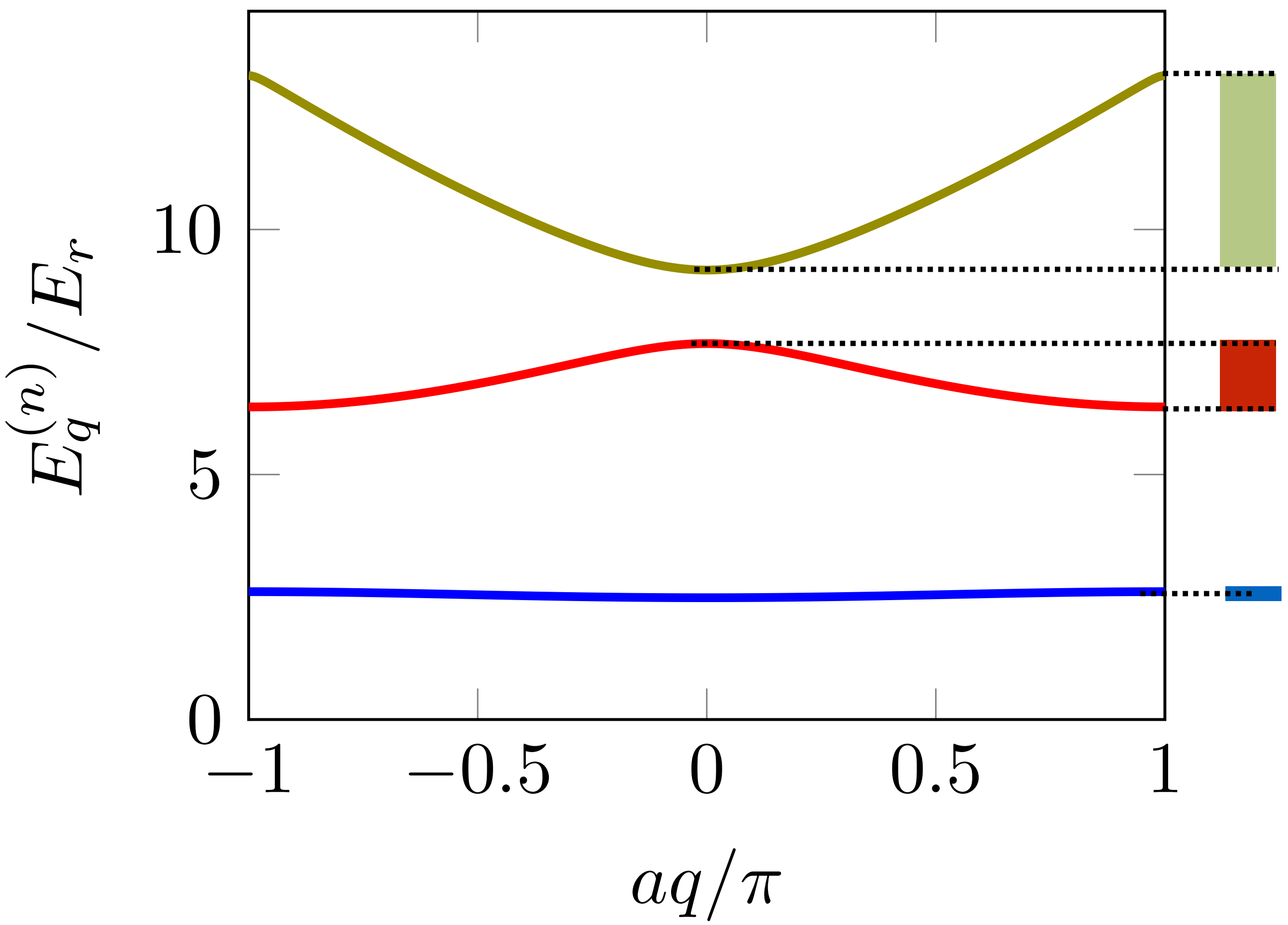
To avoid double counting, one must restrict q to a Brillouin zone:



Energy bands

Example of a sinusoidal potential created by a standing wave of light on atoms

$$V(x) = V_0 \sin^2(\pi x/a)$$



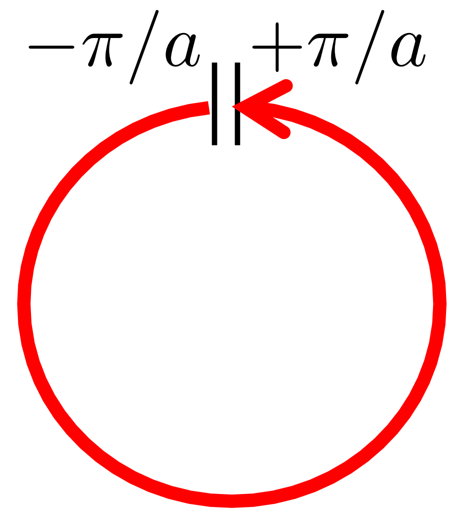
For each value of q one gets a discrete set of energy eigenvalues:

$$E_q^{(0)} \leq E_q^{(1)} \leq E_q^{(2)} \leq \dots$$

When one groups the energies found for all values of q, one obtains allowed energy bands, separated by forbidden gaps.

Zak phase

A first step towards topology



The one-way path from $-\pi/a$ to $+\pi/a$ corresponds to a closed loop in parameter space

What happens to Berry phase in this case? $\Phi_{\rm geom} = \int_{-\pi/a}^{+\pi/a} \mathrm{i} \langle u_q | \partial_q u_q \rangle \, \mathrm{d}q$

« we have here, for the first time, the appearance of Berry's phase for a one-dimensional parameter space »

Zak, 1989

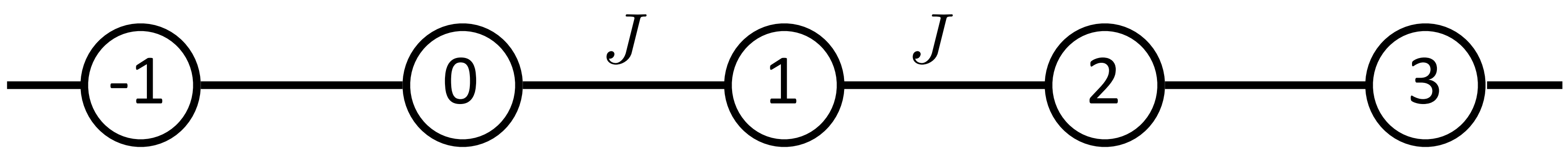
Requires that $|u_q\rangle$ has non-zero imaginary part

The SSH model

Su, Schrieffer & Heeger
Solitons in Polyacetylene
Phys. Rev. Lett. 42, 1698 (1979)

Hubbard model

The simplest model of a 1D crystal

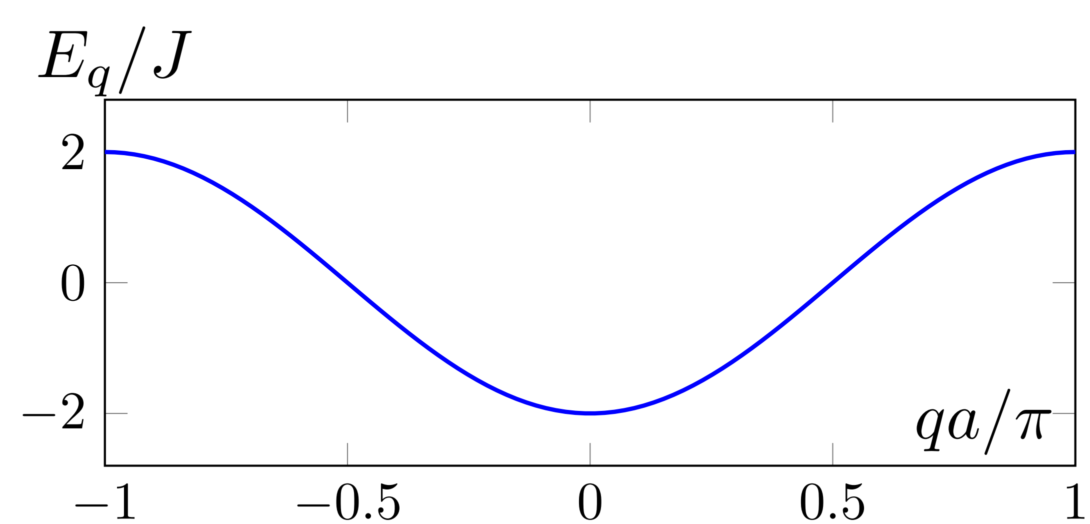


- All sites are equivalent (zero energy by convention)
- Only nearest neighbour couplings: $\hat{H} = -J \sum_j |j+1\rangle\langle j| + \mathrm{h.c.}$

Bloch theorem: a single periodic function $|u\rangle \propto \sum_j |j\rangle$

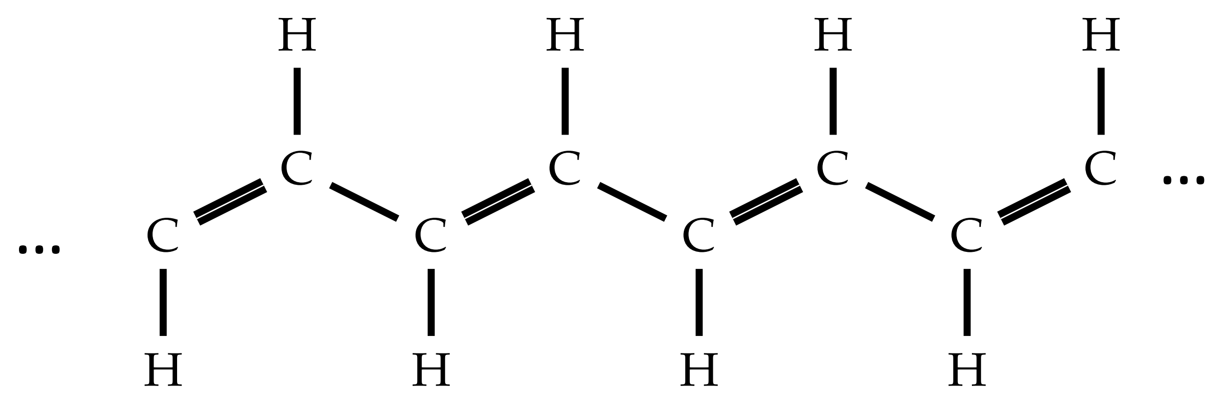
$\longrightarrow$ The function $|u\rangle$ is real: zero Zak phase, no topological properties

$$\longrightarrow \left\{ \begin{array}{l} |\psi_q\rangle = \sum_j \mathrm{e}^{\mathrm{i}jqa}\,|j\rangle \\ E_q = -2J\,\cos(qa) \\ \text{Only one energy band} \end{array} \right.$$

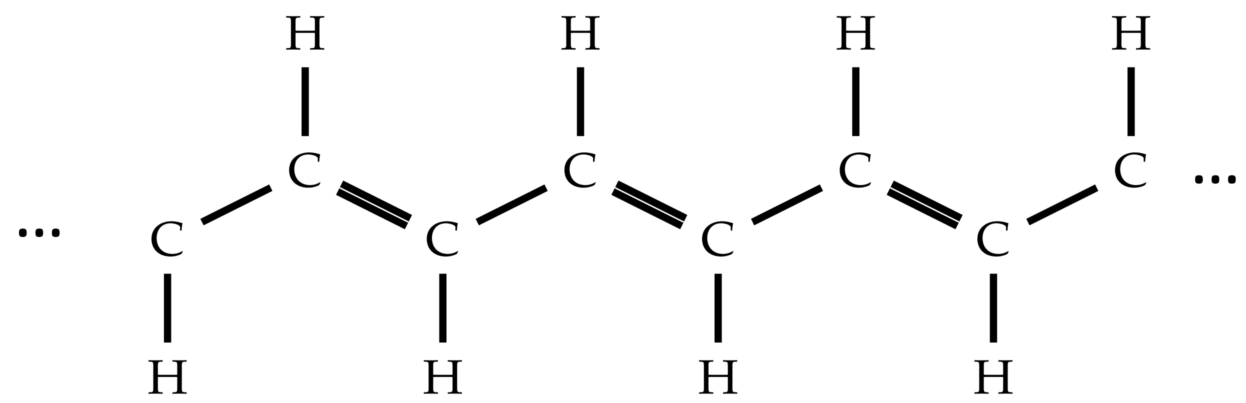


The SSH problem

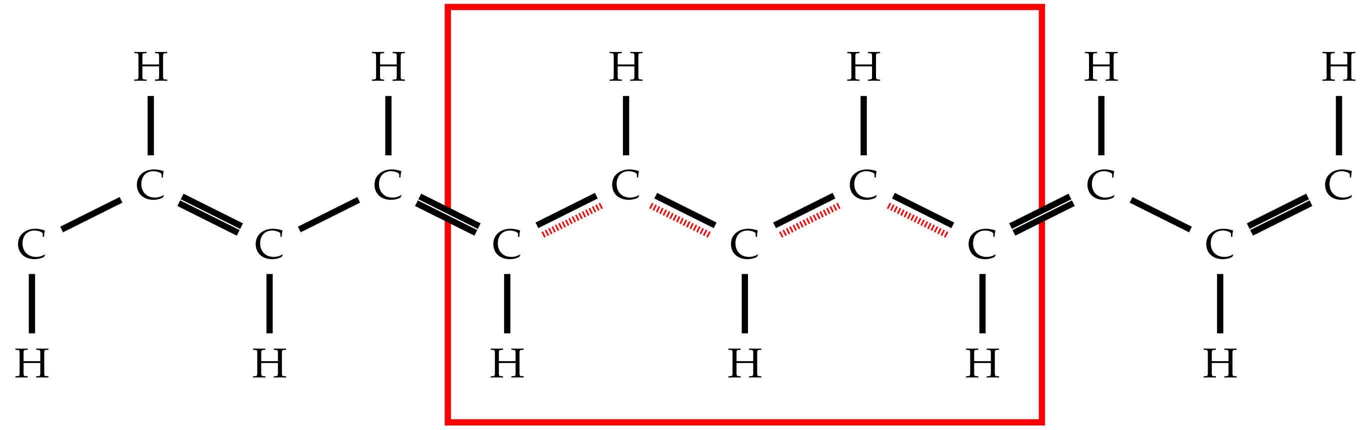
Description of density waves with
a solitonic nature along a polymer chain



Dimerized form:

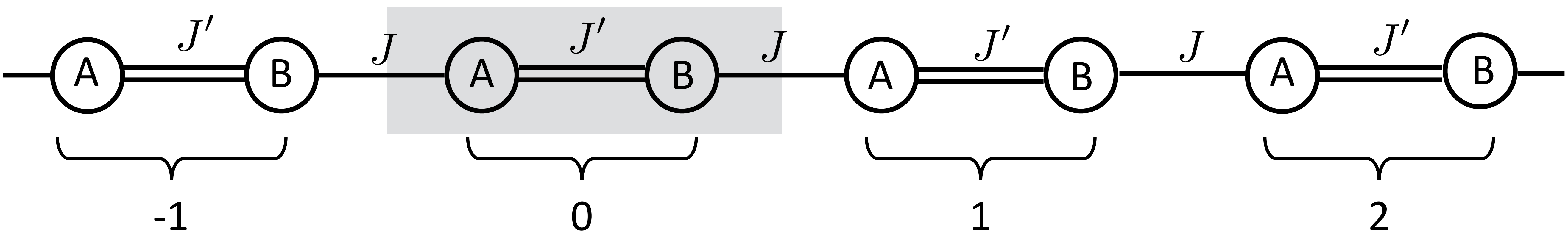


What happens at the junction between the two dimerizations?



A model for the SSH problem

Generalize the Hubbard problem with two possible values for the NN coupling



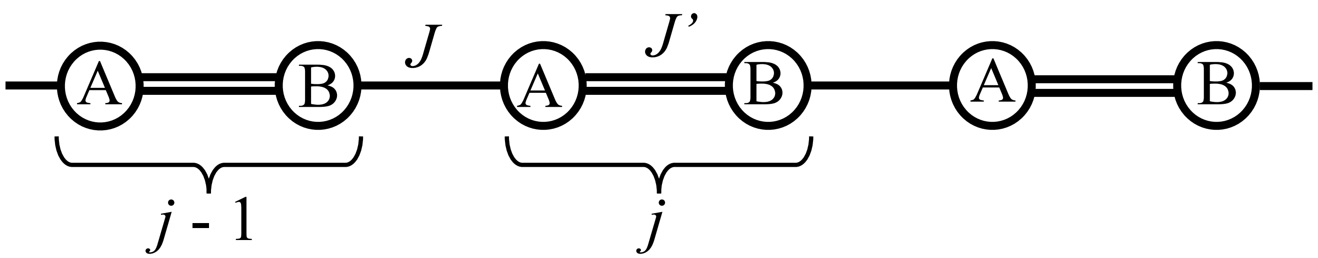
Unit cell with two sites A and B

Periodic function: $$|u_q\rangle = \alpha_q \left(\sum_j |A_j\rangle\right) + \beta_q \left(\sum_j |B_j\rangle\right) = \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix}$$

Pseudo-spin 1/2 for each value of the wavevector ***q***

Complex coefficients α_q, β_q $$|\alpha_q|^2 + |\beta_q|^2 = 1$$

The periodic Hamiltonian



$$|u_q\rangle = \alpha_q \left(\sum_j |A_j\rangle \right) + \beta_q \left(\sum_j |B_j\rangle \right)$$

Corresponding Bloch function: $|\psi_q\rangle = \sum_j \mathrm{e}^{\mathrm{i}\, jqa} \left(\alpha_q |A_j\rangle + \beta_q |B_j\rangle \right)$

Solution of $\hat{H}|\psi_q\rangle = E_q\, |\psi_q\rangle$ with

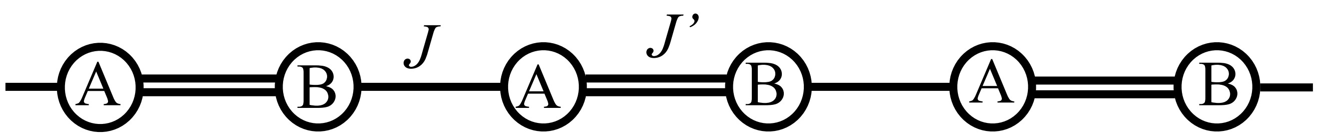
$$\hat{H} = -J' \sum_j |B_j\rangle\langle A_j| - J \sum_j |B_{j-1}\rangle\langle A_j| \; + \; \mathrm{h.c.}$$

For each ***q***, one obtains a 2x2 system for the coefficients α_q, β_q :

$$\hat{H}_q \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix} = E_q \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix} \qquad \text{with} \qquad \hat{H}_q = - \begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix}$$

Periodic Hamiltonian of the generic type for the spin 1/2 case

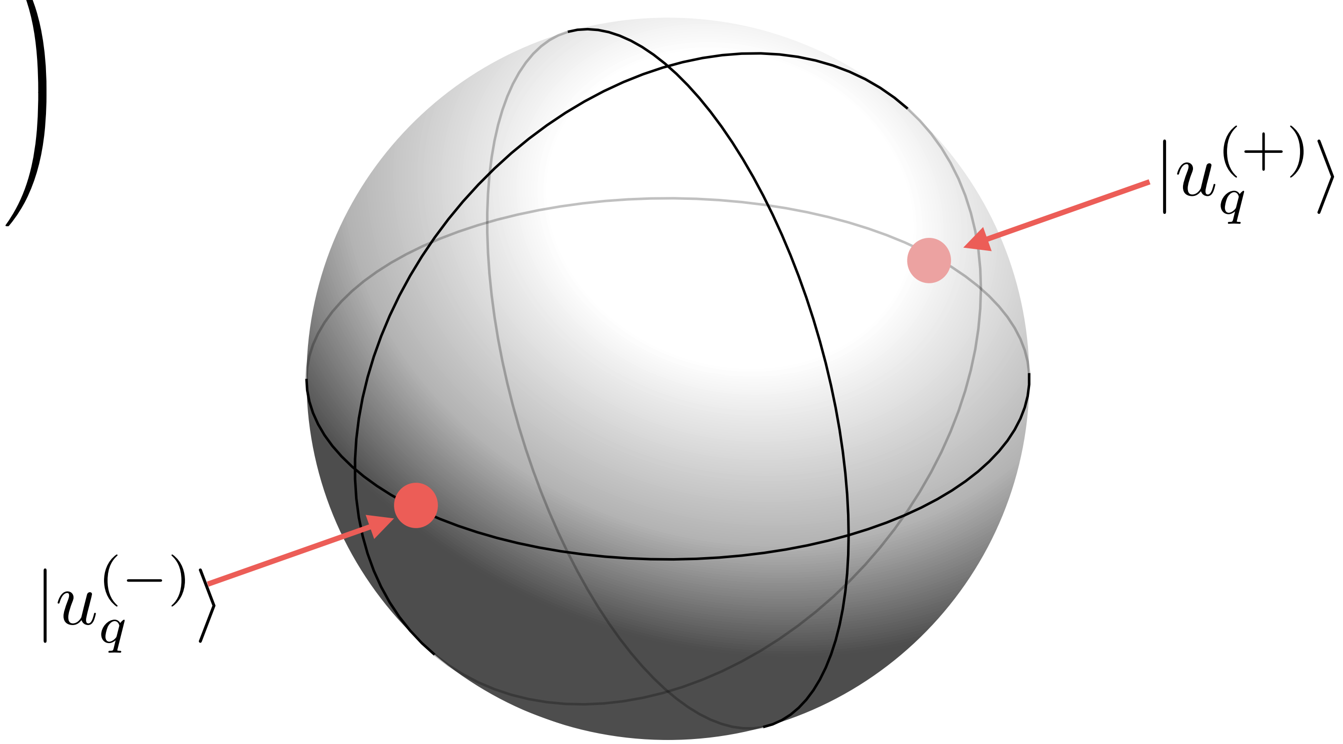
Structure of the periodic Hamiltonian



Periodic Hamiltonian: $$\hat{H}_q = -\begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix} = -\boldsymbol{h}(q)\cdot\hat{\boldsymbol{\sigma}}$$

with the vector $$\boldsymbol{h}(q) = \begin{pmatrix} J' + J\cos(qa) \\ J\sin(qa) \\ 0 \end{pmatrix}$$

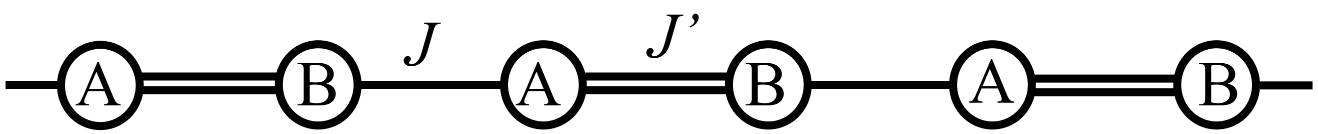
$h_z(q) = 0$: stays on the equator of Bloch sphere



Energies : $E_q^{(\pm)} = \pm|J' + J\,\mathrm{e}^{\mathrm{i}qa}|$

Two non-touching energy bands, except for $J = J'$

Energy bands for the SSH model

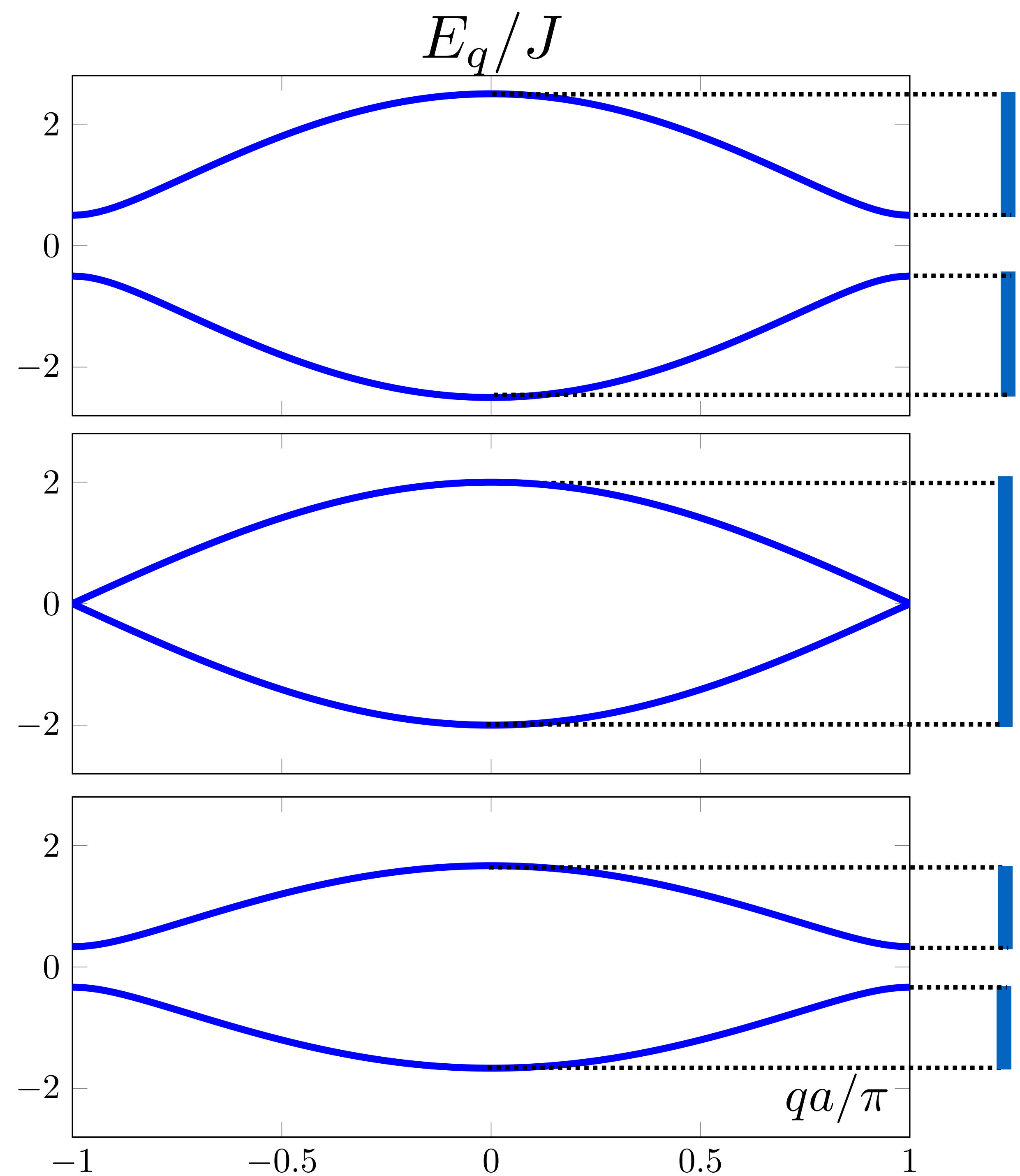


$$E_q^{(\pm)} = \pm|J' + J\,\mathrm{e}^{\mathrm{i}qa}|$$

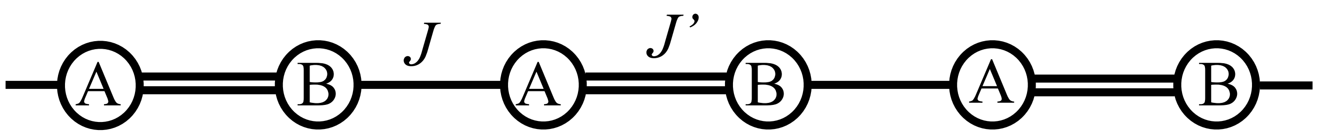
$$J' = \frac{3}{2}J$$

$$J' = J$$

$$J' = \frac{2}{3}J$$

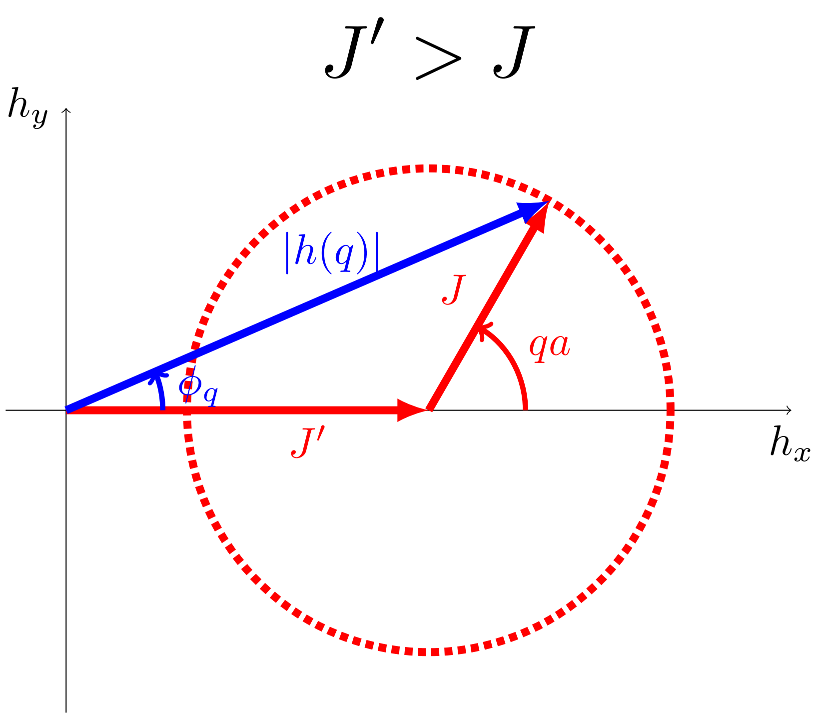


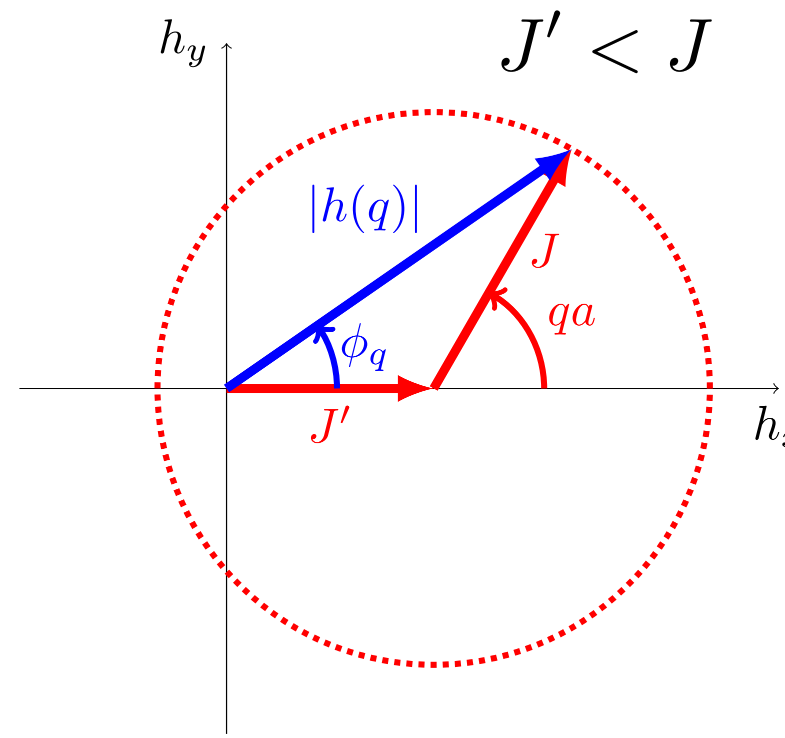
Eigenstates and topology for the SSH



$$\hat{H}_q = -\begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix} \qquad |u_q^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp\mathrm{e}^{\mathrm{i}\phi_q} \end{pmatrix}$$

where the phase ϕ_q is defined by $\mathrm{e}^{\mathrm{i}\phi_q} = \dfrac{J' + J\,\mathrm{e}^{\mathrm{i}qa}}{|J' + J\,\mathrm{e}^{\mathrm{i}qa}|}$

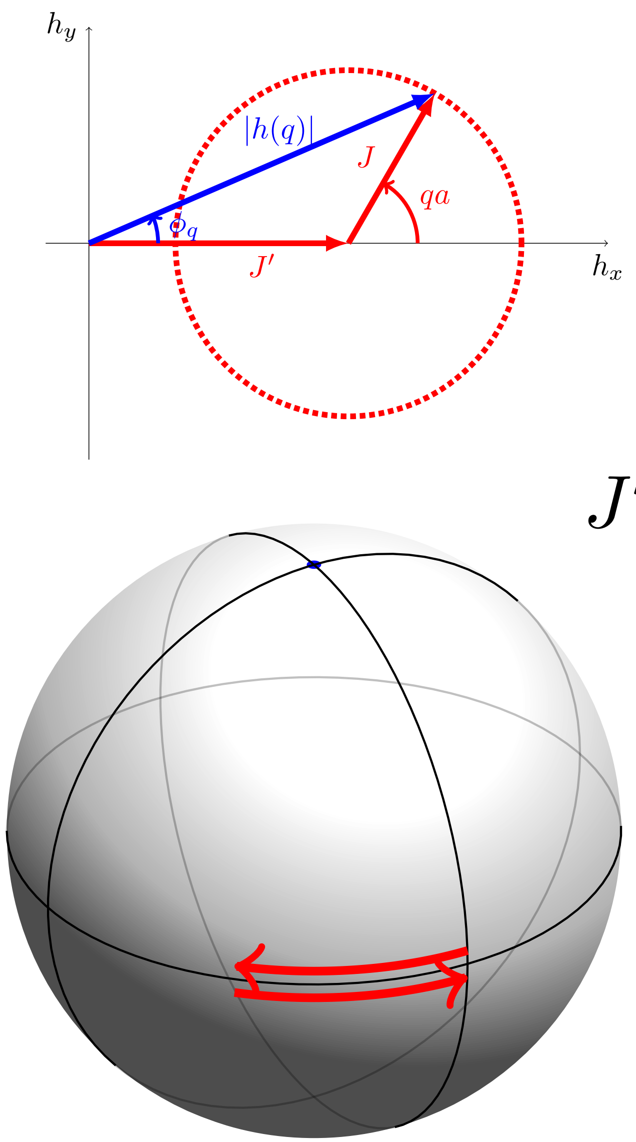




The geometrical Berry - Zak phase

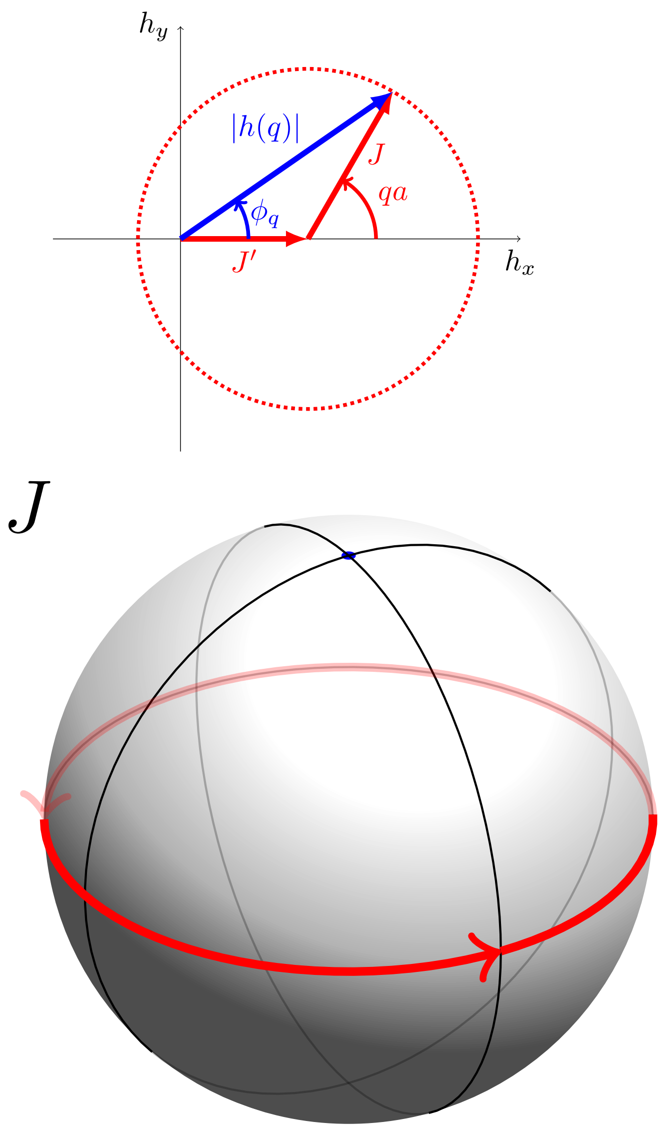
Trajectory of one of the two eigenstates, say $|u_q^{(-)}\rangle$, on the Bloch sphere

$$|u_q^{(-)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\phi_q} \end{pmatrix}$$



$J' > J$

$\Phi_{\mathrm{Zak}} = 0$



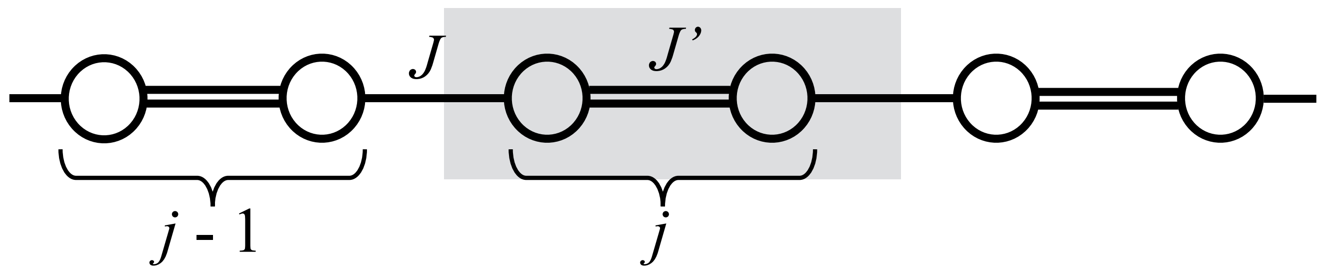
$J' < J$

$\Phi_{\mathrm{Zak}} = \pi$

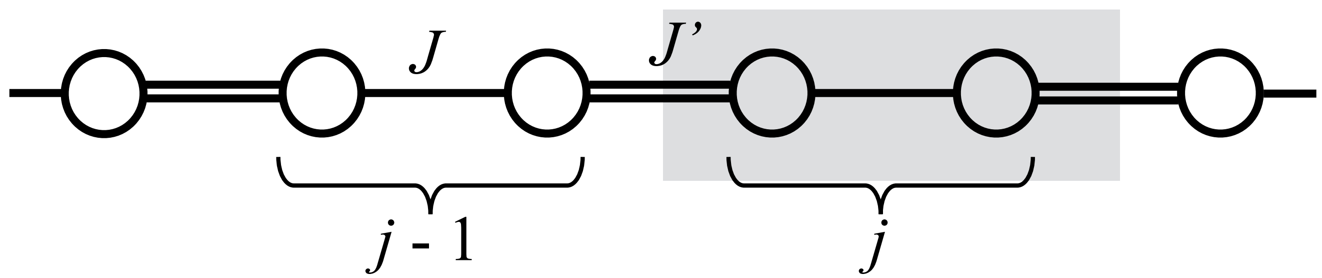
The winding number Φ_{Zak}/π is a topological invariant

What is the physical reality of this topology?

In what precedes, *J'* stands for the intra-cell coupling and *J* for the coupling between adjacent cells



For an infinite chain, one can change the labelling and exchange the roles of *J* and *J'*



Is the distinction between $[J' < J]$ and $[J' > J]$ a mere mathematical artefact?

No! But it acquires a physical meaning only when one considers the border between two zones with different topologies.

Crucial role of edge states in the link between Physics and Topology

3.

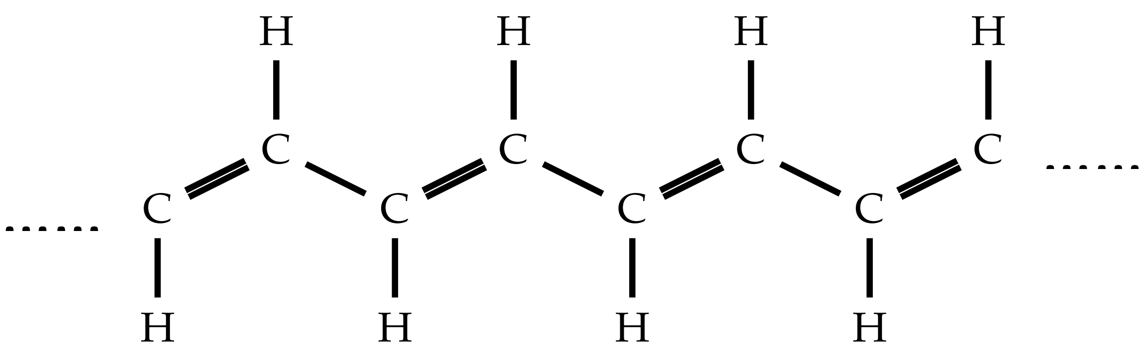
Edge states of the SSH chain

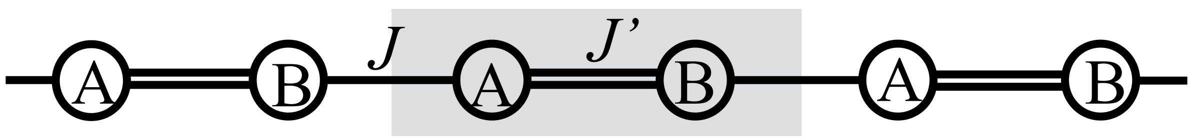
The *bulk — edge* correspondance

Finite and semi-infinite chain

Where we stand for now

Su, Schrieffer & Heeger, *Solitons in Polyacetylene*, Phys. Rev. Lett. 42, 1698 (1979)





Nearest neigbour coupling

Unit cell with two sites: We look for the energy eigenstates with the Bloch form

wavevector q, energy E_q

$$|\psi_q\rangle = \sum_j \mathrm{e}^{\mathrm{i}\,jqa}\left(\alpha_q|A_j\rangle + \beta_q|B_j\rangle\right) \qquad |u_q\rangle = \begin{pmatrix}\alpha_q\\ \beta_q\end{pmatrix}$$

The coefficients α_q, β_q correspond to a pseudo-spin 1/2 and are obtained from:

$$\hat{H}_q \begin{pmatrix}\alpha_q\\ \beta_q\end{pmatrix} = E_q \begin{pmatrix}\alpha_q\\ \beta_q\end{pmatrix} \qquad \text{with} \qquad \hat{H}_q = -\begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa}\\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0\end{pmatrix}$$

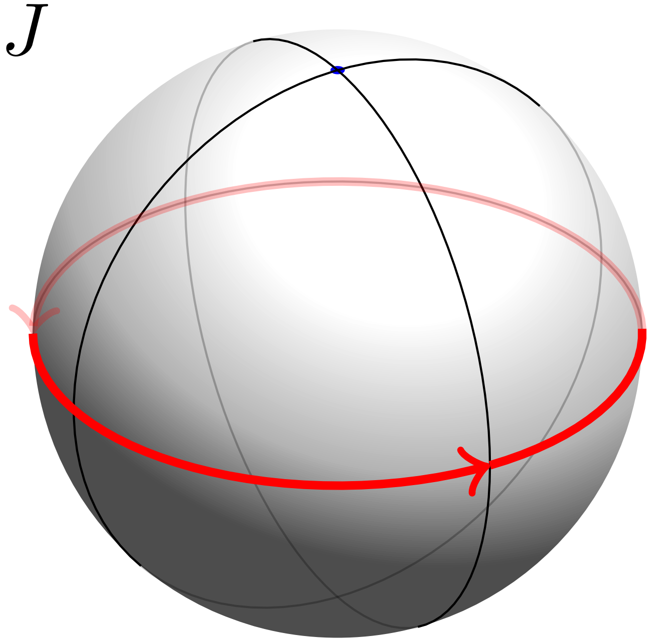
Eigenstates of the SSH

$$\hat{H}_q = -\begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix} = -\boldsymbol{h}(q)\cdot\hat{\boldsymbol{\sigma}} \qquad \boldsymbol{h}(q) = \begin{pmatrix} J' + J\cos(qa) \\ J\sin(qa) \\ 0 \end{pmatrix}$$

Two-site unit cell: two eigenstates $|u_q^{(\pm)}\rangle$ for each value of q in the Brillouin zone

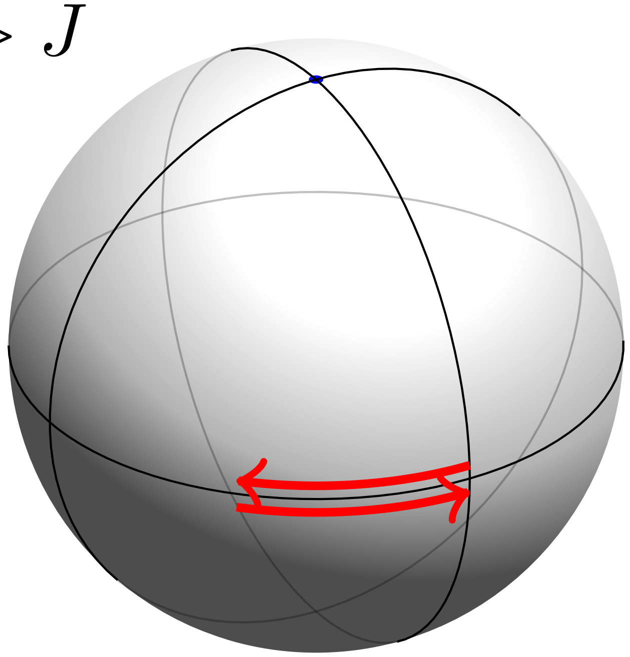
$$E_q^{(\pm)} = \pm|J' + J\,\mathrm{e}^{\mathrm{i}qa}| \qquad |u_q^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp\mathrm{e}^{\mathrm{i}\phi_q} \end{pmatrix} \qquad \mathrm{e}^{\mathrm{i}\phi_q} = \frac{J' + J\,\mathrm{e}^{\mathrm{i}qa}}{|J' + J\,\mathrm{e}^{\mathrm{i}qa}|}$$

$J' < J$



Topological case

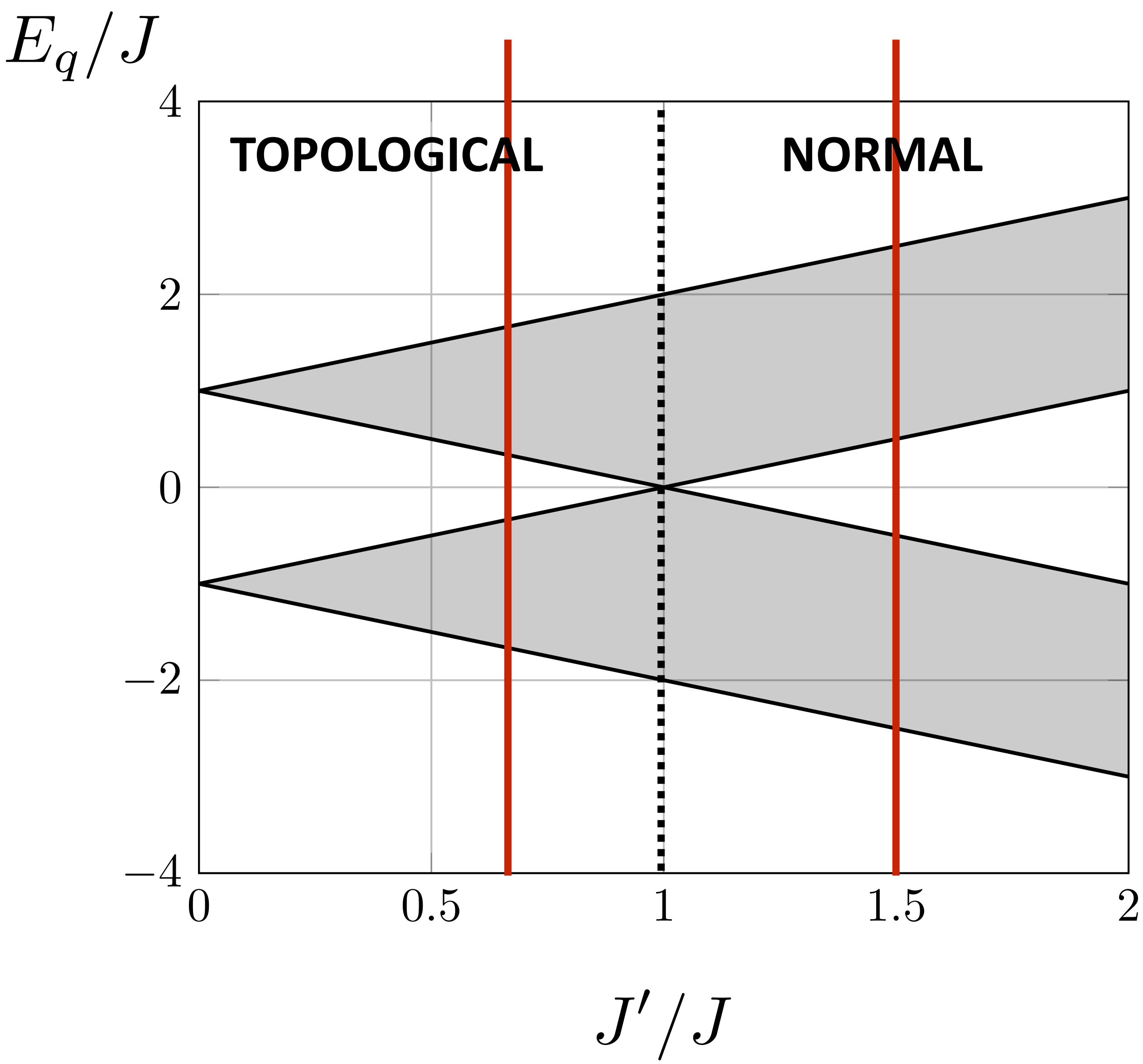
$J' > J$

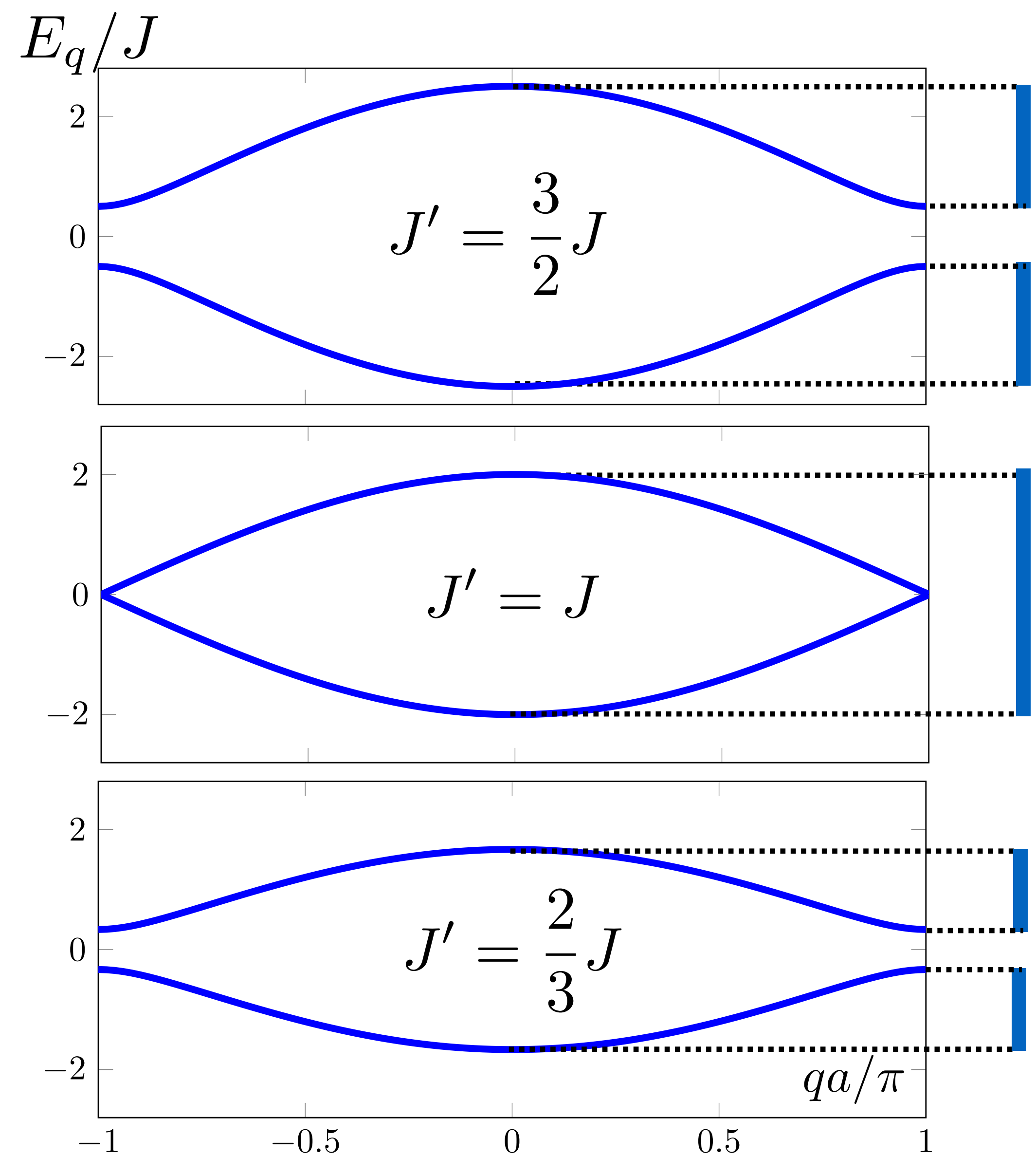


Normal case

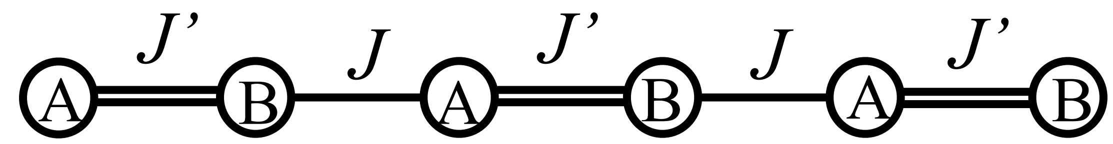
Energy band for the SSH model

$$\hat{H}_q = -\begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix} \longrightarrow E_q^{(\pm)} = \pm|J' + J\,\mathrm{e}^{\mathrm{i}qa}|$$

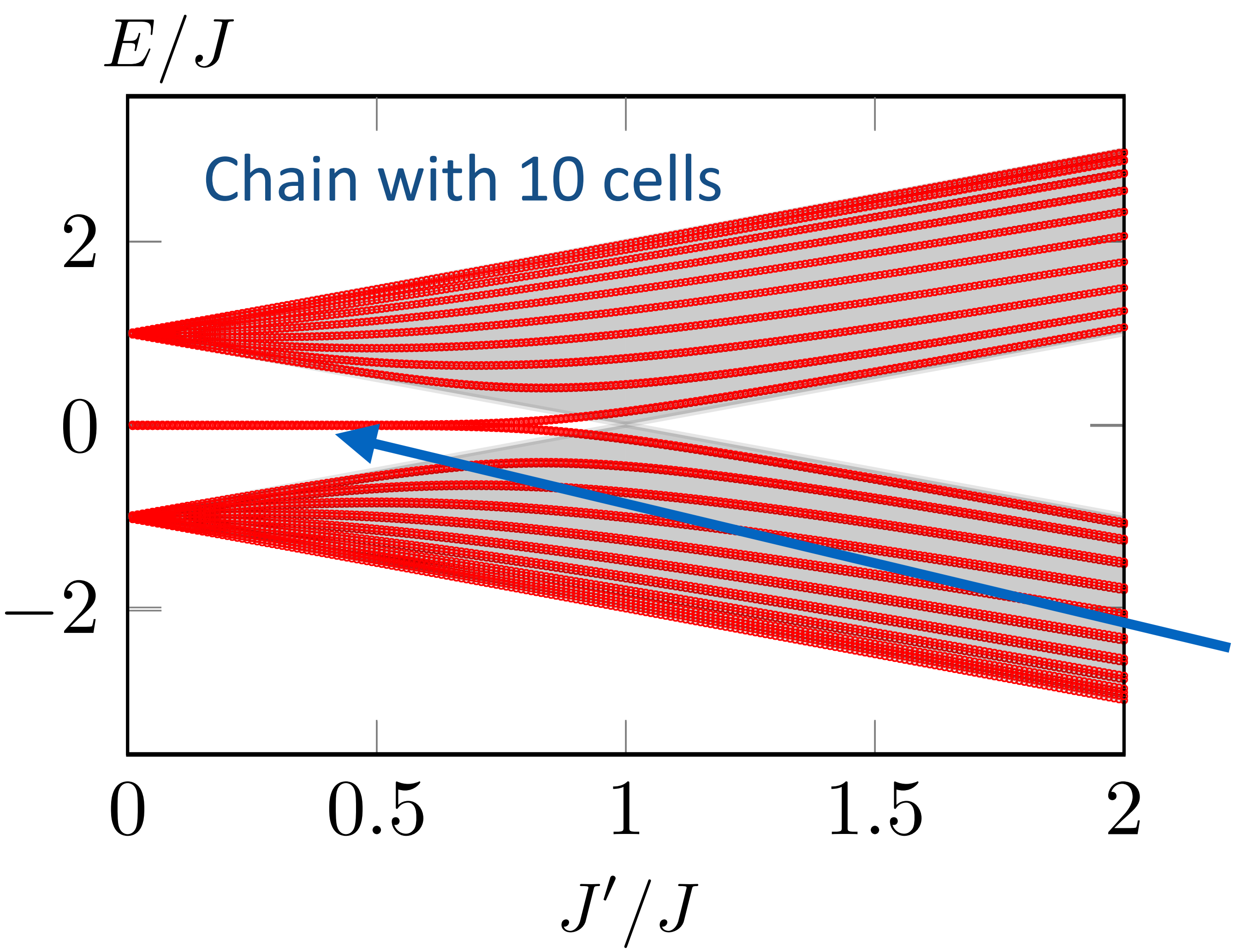




The finite SSH chain



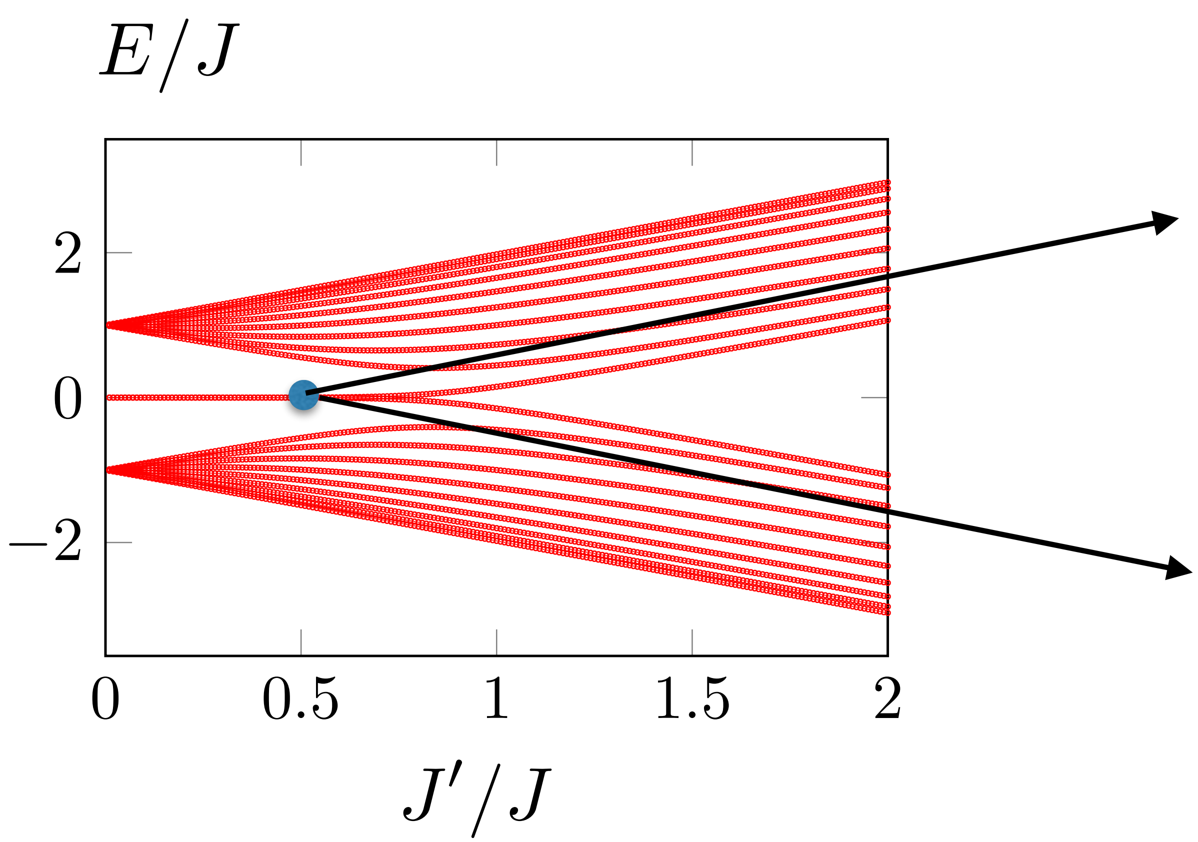
Numerical determination of eigenstates (no Bloch theorem anymore!)

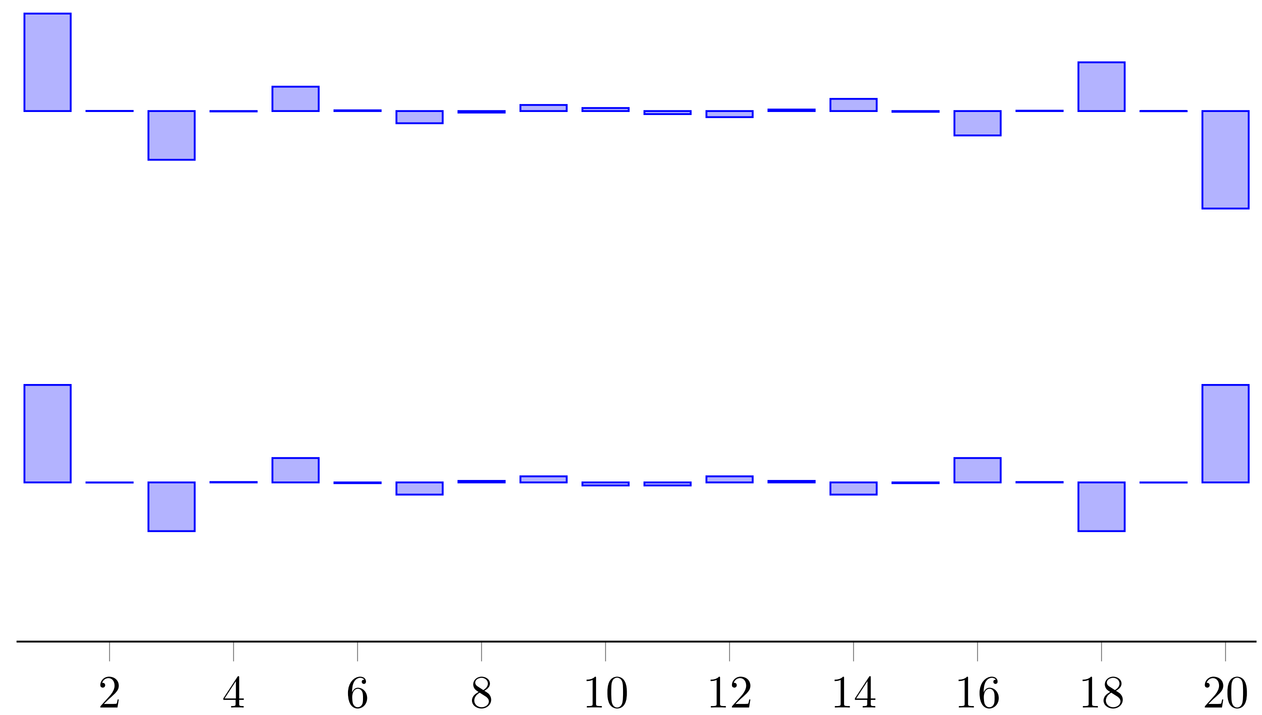


- One recovers the bands of the infinite chain
- One finds a symmetric spectrum with respect to $E = 0$
- In the topological zone $J' < J$, states with a quasi-null energy appear

These states are localized on the edge

Edge states for the finite SSH chain (20 sites = 10 cells)

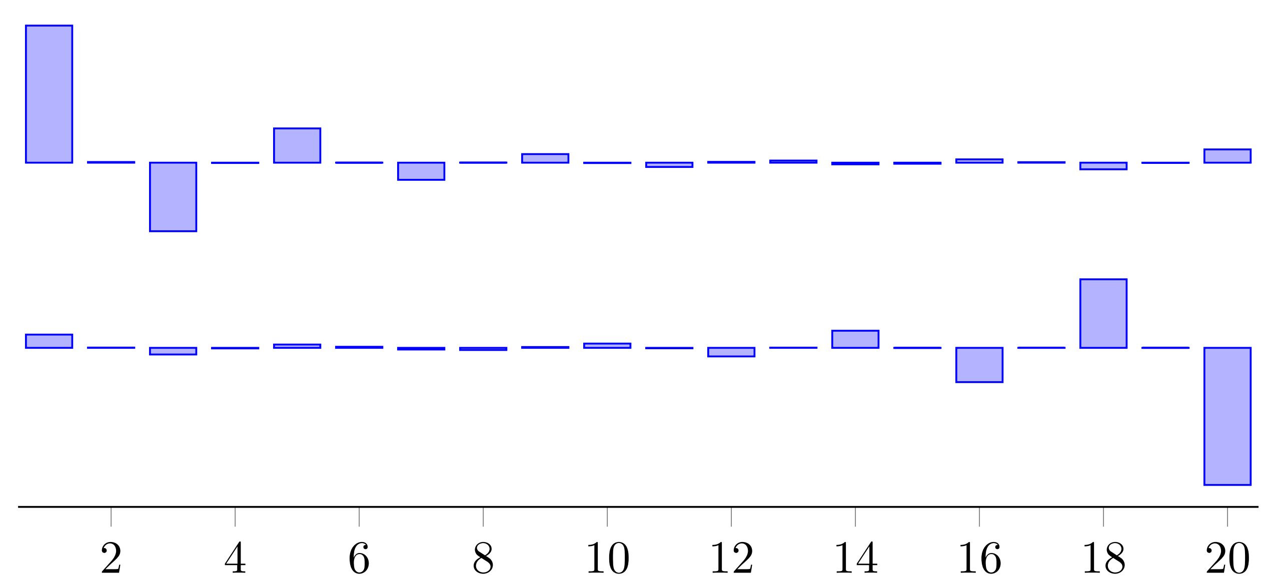




Symmetric probability distribution, originating from the symmetry of the chain

But this symmetry is fragile!

Response to an energy shift J/100 of the site A_1



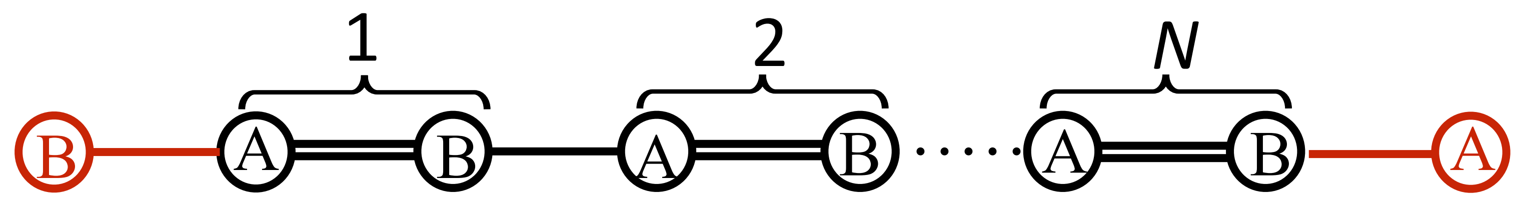
Edge states and phase winding

Delplace, Ullmo & Montambaux (2011)

Look for the edge states of the finite chain as standing waves

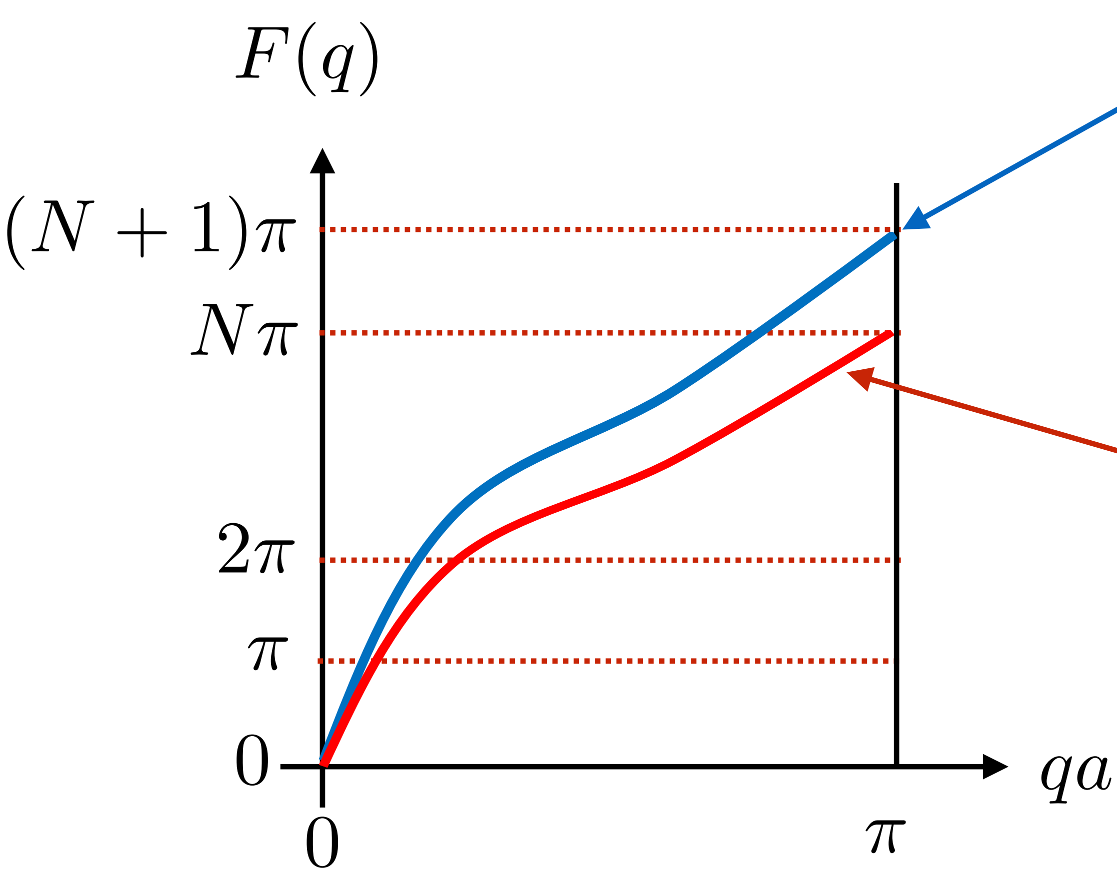
$$\alpha|\psi_q^{(-)}\rangle \;+\; \beta|\psi_{-q}^{(-)}\rangle$$

$$\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\phi_q} \end{pmatrix} \qquad \begin{pmatrix} 1 \\ \mathrm{e}^{-\mathrm{i}\phi_q} \end{pmatrix}$$



Boundary conditions: zero amplitude on sites B_0 et A_{N+1}

$$q \geq 0 \qquad F(q) \equiv (N+1)qa - \phi_q = 0 \;\; \text{modulo } \pi$$



Normal case (no phase winding)

$\phi_q = 0$ for $qa = 0$ and $qa = \pi$

N solutions: no state is missing

Topological case (phase winding)

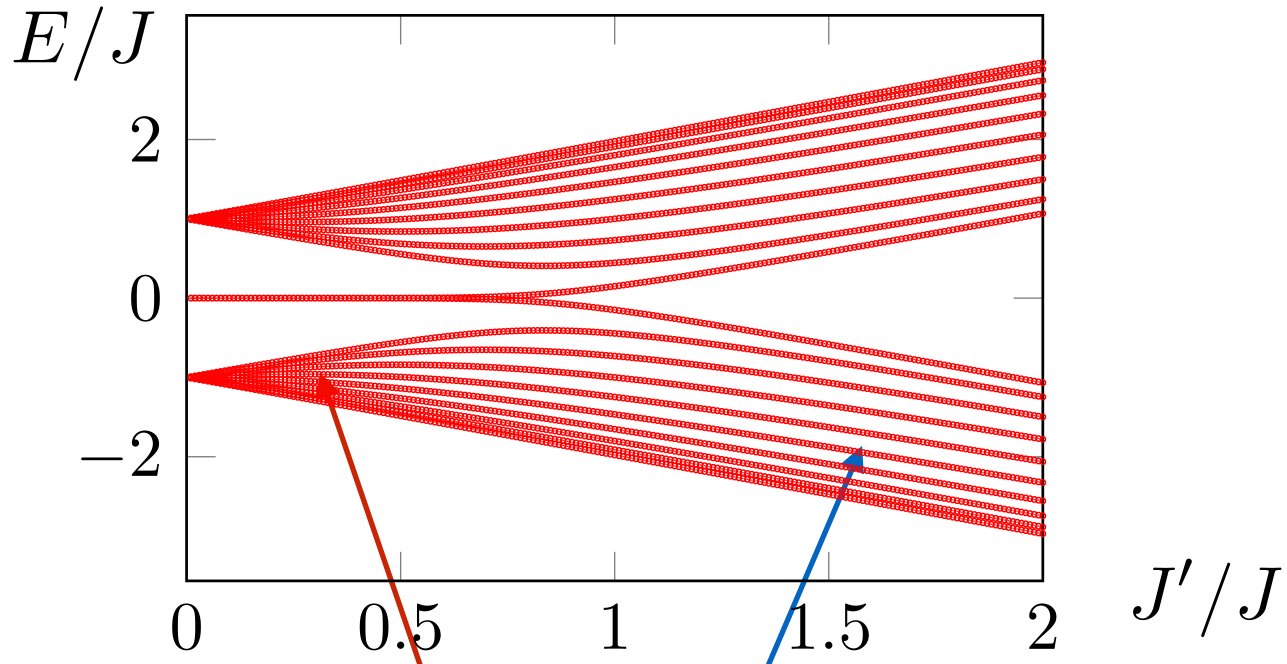
$\phi_q = 0$ for $qa = 0$

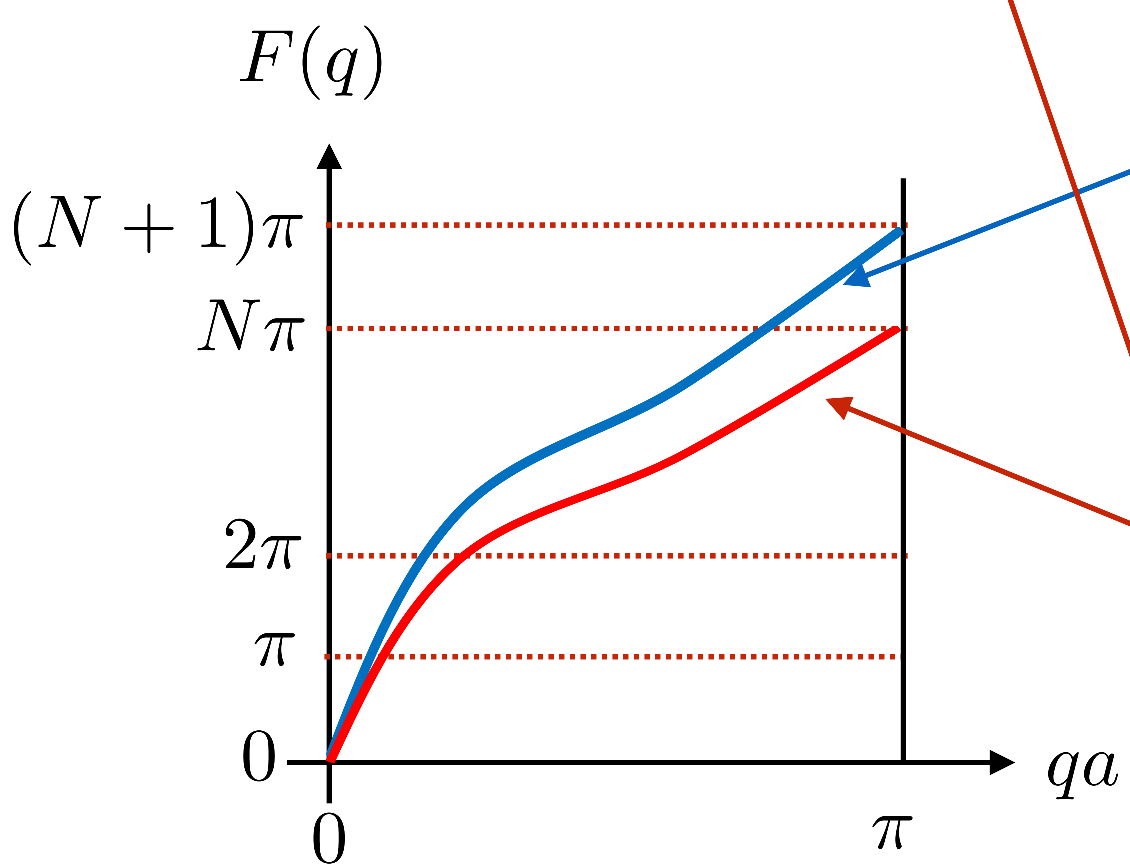
$\phi_q = \pi$ for $qa = \pi$

N-1 solutions: One state is missing!

Edge states and phase winding

Delplace, Ullmo & Montambaux (2011)





Normal case (no phase winding)

$\phi_q = 0$ for $qa = 0$ and $qa = \pi$

N solutions: no state is missing

Topological case (phase winding)

$\phi_q = 0$ for $qa = 0$

$\phi_q = \pi$ for $qa = \pi$

The symmetry of the spectrum $E \leftrightarrow -E$

Sublattice symmetry (or chiral symmetry) described by the unitary operator

$\hat{\mathcal{S}} = \hat{P}_A - \hat{P}_B$ $\quad$ $\hat{P}_A, \hat{P}_B$: projectors on the sublattices A, B

The absence of couplings of the type $A - A$ or $B - B$ implies that

$$\hat{P}_A \hat{H} \hat{P}_A = 0 \qquad \hat{P}_B \hat{H} \hat{P}_B = 0$$

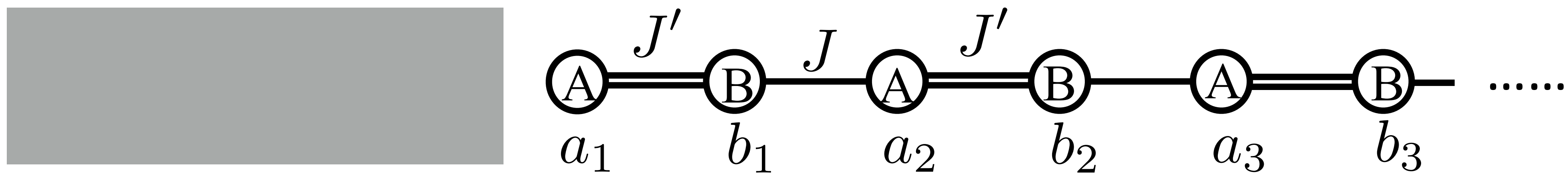
which entails $\hat{H}\hat{\mathcal{S}} = -\hat{\mathcal{S}}\hat{H}$ **Anti-commutation relation**

If $|\psi\rangle$ is an eigenstate of $\hat{H}$ with the eigenvalue E,
then $\hat{\mathcal{S}}|\psi\rangle$ is an eigenstate with the eigenvalue $-E$

$$\hat{H}\left(\hat{S}\,|\psi\rangle\right) = -\hat{S}\left(\hat{H}\,|\psi\rangle\right) = -E\left(\hat{S}\,|\psi\rangle\right)$$

Explain the observed symmetry

The semi-infinite chain



Determination of eigenstates: one expects a spectrum equal to the infinite case + possibly one edge state

$$|\psi\rangle = \sum_{j=1}^{+\infty} (a_j\,|A_j\rangle \;+\; b_j\,|B_j\rangle) \qquad \text{inserted in} \qquad \hat{H}|\psi\rangle = E\,|\psi\rangle$$

$$-J'b_1 = E\,a_1$$
$$-J'a_1 - Ja_2 = E\,b_1$$

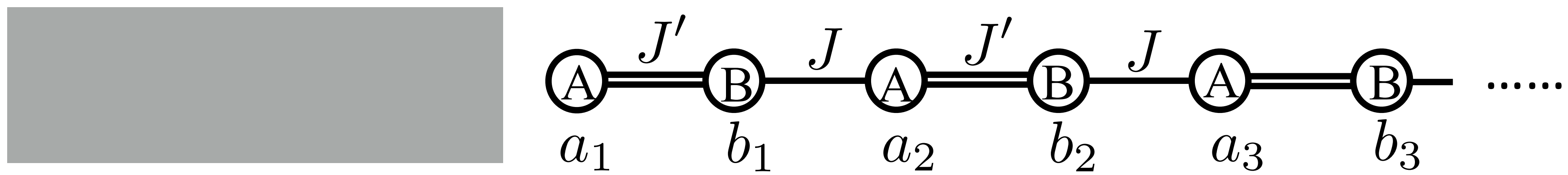
$$\left.\begin{aligned} -J'b_j - Jb_{j-1} &= E\,a_j \\ -J'a_j - Ja_{j+1} &= E\,b_j \end{aligned}\right. \qquad j \geq 2$$

Solution using a transfert matrix:

$$\begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix} = \hat{M} \begin{pmatrix} a_j \\ b_j \end{pmatrix}$$

$\hat{M}$: real matrix with determinant 1

The edge state for the semi-infinite chain



$$-J'b_1 = E\,a_1$$
$$-J'a_1 - Ja_2 = E\,b_1$$

$$-J'b_j - Jb_{j-1} = E\,a_j$$
$$-J'a_j - Ja_{j+1} = E\,b_j$$

If there is only one edge state, the symmetry of the spectrum imposes that $E = 0$

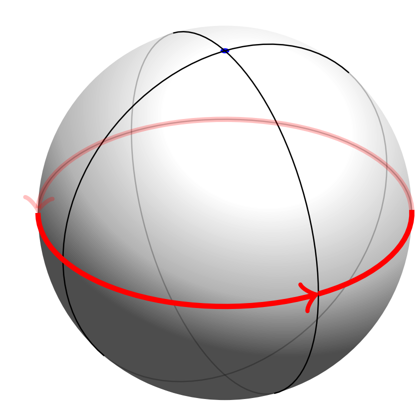
Inject $E = 0$ in this set of equations:

$\longrightarrow$ $b_1 = 0$ then by recursion $b_j = 0 \quad \forall j$

$\longrightarrow$ $a_{j+1} = -\dfrac{J'}{J} a_j$ then by recursion $a_j \propto \left(-\dfrac{J'}{J}\right)^j$

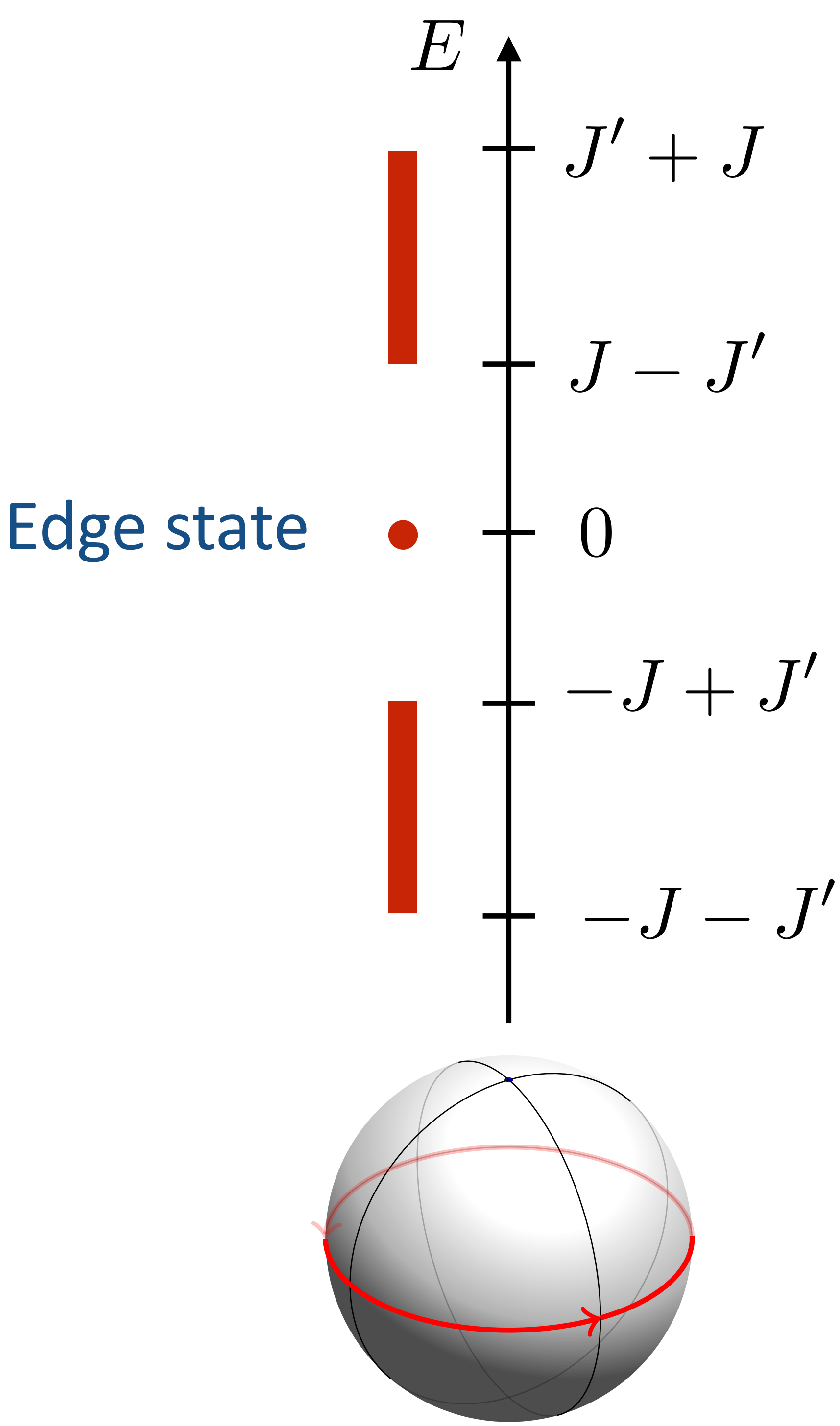
Physically acceptable result if and only if $J' < J$:

Topological region

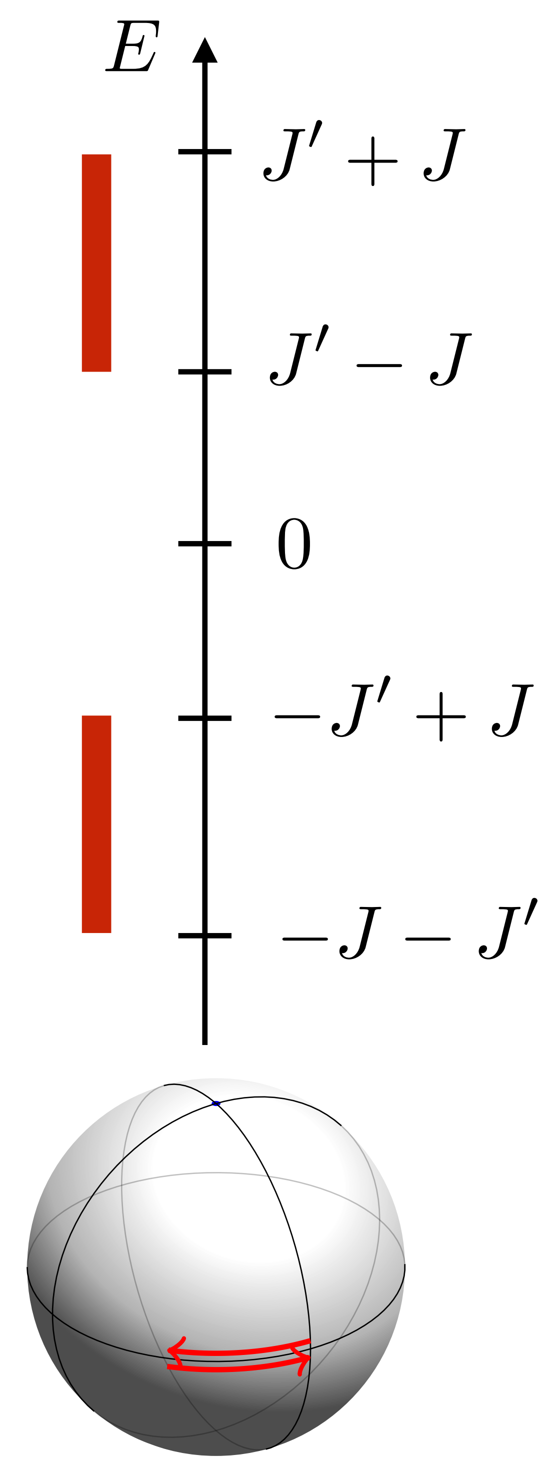


Spectrum for the semi-infinite chain

Topological case $J' < J$

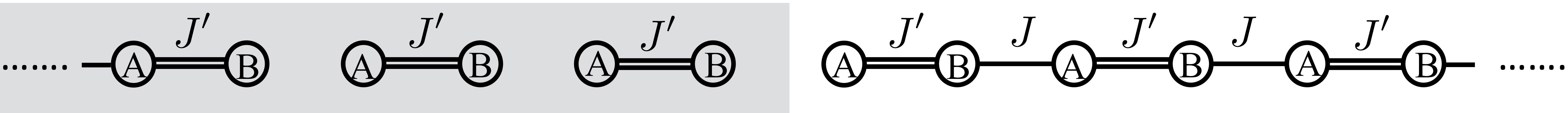


Normal case $J' > J$



Junction between two semi-infinite chains

The semi-infinite chain can be seen as a fraction of an infinite chain:



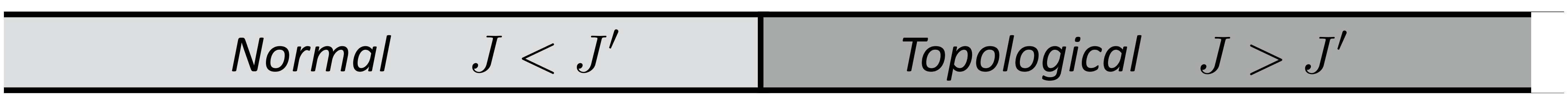
Left : $J = 0 \quad < J'$

Normal

Right: $J < J'$ or $J' < J$

Normal or topological

One can show within the SSH model that the edge state appears when one connects two chains with a different nature



and it remains true even for a soft junction (*J* and *J'* vary slowly in space)

No edge state for a *Normal-Normal* or *Topological-Topological* junction

The SSH model in photonics

P. ST-JEAN, V. GOBLOT, E. GALOPIN, A. LEMAÎTRE, T. OZAWA,
L. LE GRATIET, I. SAGNES, J. BLOCH & A. AMO
Lasing in topological edge states of a one-dimensional lattice
Nature Photonics **11**, p. 651 (2017)

see also Weimann, Kremer et al. (2017)

Haldane & Raghu (2008), Wang, Chong et al. (2009)

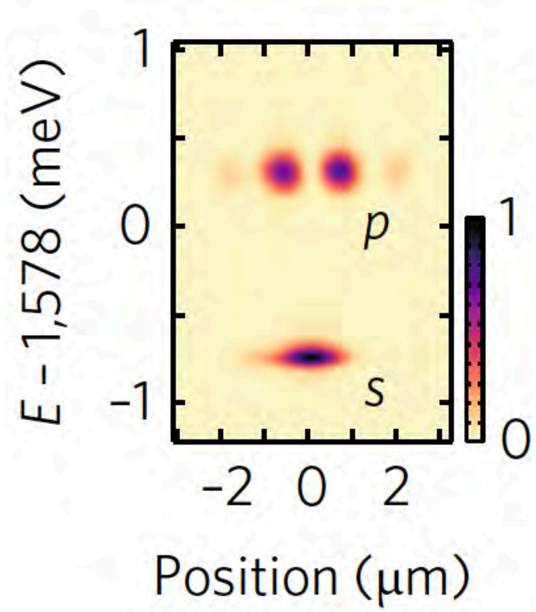
SSH in the microwave domain: Poli, Bellec et al. (2015)

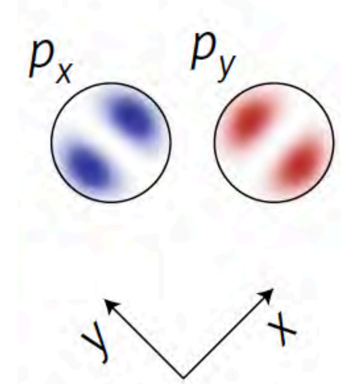
The elementary cell

Micro-pilar composed of circular GaAs quantum wells placed between two Bragg mirrors, and irradiated with non-resonant light

The amplitude of the electromagnetic field in this pilar corresponds to the coefficients a_j and b_j introduced above

Field modes (polariton) in this pilar, observed with a good energy resolution





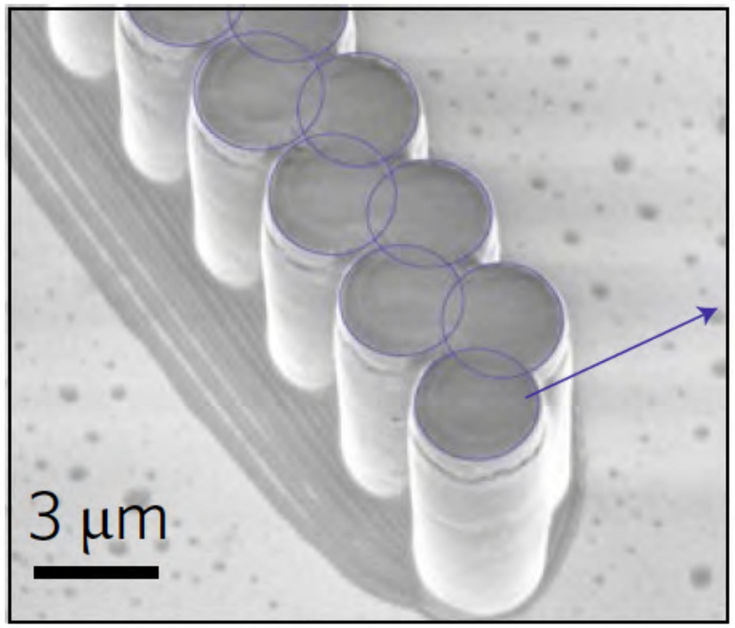
p mode with two orbitals p_x and p_y doubly degenerate

s mode with cylindrical symmetry

Mesure in real space: image of the micro-pilar

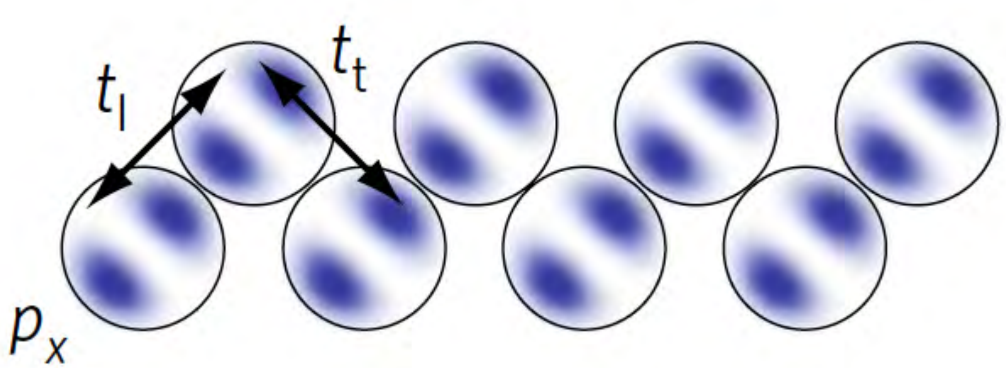
Mesure in momentum space: diffraction in various directions

Chain composed with adjacent pilars



Zig-zag disposition of the chain and coupling of the fields between adjacent micro-pilars by evanescent waves

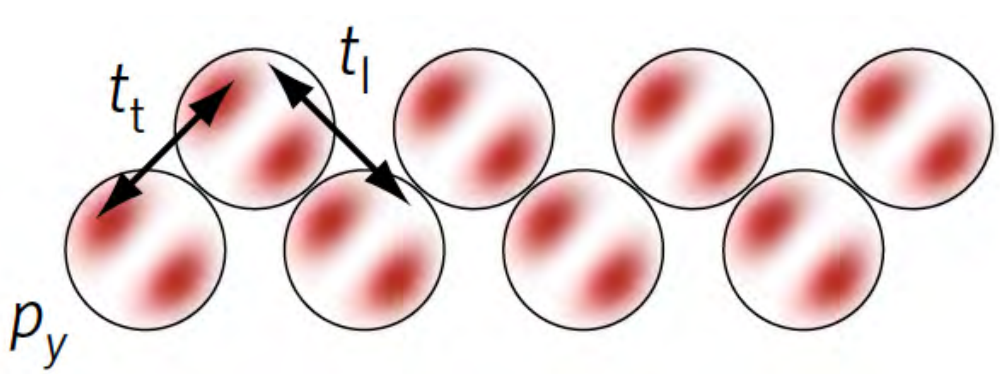
For the s mode (isotropic), all couplings are equal: Hubbard Hamiltonian



For p modes, the coupling for "aligned modes" is different from "facing modes"

$t_{trans} = 0.15\ t_{long}$

Plays the role of the coefficients *J* and *J'* of SSH

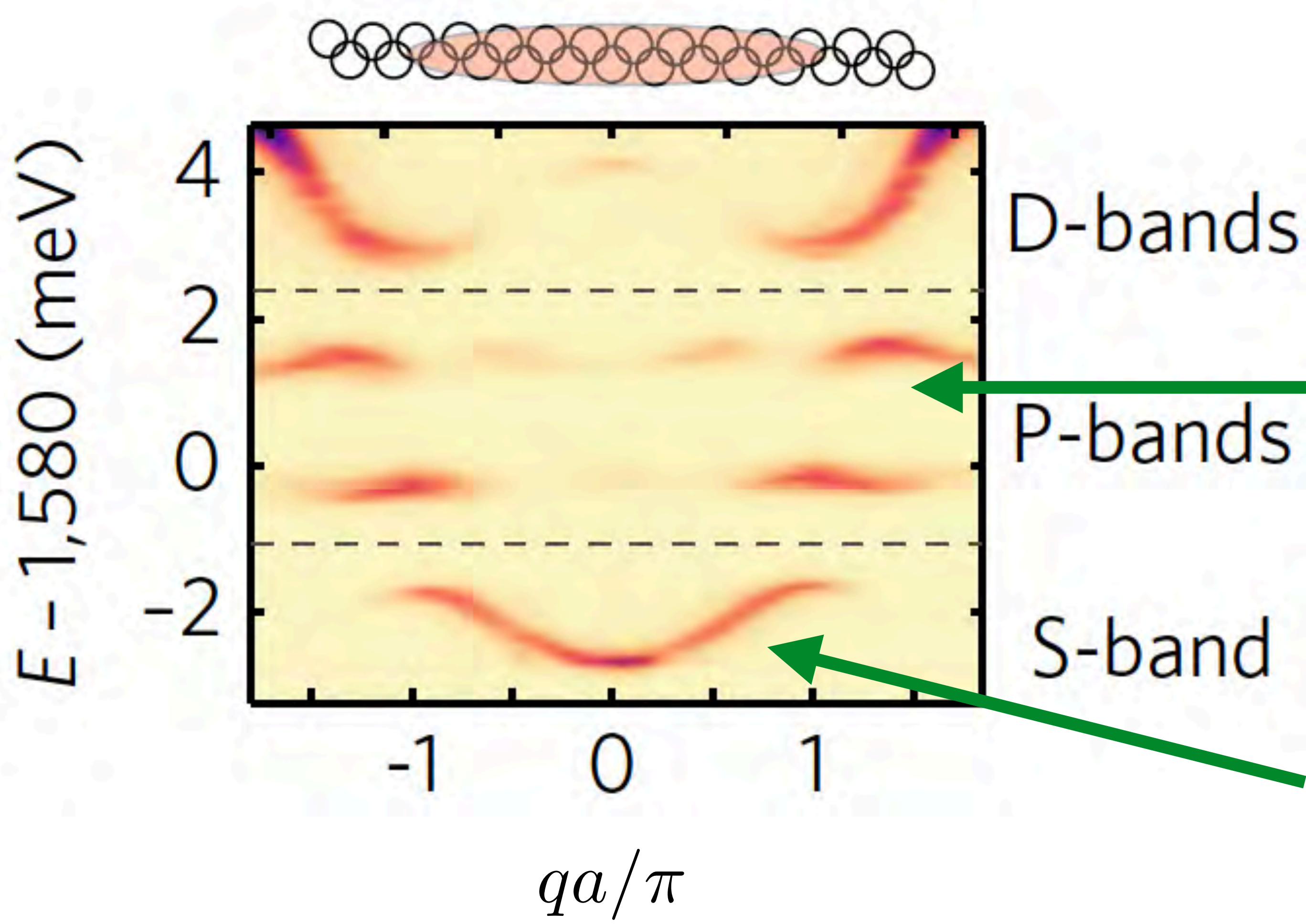


Inversion of $[J' > J]$ into $[J' < J]$ when one passes from the orbital p_x to the orbital p_y

The bulk states

Irriadiation of the center of the chain (negligible role of edge states)

Energy as a function of the momentum ***q*** :



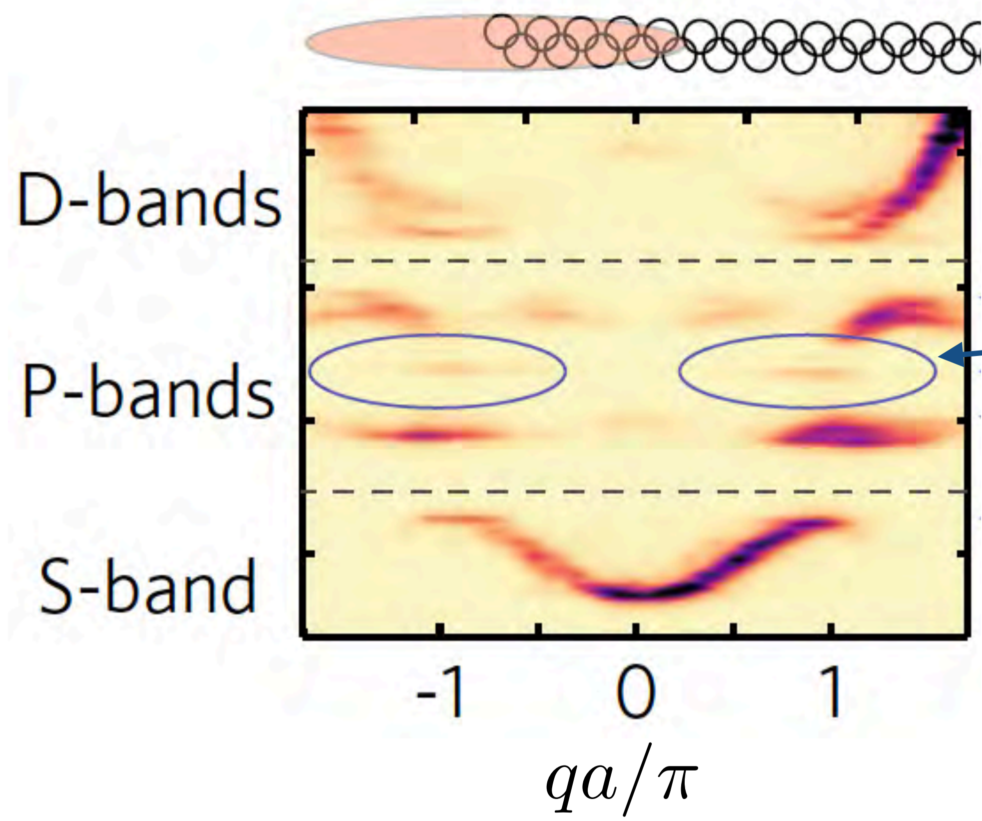
Two bands of the SSH model with a gap

Single band of the Hubbard model : $E = E_0 - 2J\cos(qa)$

Energy bands in the “unfolded” representation

Edge states

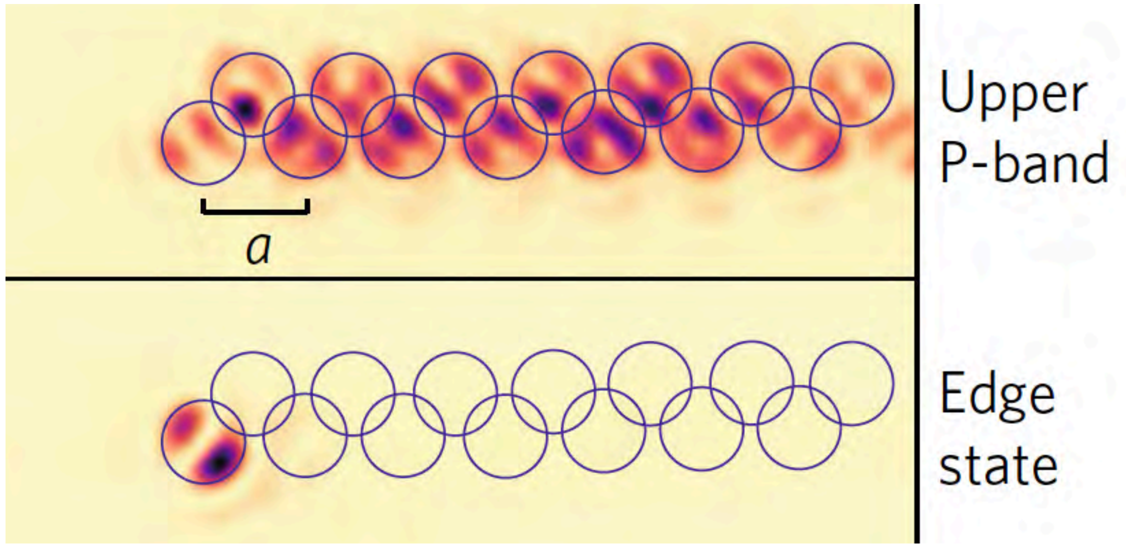
Irradiation of one end of the chain



Emergence of energy states in the SSH gap: **Edge states!**

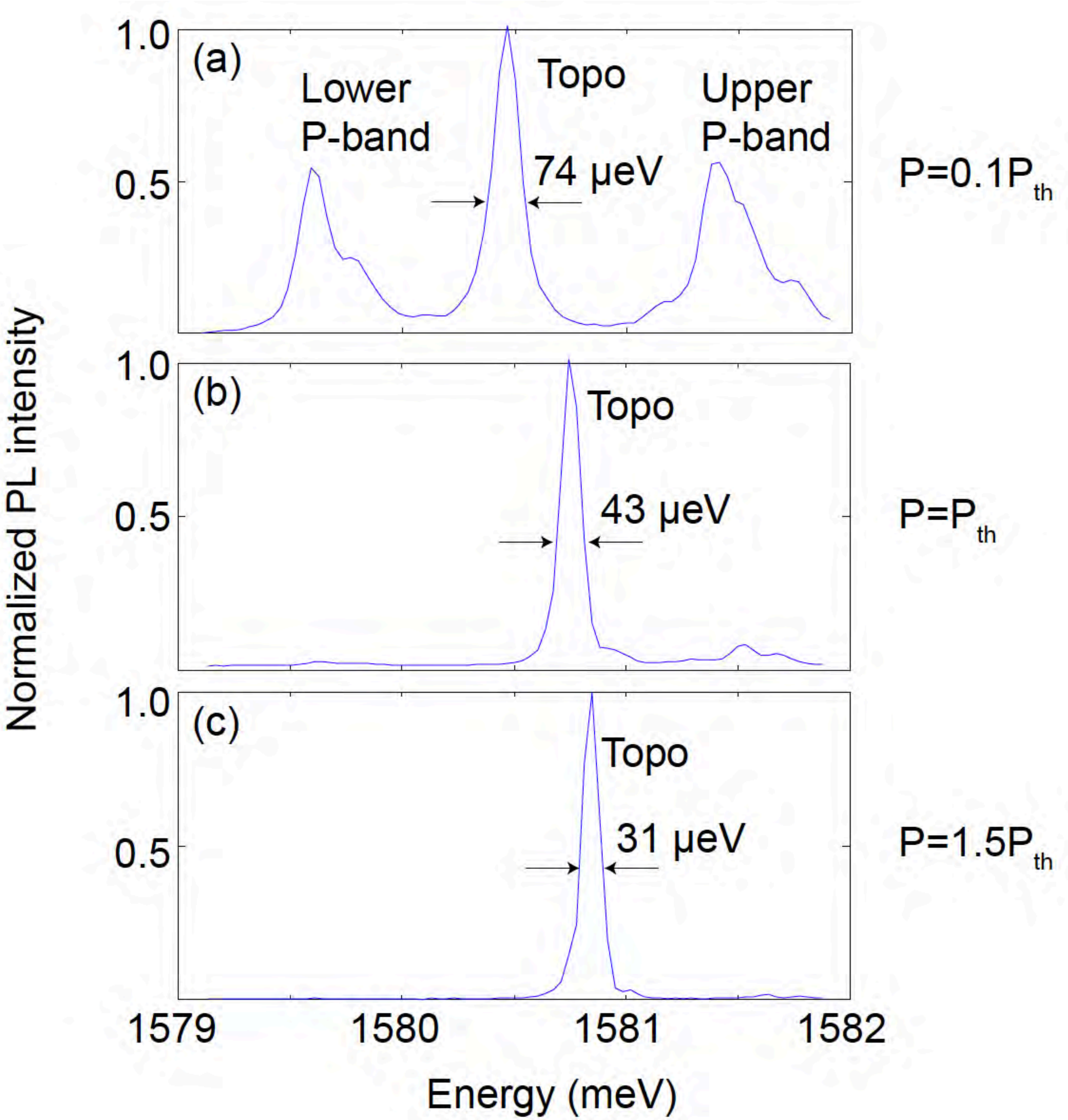
Which topology for these states?

They correspond to the orbital p_y for which $J' < J$



in-situ measurement

A topological laser



Increase the light power on the chain

Above a certain threshold, lasing on the edge state

Decrease of the spectral width

Robust lasing effect with respect to defects that may appear during the fabrication

The SSH model with cold atoms

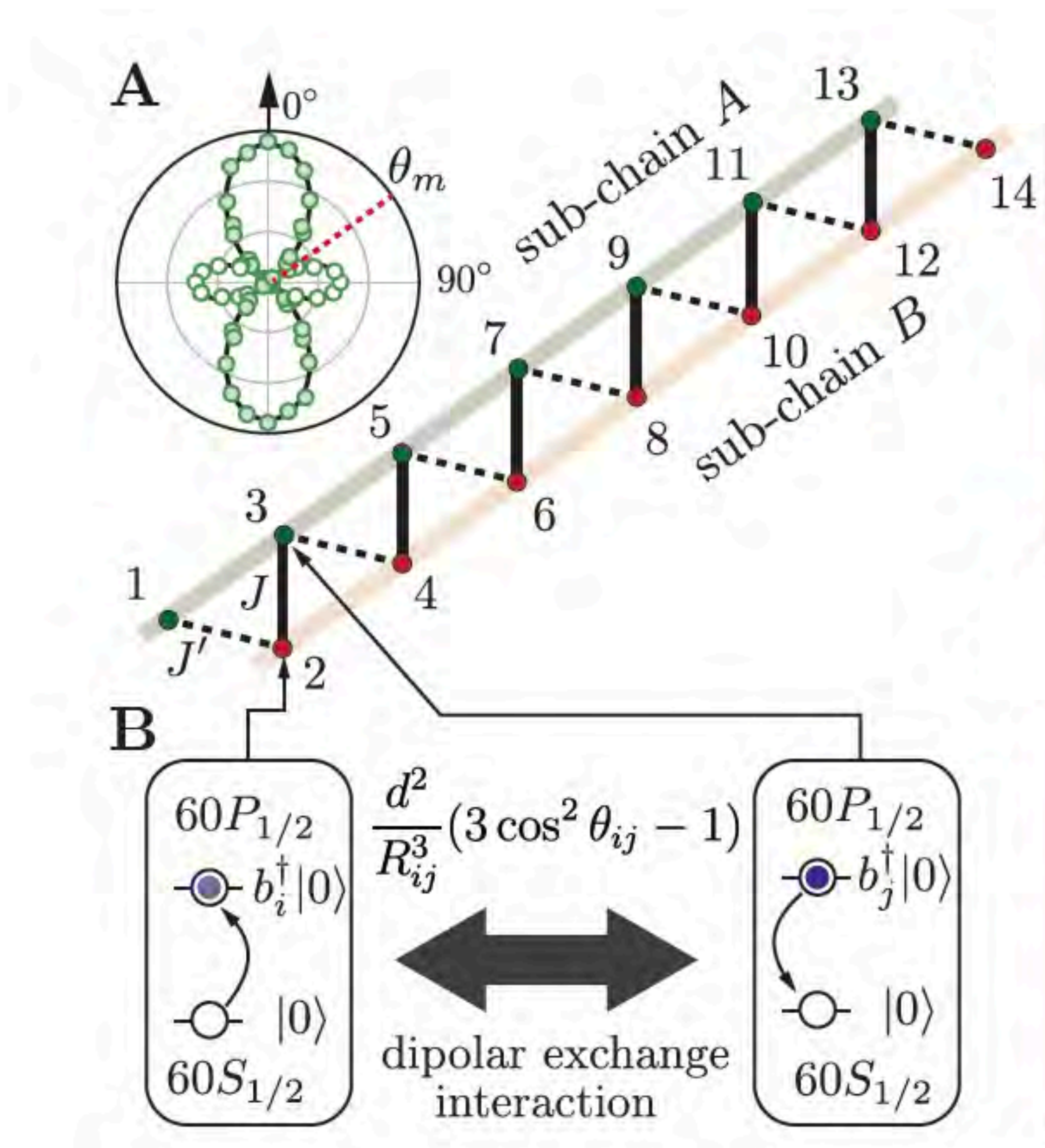
Sylvain de Léséleuc,..., Hans Peter Büchler, Thierry Lahaye, Antoine Browaeys,
Science, 23 Aug 2019, Vol. 365, pp. 775-780

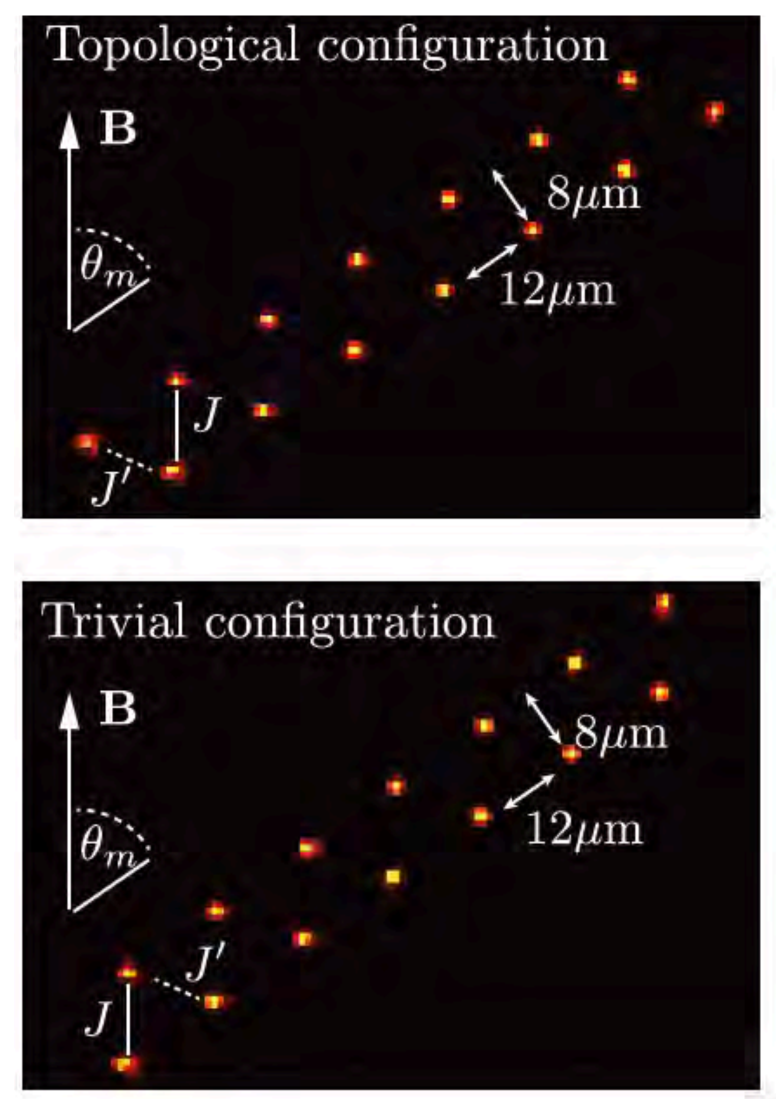
Chain of N=14 tweezers

Each tweezer contains initially one atom in the Rydberg internal state $60S_{1/2} \equiv |0\rangle$

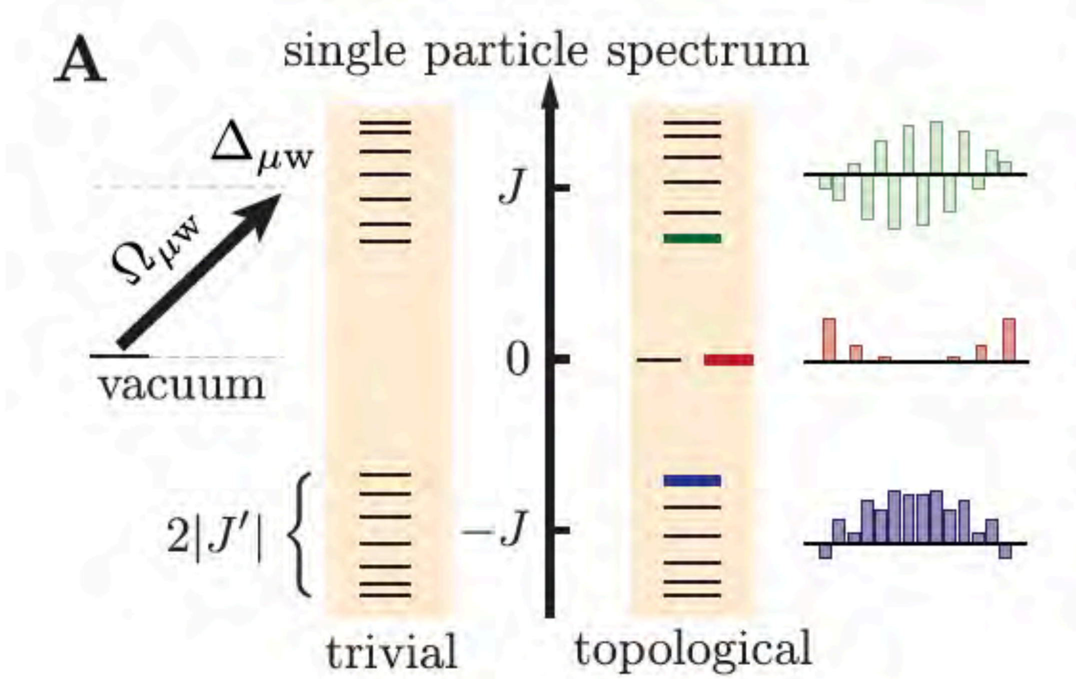
A "particle" is an excitation to the Rydberg state $60P_{1/2} \equiv |1\rangle$

The "particle" jumps between neighboring sites by resonant dipole-dipole interaction



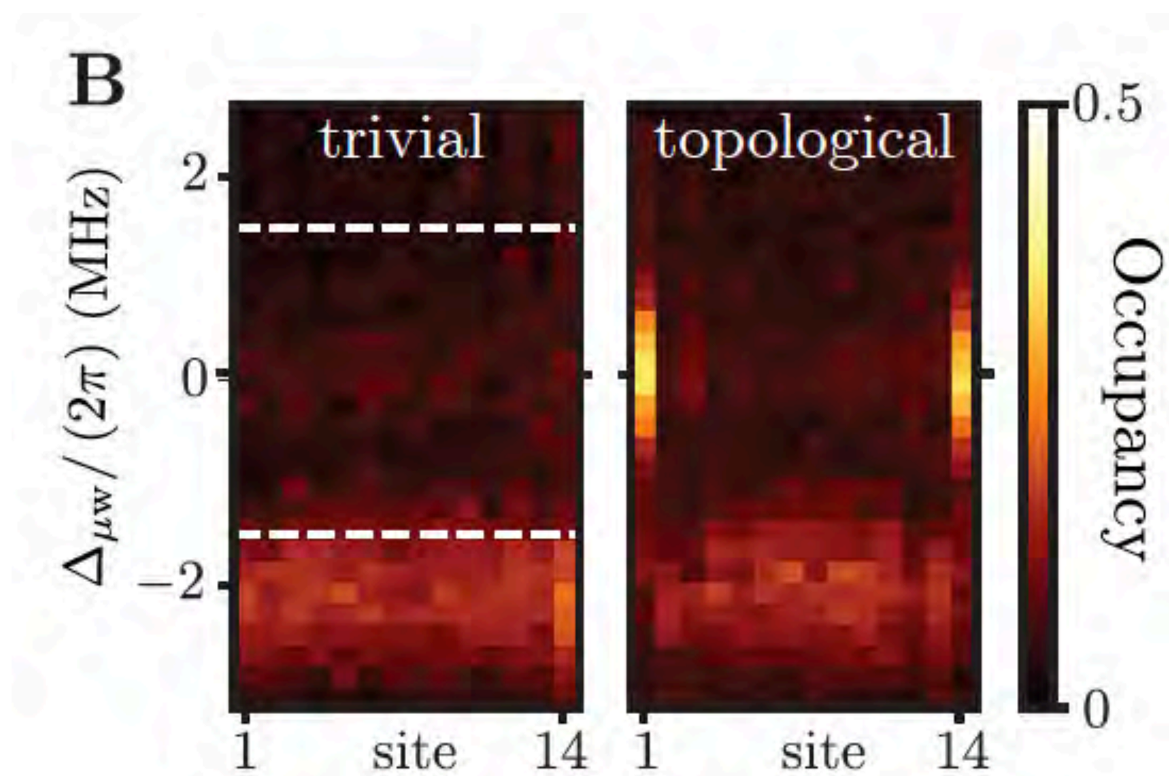


The single particle spectrum for this finite SSH chain



Microwave spectroscopy:
coherent creation of a "particle" only if an eigenstate energy matches the microwave detuning

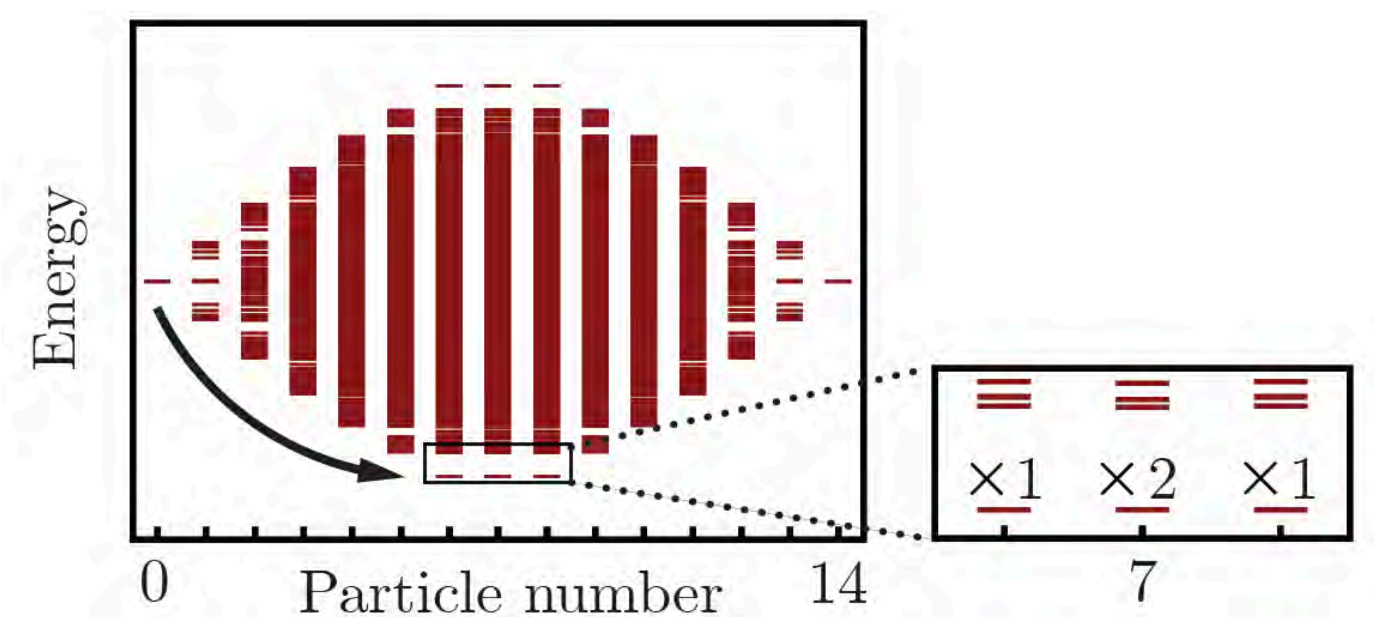
NB. The upper band is not visible due to a suppressed coupling to the microwave probe



Many-body ground state of the 14-site chain

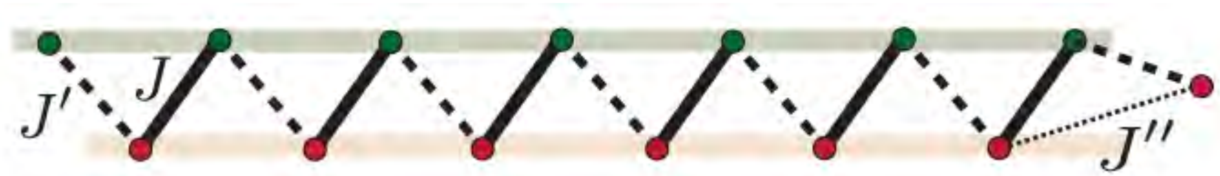
The "particles" can be viewed as hard-core bosons

For nearest neighbor hoppings: Bosonic many-body GS $\xleftrightarrow{\text{Jordan-Wigner transformation}}$ Free fermion GS



Calculated spectrum, with a 4-fold degenerate ground state

For strongly interacting bosons, the degeneracy remains, even in the presence of a coupling violating the sub-lattice symmetry (not true for free fermions)



4.

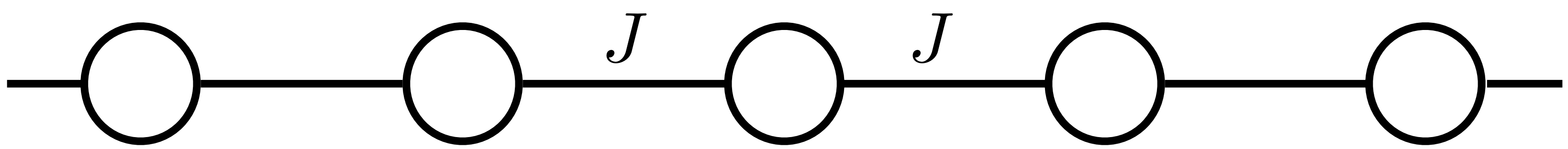
Majorana states in a 1D chain

A.Y. Kitaev
Unpaired Majorana fermions in quantum wires
Physics Uspekhi **44**, 131(2001)

The Kitaev model

The simplest model for a topological superconductor

1D chain, spin plays no role



Fermions : at most one particle per site

Emergence of edge states in a finite chain at *E* = 0

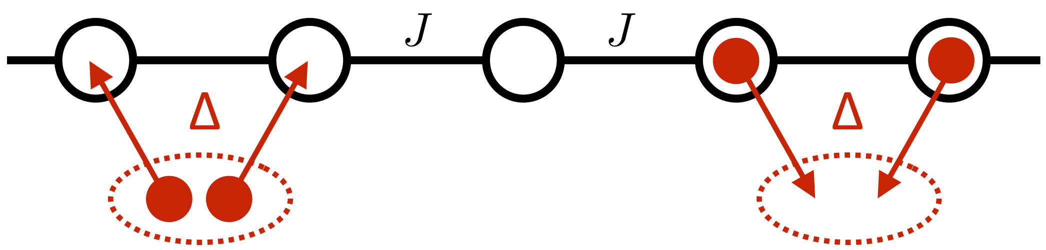
Same as the SSH model for non-interacting particles?

No: additional robustness due to the particle-hole symmetry, originating from the interaction in a superconductor

Representation as Majorana modes:

creation operator= annihilation operator

Kitaev Hamiltonian



Fermions with a frozen spin degree of freedom

Relevant operators: $\hat{a}_j, \hat{a}_j^\dagger$ annihilates and creates a particle on site ***j***

- 1D Hubbard chain, with nearest-neighbour hopping:

$$-J \sum_j \hat{a}_{j+1}^\dagger \hat{a}_j + \mathrm{h.c.}$$

- Coupling to a superconductor reservoir which injects or removes pairs of particles on the chain on neighbouring sites:

$$\Delta \sum_j \left(\hat{a}_{j+1}^\dagger \hat{a}_j^\dagger \ + \ \hat{a}_j \hat{a}_{j+1} \right)$$

Δ : superconducting gap

- Particle number is not conserved in this model: one adds a chemical potential term, which controls the average number of particles on the chain

$$-\mu \hat{N}_\mathrm{p} \quad \text{avec} \quad \hat{N}_\mathrm{p} = \sum_j \hat{a}_j^\dagger \hat{a}_j$$

The Bogoliubov - de Gennes approach

N sites problem, Hilbert space with dimension 2^N : exponential increase with N

For a Hamiltonian $\hat{H}$ that is quadratic in $\hat{a}_j, \hat{a}_j^\dagger$, considerable reduction of the size:

$$\hat{H} = \frac{1}{2} \hat{A}^\dagger \hat{H}_{\mathrm{BdG}} \hat{A}$$

$$\hat{A}^\dagger = \begin{pmatrix} \hat{a}_1^\dagger & \ldots & \hat{a}_N^\dagger & \hat{a}_1 & \ldots & \hat{a}_N \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_N \\ \hat{a}_1^\dagger \\ \vdots \\ \hat{a}_N^\dagger \end{pmatrix}$$

Nambu spinor

The Bogoliubov-de Gennes Hamiltonian is a $2N \times 2N$ square matrix

$$\hat{H}_{\mathrm{BdG}} = \begin{pmatrix} \hat{M}_1 & \hat{M}_2 \\ -\hat{M}_2^* & -\hat{M}_1^* \end{pmatrix}$$

Once the energies E_j and the eigenvectors of $\hat{H}_{\mathrm{BdG}}$ are known, one can write $\hat{H}$ as

$$\hat{H} = \frac{1}{2} \sum_{j=1}^{2N} E_j \, \hat{b}_j^\dagger \hat{b}_j$$

Particle hole symmetry

We started with a N sites problem, hence a Hilbert space with dimension 2^N

We diagonalized $\hat{H}_{\mathrm{BdG}}$ by introducing $2N$ modes (fermionic quasi-particles):

$$\hat{H} = \frac{1}{2} \sum_{j=1}^{2N} E_j \, \hat{b}_j^\dagger \hat{b}_j$$

Each mode can be empty or occupied:
2^{2N} independent states???

No! The spectrum $\{E_j\}$ is symmetric and each excitation appears twice

mode with energy *E* : $\hat{b}_E = \sum_{i=1}^{N} u_i \hat{a}_i + v_i \hat{a}_i^\dagger$ $\qquad$ $\hat{b}_E^\dagger = \sum_{i=1}^{N} u_i^* \hat{a}_i^\dagger + v_i^* \hat{a}_i$

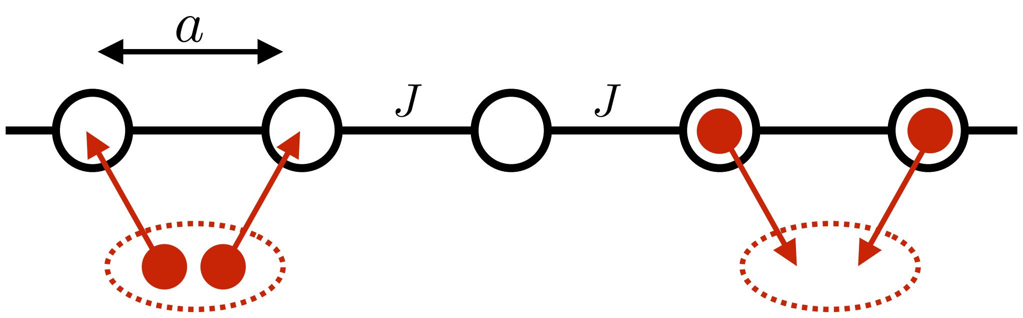
mode with energy -*E* : $\hat{b}_{-E} = \hat{b}_E^\dagger$ $\qquad$ $\hat{b}^\dagger_{-E} = \hat{b}_E$

Annihilation of a quasi-particle -*E* $\iff$ Creation of a quasi-particle +*E*

In summary:

$$\hat{H} = \frac{1}{2} \sum_{j=1}^{2N} E_j \, \hat{b}_j^\dagger \hat{b}_j \longrightarrow \hat{H} = \sum_{j=1}^{N} E_j \, \left(\hat{b}_j^\dagger \hat{b}_j - 1/2 \right) \qquad E_j \geq 0$$

The infinite Kitaev chain



Translation invariant system: Bloch theorem

$$\hat{a}_q^\dagger \propto \sum_j \mathrm{e}^{\mathrm{i}qja}\, \hat{a}_j^\dagger$$

: creates a particle with momentum ***q***

$$-\pi/a \;\le\; q \;<\; \pi/a$$

B-dG formalism reduces in this case to a series of 2×2 matrices

$$\hat{H} = \frac{1}{2}\sum_q \begin{pmatrix} \hat{a}_q^\dagger & \hat{a}_{-q} \end{pmatrix} \hat{H}_q \begin{pmatrix} \hat{a}_q \\ \hat{a}_{-q}^\dagger \end{pmatrix}$$

$$\hat{H}_q = \begin{pmatrix} -\left[2J\cos(qa)+\mu\right] & -2\mathrm{i}\Delta\,\sin(qa) \\ 2\mathrm{i}\Delta\,\sin(qa) & 2J\cos(qa)+\mu \end{pmatrix}$$

$$\hat{H}_q = -\boldsymbol{h}(q)\cdot\hat{\boldsymbol{\sigma}}$$

$$\boldsymbol{h} = \begin{pmatrix} 0 \\ -2\Delta\,\sin(qa) \\ 2J\cos(qa)+\mu \end{pmatrix}$$

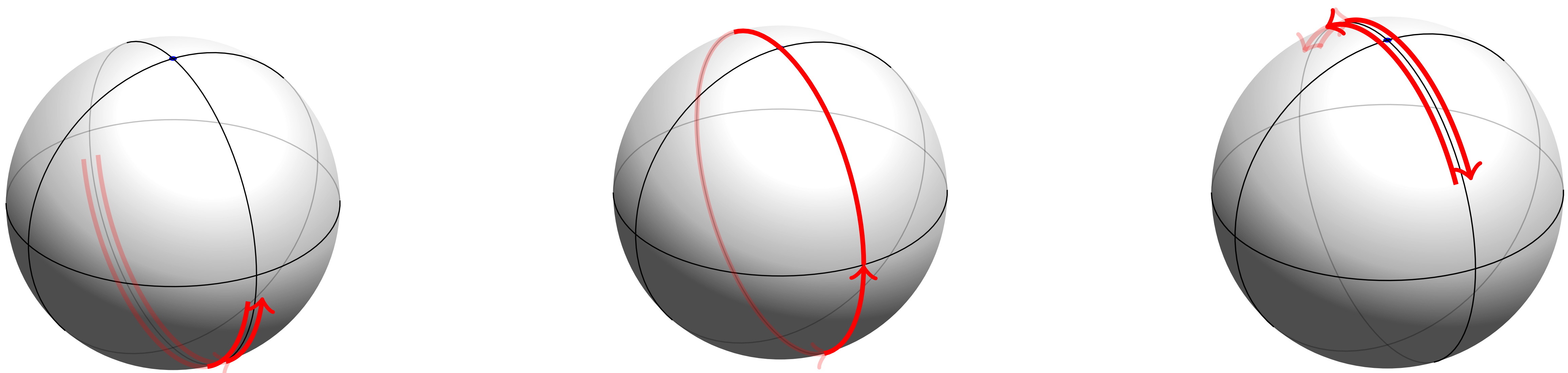
Pseudo-spin 1/2 problem, as for SSH model

What is the trajectory of $\boldsymbol{n} = \dfrac{\boldsymbol{h}}{|\boldsymbol{h}|}$ on the Bloch sphere?

Possible topologies for Kitaev chain

$$\boldsymbol{h} = \begin{pmatrix} 0 \\ -2\Delta\,\sin(qa) \\ 2J\cos(qa) + \mu \end{pmatrix}$$: always on the meridian $h_x = 0$

$\mu < -2J$	$\lvert\mu\rvert < 2J$	$\mu > 2J$



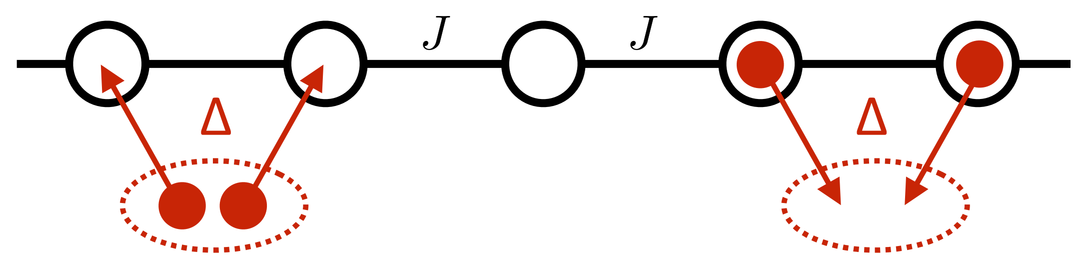
normal	topological	normal
small filling	intermediate filling	large filling

Energies of $\hat{H}_{\mathrm{BdG}}$: $E_q = |\boldsymbol{h}(q)| = \left\{4\Delta^2 \sin^2(qa) + [2J\cos(qa) + \mu]^2\right\}^{1/2}$

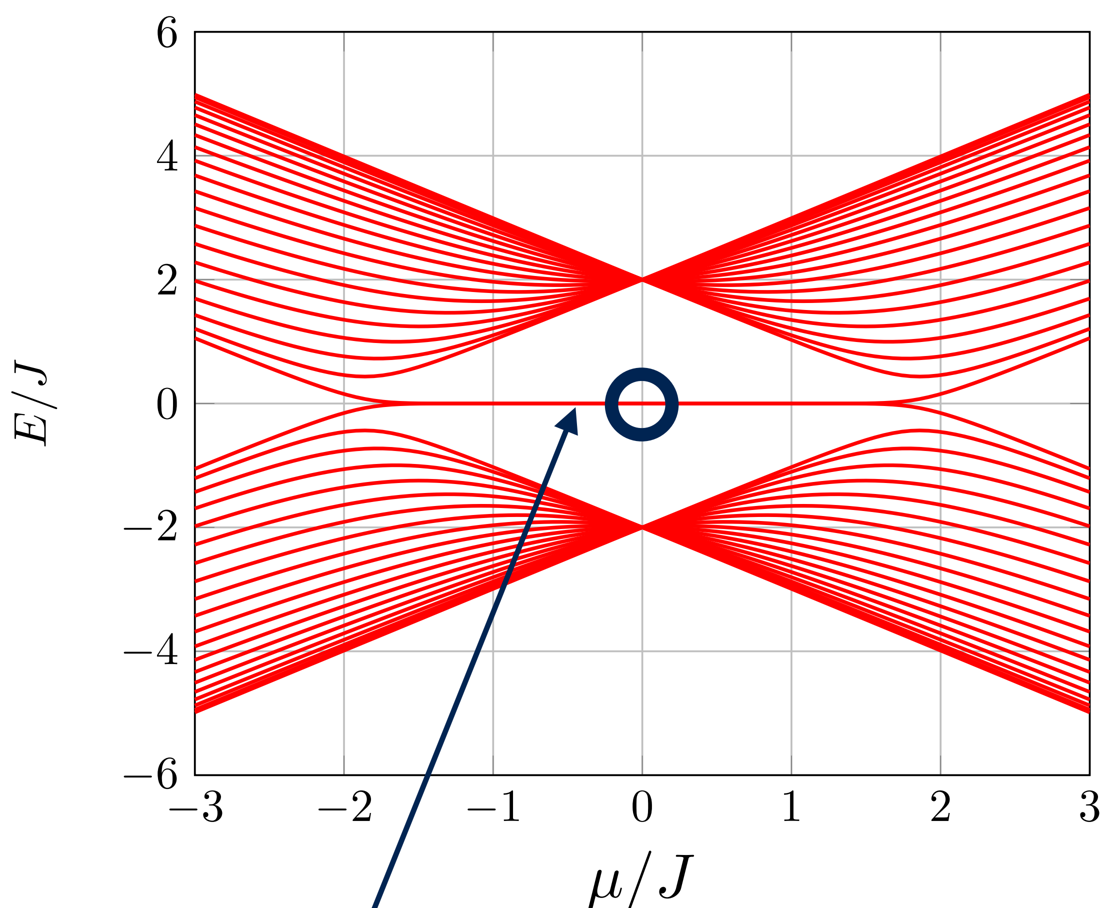
Majorana modes in Kitaev chain

Finite vs. infinite Kitaev chain

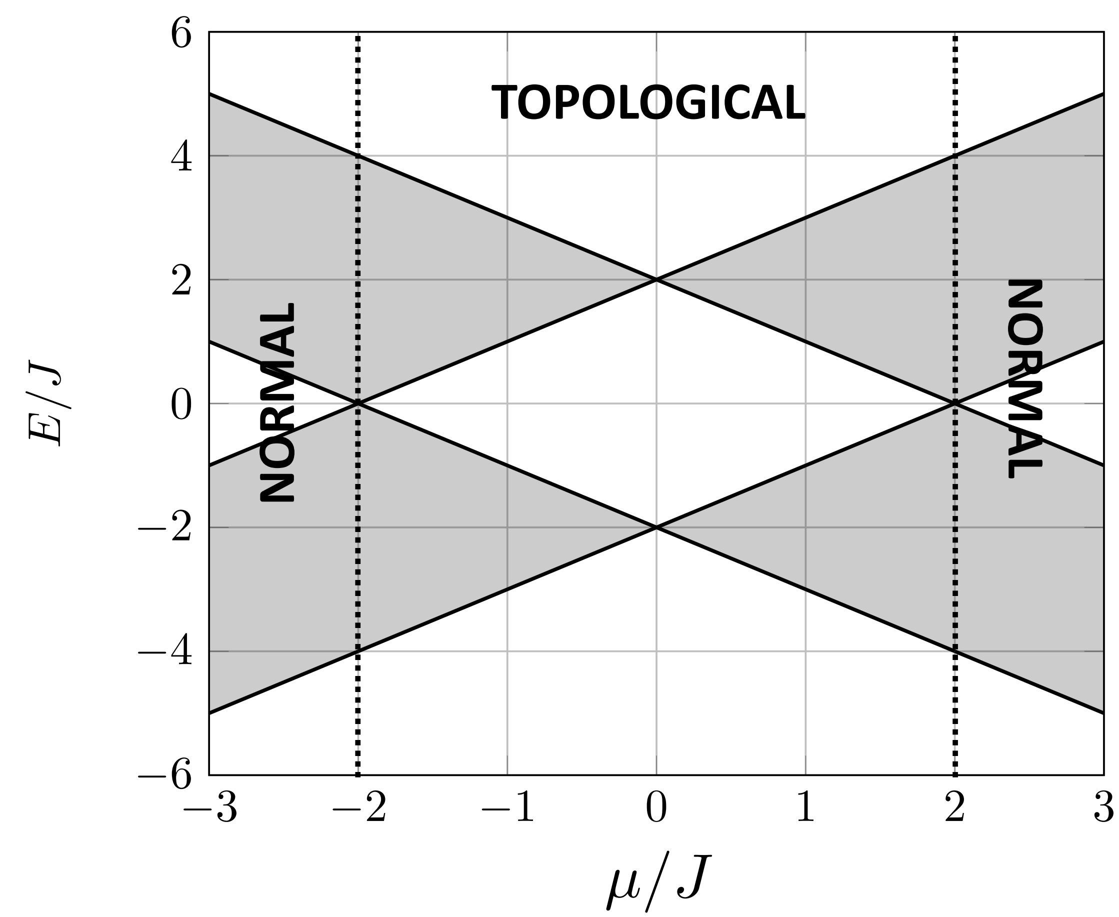
Particular case : $\Delta = J$



20 sites chain



Bands for an infinite chain

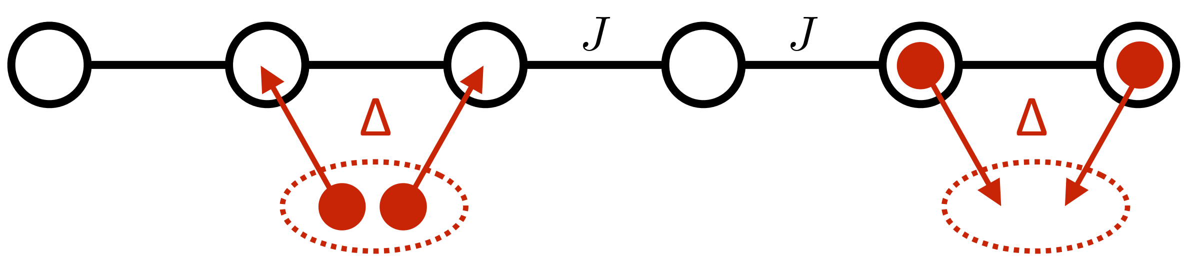


A mode $(E, -E)$ with $E \approx 0$ appears in the topological region $|\mu| < 2J$

$$\pm\left\{4\Delta^2 \sin^2(qa) + [2J\cos(qa) + \mu]^2\right\}^{1/2}$$
$$= \pm\left\{4J^2 + 4J\mu\cos(qa) + \mu^2\right\}^{1/2}$$

Edge state

The case μ = 0



N sites segment

The choice $\mu = 0,\ \Delta = J$ leads to a Hamiltonian that can be analytically handled:

$$\hat{H} = J \sum_{j=1}^{N-1} \left(-\hat{a}_j^\dagger \hat{a}_{j+1} - \hat{a}_{j+1}^\dagger \hat{a}_j + \hat{a}_j \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j^\dagger \right) = \sum_{j=1}^{N} E_j \left(\hat{b}_j^\dagger \hat{b}_j - \frac{1}{2} \right)$$

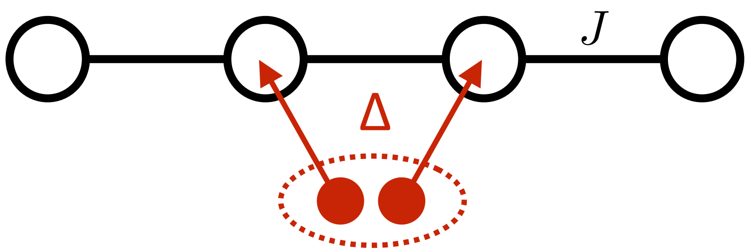
Quasi-particle energies:

$$\begin{cases} E_j = 2J & j = 1, \ldots, N-1 \\ E_N = 0 \end{cases}$$

Eigenmodes:

$$\begin{cases} \hat{b}_j = \frac{\mathrm{i}}{2} \left(\hat{a}_j^\dagger - \hat{a}_j + \hat{a}_{j+1}^\dagger + \hat{a}_{j+1} \right) & j = 1, \ldots, N-1 \\ \hat{b}_N = \frac{\mathrm{i}}{2} \left(\hat{a}_N^\dagger - \hat{a}_N + \hat{a}_1^\dagger + \hat{a}_1 \right) \end{cases}$$

The ground state



$$\left.\begin{aligned}\hat{H} &= \sum_{j=1}^{N} E_j \left(\hat{b}_j^\dagger \hat{b}_j - \frac{1}{2}\right) \\ E_j &\geq 0\end{aligned}\right\}$$

Ground state obtained by solving

$$\hat{b}_j|\psi_0\rangle = 0, \qquad j = 1, \ldots, N$$

A unique solution but…

The mode ***j* = *N*** has a zero energy: $|\psi_1\rangle = \hat{b}_N^\dagger|\psi_0\rangle$ has the same energy as $|\psi_0\rangle$

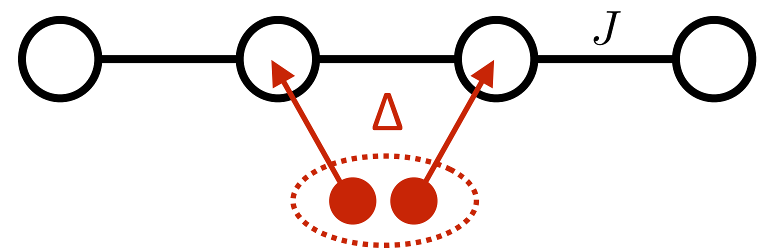
Doubly degenerate ground state: $\{|\psi_0\rangle, |\psi_1\rangle\}$

Example of the 4 site chain (dimension of Hilbert space 2^4=16)

$$|\psi_0\rangle = \underbrace{|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle}_{\textbf{1 part.}} + \underbrace{|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle}_{\textbf{3 part.}}$$

$$|\psi_1\rangle = \underbrace{|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle}_{\textbf{2 part.}} + \underbrace{|0000\rangle}_{\textbf{0 part.}} + \underbrace{|1111\rangle}_{\textbf{4 part.}}$$

The ground state



$$\left.\begin{array}{l}\hat{H} = \sum_{j=1}^{N} E_j \left(\hat{b}_j^\dagger \hat{b}_j - \frac{1}{2}\right) \\ \\ E_j \geq 0\end{array}\right\}$$

Ground state obtained by solving

$$\hat{b}_j|\psi_0\rangle = 0, \qquad j = 1, \ldots, N$$

A unique solution but…

The mode ***j* = *N*** has a zero energy: $|\psi_1\rangle = \hat{b}_N^\dagger|\psi_0\rangle$ has the same energy as $|\psi_0\rangle$

Doubly degenerate ground state: $\{|\psi_0\rangle, |\psi_1\rangle\}$

Description in terms of « Majorana » operators:

$$\hat{b}_N = \frac{\mathrm{i}}{2}\left(\hat{a}_N^\dagger - \hat{a}_N + \hat{a}_1^\dagger + \hat{a}_1\right)$$

$$\hat{\gamma}_{1,+} = \hat{a}_1 + \hat{a}_1^\dagger$$

$$\hat{\gamma}_{N,-} = \mathrm{i}\left(\hat{a}_N^\dagger - \hat{a}_N\right)$$

$$\hat{b}_N^\dagger \hat{b}_N = \frac{1}{2} + \mathrm{i}\,\hat{\gamma}_{N,-}\,\hat{\gamma}_{1,+}$$

Majorana operators

Ettore Majorana
1906-19??

Majorana (1937): Dirac equation is written for a complex field, where particles and antiparticles play distinct roles

Majorana proposes to split it into two equations for real fields, in which particles et antiparticles are the same

Do such elementary particles exist??? Open question…

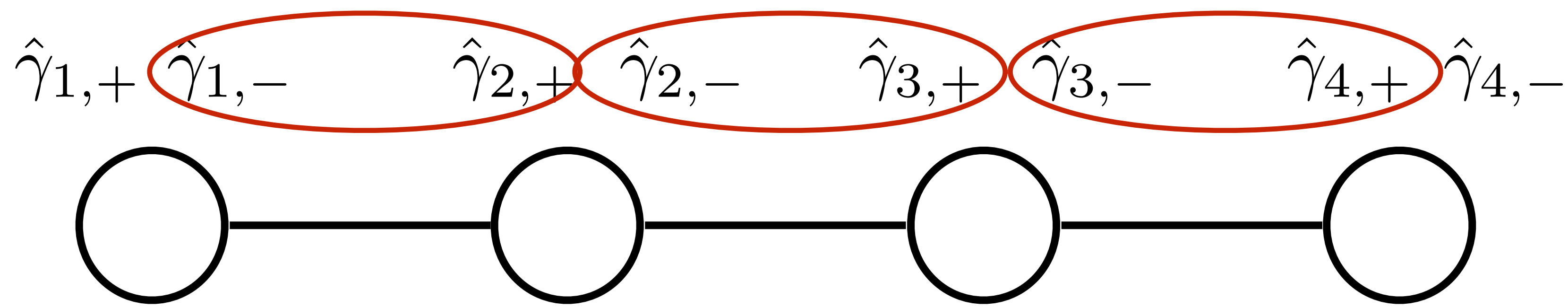
In condensed matter or AMO physics, one works with "real" fermions, but one can find emergent quasi-particles or modes

Starting from a mode $\hat{a}, \hat{a}^\dagger$, one can introduce the operators $\begin{cases} \hat{\gamma}_+ = \hat{a} + \hat{a}^\dagger \\ \hat{\gamma}_- = \mathrm{i}\left(\hat{a} - \hat{a}^\dagger\right) \end{cases}$

- Hermitian, satisfy canonical anti-commutation rules for fermions
- "Majorana-type" behavior: $\hat{\gamma}_-^2 = \hat{\gamma}_-^\dagger \hat{\gamma}_- = \hat{1}$ and the same for $\hat{\gamma}_+$

Use of Majorana opertors

Often a mere useless rewriting... but they are really relevant for the Kitaev problem



$$\hat{H} = J \sum_{j=1}^{N-1} \left(-\hat{a}_j^\dagger \hat{a}_{j+1} - \hat{a}_{j+1}^\dagger \hat{a}_j + \hat{a}_j \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j^\dagger \right)$$

$$= J \sum_{j=1}^{N-1} \left(\hat{a}_j - \hat{a}_j^\dagger \right) \, \left(\hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \right) \; = \mathrm{i}\, J \sum_{j=1}^{N} \hat{\gamma}_{j,-} \, \hat{\gamma}_{j+1,+}$$

and the delocalized mode $\hat{b}_N^\dagger \hat{b}_N = \dfrac{1}{2} + \mathrm{i}\, \hat{\gamma}_{N,-} \, \hat{\gamma}_{1,+}$ stays at zero energy

In order to give a non-zero energy to this mode and lift the ground state degeneracy, one needs a non-local perturbation that couples the two ends of the chain

Additional remarks

The two Majorana modes correspond to a single fermionic excitation, with the mode $\hat{b}_N^\dagger$ which can be empty or occupied

Fractional quasi-particle

To be contrasted with the SSH model, where each of the two edge states can be empty or occupied (i.e. 4-fold degeneracy for polarized fermions)

The fact that we considered here "spinless" fermions is essential.
For spin 1/2 particles, the number of modes is doubled

$\longrightarrow$ *Two Majorana modes at each end*

Then a spin-orbit perturbation could couple these two modes, making it as a normal fermionic excitation: back to a SSH-type problem !

Majorana modes in quantum information

Degree of freedom protected from decoherence

$\hat{\gamma}_{1,+}$ $\qquad\qquad\qquad$ $\hat{\gamma}_{N,-}$

- First approach: use the two ground states $|\psi_0\rangle, |\psi_1\rangle$ of the chain to encode a qbit

  Problem: Difficult to manipulate because they have number of particles with opposite parities

- Second approach: use (at least) two chains and braid them to manipulate the states

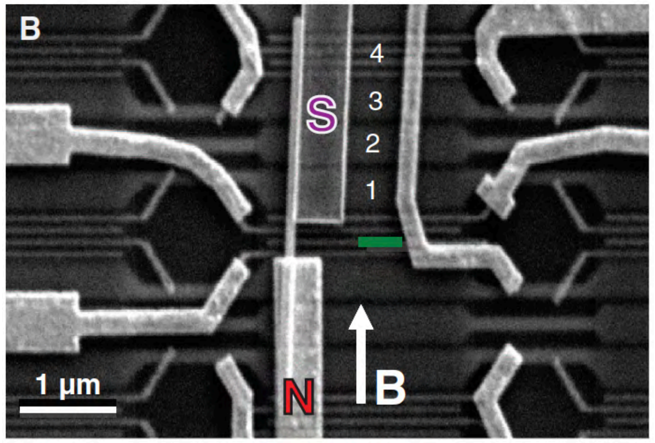
$\hat{\gamma}_{1,+}$ $\qquad$ $\hat{\gamma}_{N,-}$ $\qquad\qquad$ $\hat{\gamma}'_{1,+}$ $\qquad$ $\hat{\gamma}'_{N,-}$

One can then work in a degenerate subspace with a well defined and fixed parity:

Linear combinations of $|\psi_0, \psi'_0\rangle$ and $|\psi_1, \psi'_1\rangle$

Majorana modes in the lab

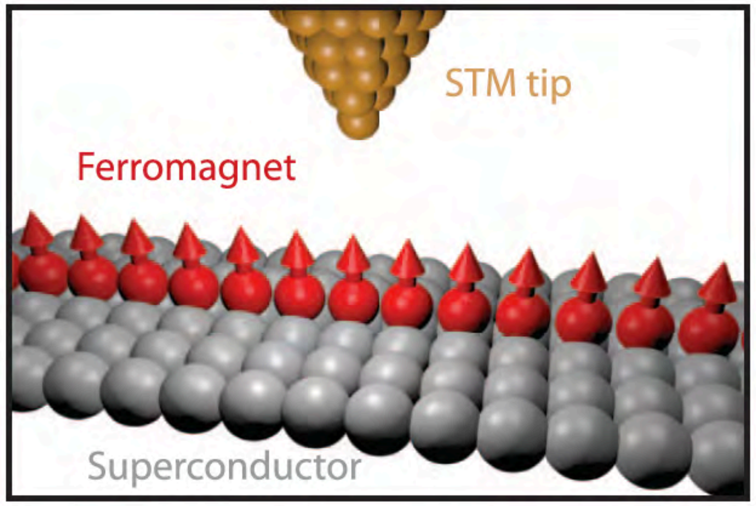
Delft



V. Mourik et al, Science 336, 1003 (2012)

Semiconducting nanowire (InGaAs) in contact with two electrodes, one normal and one superconductor

Princeton



S. Nadj-Perge et al., Science 346, 602 (2014)

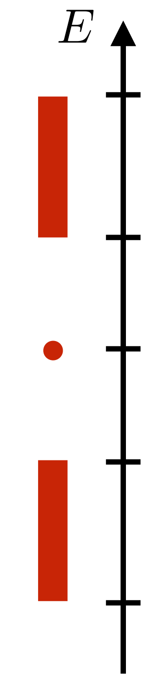
Chain of iron atoms deposited on the surface of a superconductor

Wang et al., Science 362, 333 (2018), Liu et al, Phys. Rev. X 8, 041056 (2018), Yuan et al., Nat. Phys. 15, 1046 (2019),…

And cold atoms? Several projects underway…

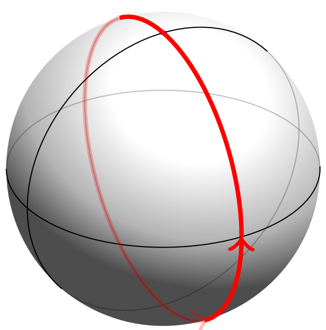
S. Nascimbene, J. Phys. B 46, 134005 (2013)

Summary

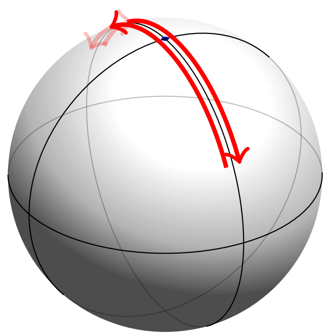


It is difficult to distinguish a normal from a superfluid phase by looking only at the bulk of the material: same band structure

The topology is revealed when looking at the edges



The classification of topological matter is not performed as in Landau theory, where one looks for the breaking of a spatial symmetry (e.g. ferromagnetism)



One must look at the topology associated to eigenstates

Nevertheless some symmetries play a crucial role

Sub-lattice (chiral) symmetry, particle hole symmetry

The presence of these symmetries guarantees the robustness of the topological phase

When happens when this symmetry is lost?