Topological matter and its exploration with quantum gases

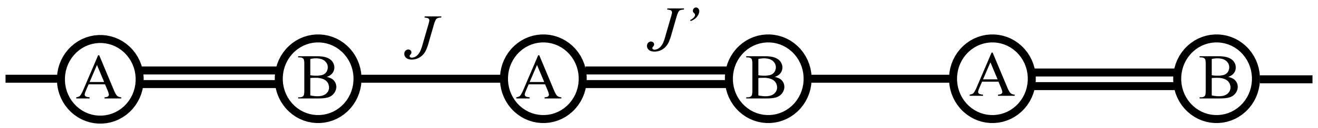
Jean Dalibard

Lectures at EPFL
November 2019

COLLÈGE
DE FRANCE
1530

A short summary of last week lecture

Simple 1D periodic problems, like the SSH model



$$E_A = E_B = 0$$

Identification of a topological classification

Principle behind this classification:

Two-site unit cell

\+

Bloch theorem

→

Representation

by a

pseudo-spin 1/2

$$|u_q^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp e^{i\phi_q} \end{pmatrix}$$

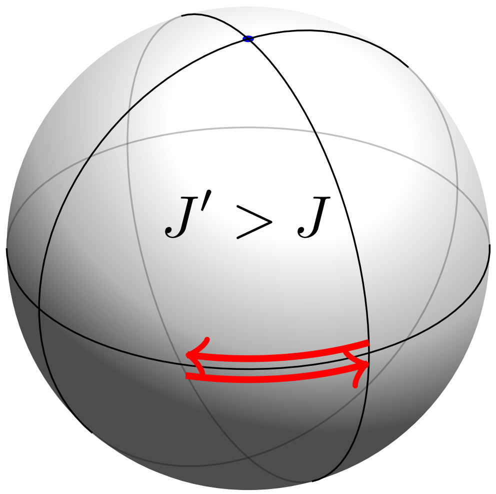
Eigenstates $e^{iqx}\underbrace{u_q(x)}_{\text{periodic}}$

Bloch momentum ***q***: $-\pi/a \;\leq\; q \;<\; \pi/a$

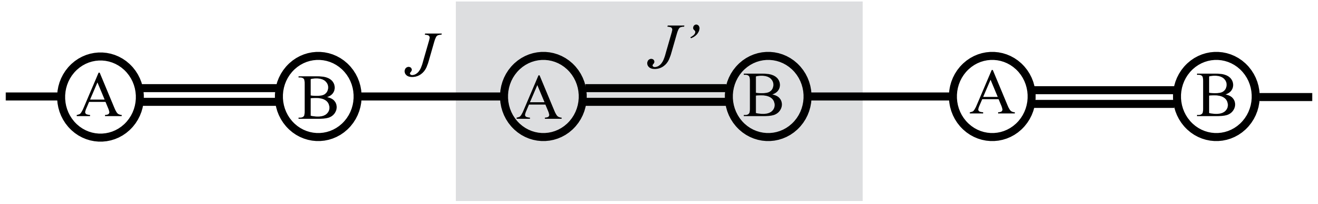
A short summary of last week lecture (2)

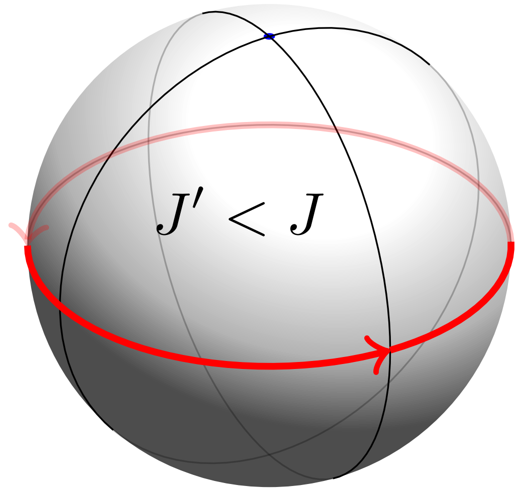
Topological characterization of an energy band based on the winding of its eigenstates on the Bloch sphere when the quasi-momentum q spans the Brillouin zone

For the SSH model, $|u_q^{(-)}\rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1 \\ \mathrm{e}^{\mathrm{i}\phi_q}\end{pmatrix}$ remains on the equator of the sphere



Normal case





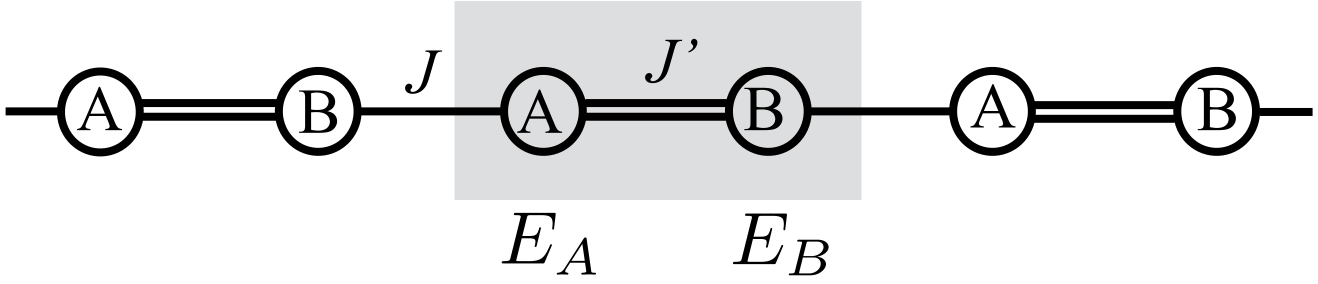
Topological case

Physical manifestation of this topological classification: robust edge states

Next goal

- What happens when we loose the symmetry that protects the topological classification, i.e. the restriction to the equator of the Bloch sphere ?

We will slightly enrich the 1D SSH model: We release the constraint $E_A = E_B$



1D Rice-Mele model

We keep a unit cell with two sites: the pseudo spin 1/2 approach remains valid

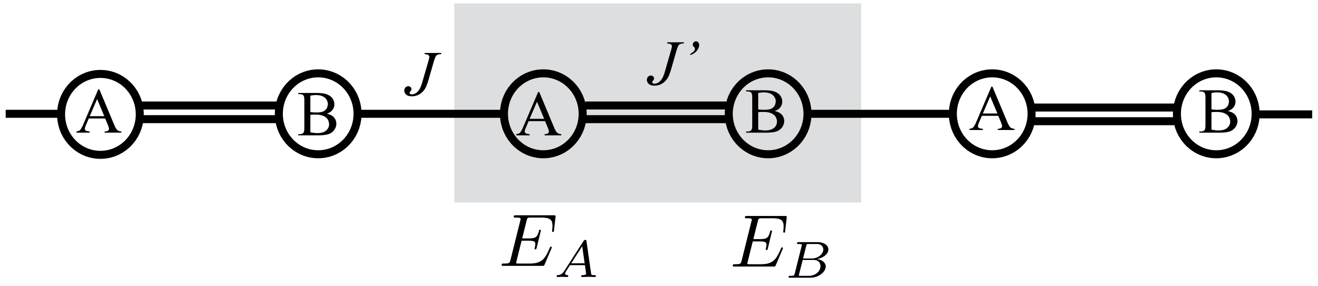
- Switch to a time-dependent problem for Rice-Mele model; Time plays the role of a synthetic dimension, leading to an effective 2D problem: new topological classification!

Adiabatic pump and quantization of the displacement in a periodic evolution

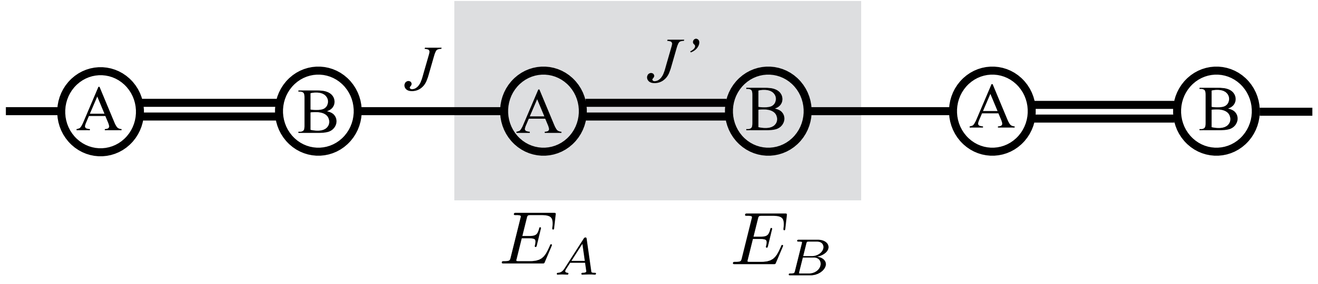
1.

Beyond SSH: the Rice-Mele model

M.J. Rice and E.J. Mele
Elementary excitations of a linearly conjugated diatomic polymer
Phys. Rev. Lett. **49**, 1455 (1982)



Reminder on two-site Hamiltonians



Infinite periodic chain described by a tight-binding model

$$|\psi_q\rangle = \sum_j \mathrm{e}^{\mathrm{i}\,jqa}\,(\alpha_q|A_j\rangle + \beta_q|B_j\rangle)$$

$$|u_q\rangle = \alpha_q\Big(\sum_j |A_j\rangle\Big) \; + \; \beta_q\Big(\sum_j |B_j\rangle\Big) \quad \longrightarrow \quad |u_q\rangle = \begin{pmatrix}\alpha_q\\ \beta_q\end{pmatrix}$$

Periodic Hamiltonian allowing one to find $|u_q\rangle$: 2x2 hermitian matrix

$$\hat{H}_q = E_0(q)\;\hat{1} \; - \; \boldsymbol{h}(q)\cdot\hat{\boldsymbol{\sigma}}$$

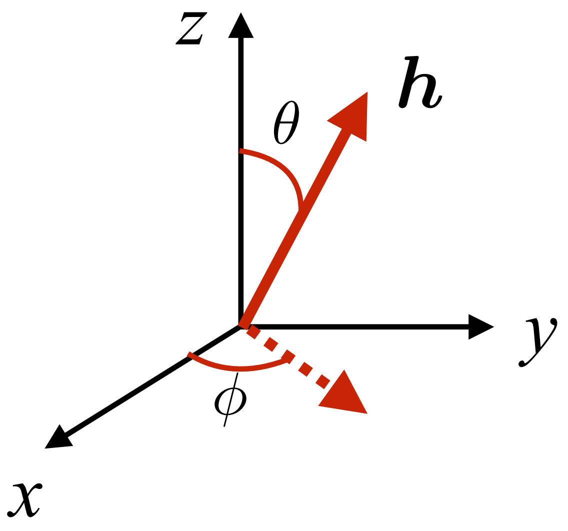
$\hat{\boldsymbol{\sigma}} = \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$: Pauli matrices

$\boldsymbol{h} = \begin{pmatrix}h_x\\ h_y\\ h_z\end{pmatrix}$: three real components

Parametrization in terms of energy and angles

For the pseudo-spin 1/2 the problem is fully characterized by $E_0(q), \boldsymbol{h}(q)$

Parametrize the vector $\boldsymbol{h}$ by its modulus $|\boldsymbol{h}|$
and the angles in spherical coordinates θ, ϕ

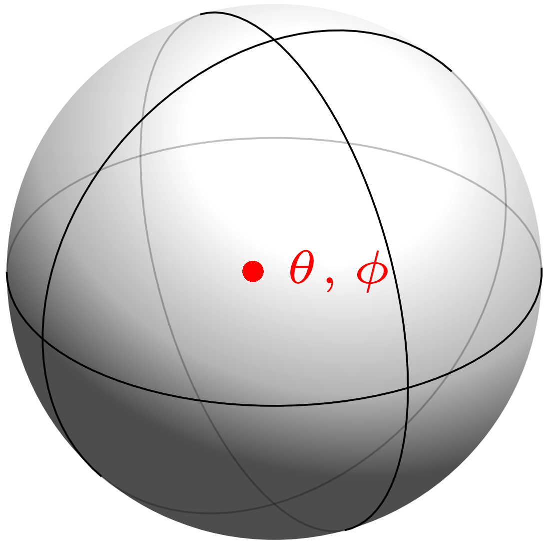


The periodic Hamiltonian reads

$$\hat{H}_q = E_0(q)\,\hat{1} \;-\; |\boldsymbol{h}(q)| \begin{pmatrix} \cos\theta_q & \mathrm{e}^{-\mathrm{i}\phi_q}\,\sin\theta_q \\ \mathrm{e}^{\mathrm{i}\phi_q}\,\sin\theta_q & -\cos\theta_q \end{pmatrix}$$

Energies: $E_0 \pm |\boldsymbol{h}|$

Eigenstates:
$$\begin{cases} |u^{(-)}\rangle = \begin{pmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{pmatrix} \\ |u^{(+)}\rangle = \begin{pmatrix} \sin(\theta/2) \\ -\mathrm{e}^{\mathrm{i}\phi}\cos(\theta/2) \end{pmatrix} \end{cases}$$



Berry connection

Tool to calculate the geometrical phase on a closed contour

For the lower band:

$$|u^{(-)}\rangle = \begin{pmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{pmatrix} \longrightarrow \quad \mathcal{A}^{(-)}(q) = \mathrm{i}\langle u_q^{(-)}|\partial_q u_q^{(-)}\rangle = -\frac{\mathrm{d}\phi_q}{\mathrm{d}q}\sin^2(\theta_q/2)$$

$$= \frac{1}{2}\,\frac{\mathrm{d}\phi_q}{\mathrm{d}q}\,(-1+\cos\theta_q)$$

For the upper band:

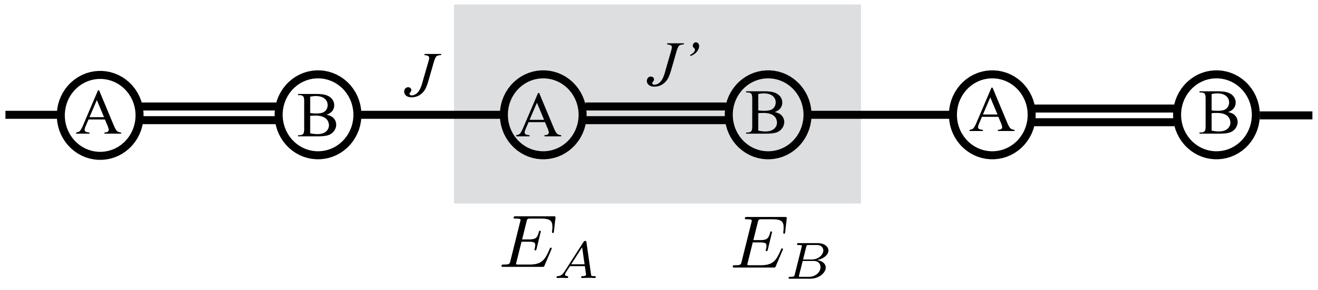
$$|u^{(+)}\rangle = \begin{pmatrix} \sin(\theta/2) \\ -\mathrm{e}^{\mathrm{i}\phi}\cos(\theta/2) \end{pmatrix} \longrightarrow \quad \mathcal{A}^{(+)}(q) = -\frac{1}{2}\,\frac{\mathrm{d}\phi_q}{\mathrm{d}q}\,(1+\cos\theta_q)$$

Result that depends on the choice of the gauge:

$$|\tilde{u}^{(-)}\rangle = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi}\ \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \longrightarrow \quad \tilde{\mathcal{A}}^{(-)}(q) = \frac{1}{2}\,\frac{\mathrm{d}\phi_q}{\mathrm{d}q}\,(1+\cos\theta_q)$$

The Rice-Mele problem

Enriched SSH model:



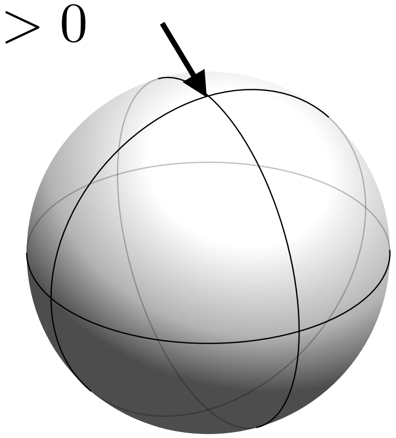
Periodic Hamiltonian:

$$\hat{H}_q = \begin{pmatrix} E_A & -(J' + J\,\mathrm{e}^{-\mathrm{i}qa}) \\ -(J' + J\,\mathrm{e}^{\mathrm{i}qa}) & E_B \end{pmatrix}$$

$$= \frac{1}{2}(E_A + E_B)\,\hat{1} \; - \; \begin{pmatrix} \Delta & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & -\Delta \end{pmatrix}$$

We will set $E_A + E_B = 0$ *,* $2\Delta = E_B - E_A$ *can be positive or negative*

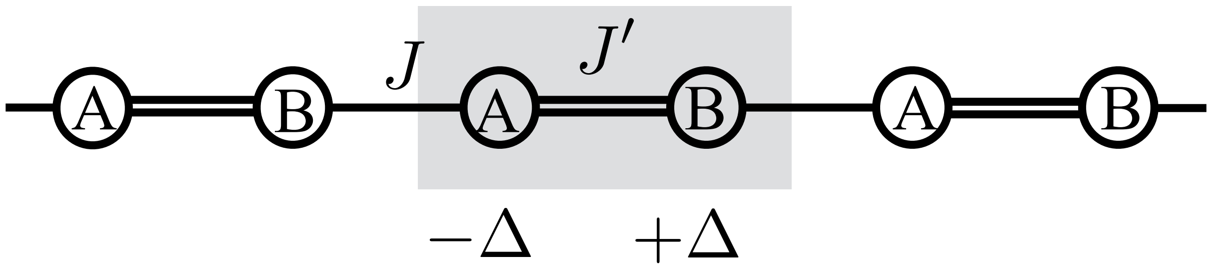
$$\hat{H}_q = -\,\boldsymbol{h}(q)\cdot\hat{\boldsymbol{\sigma}} \qquad \text{with} \quad \boldsymbol{h}(q) = \begin{pmatrix} J' + J\cos(qa) \\ J\sin(qa) \\ \Delta \end{pmatrix}$$



$qa = \pi, J' = J$
$\Delta > 0$

Any point on the Bloch sphere can now be reached

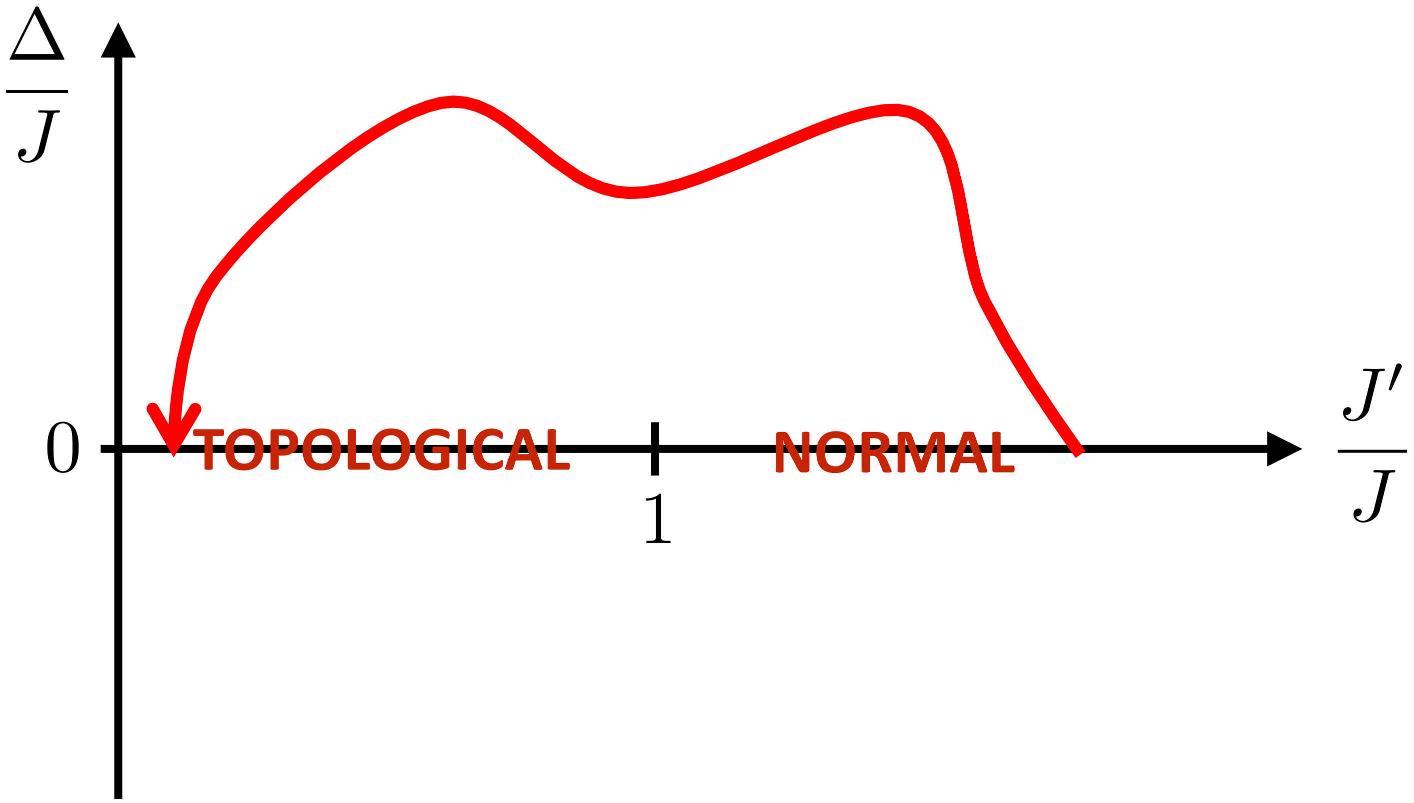
Which phase diagram for the Rice-Mele model?



$$\boldsymbol{h}(q) = \begin{pmatrix} J' + J\cos(qa) \\ J\sin(qa) \\ \Delta \end{pmatrix}$$

$$E_q^{(\pm)} = \pm|\boldsymbol{h}(q)|$$

Two dimensionless parameters: $\frac{J'}{J}$ and $\frac{\Delta}{J}$



SSH model: $\Delta = 0$

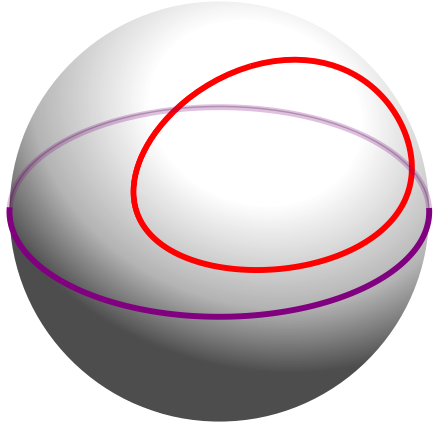
One goes from the normal phase to the topological phase across the singular point $J' = J$ where the gap vanishes

sublattice symmetry

RM model: the gap remains except in $J' = J,\ \Delta = 0$

The loss of the sublattice symmetry entails the loss of the topological robustness

Zak phase for the Rice-Mele model



$$\Phi_{\text{Zak}}^{(-)} = \int_{\text{BZ}} \mathcal{A}^{(-)}(q)\,\mathrm{d}q \quad \text{with} \quad \mathcal{A}^{(-)}(q) = \frac{1}{2}\,\frac{\mathrm{d}\phi_q}{\mathrm{d}q}\,(-1+\cos\theta_q)$$

The angles θ_q, ϕ_q are obtained from

$$\boldsymbol{h}(q) = \begin{pmatrix} J' + J\cos(qa) \\ J\sin(qa) \\ \Delta \end{pmatrix}$$

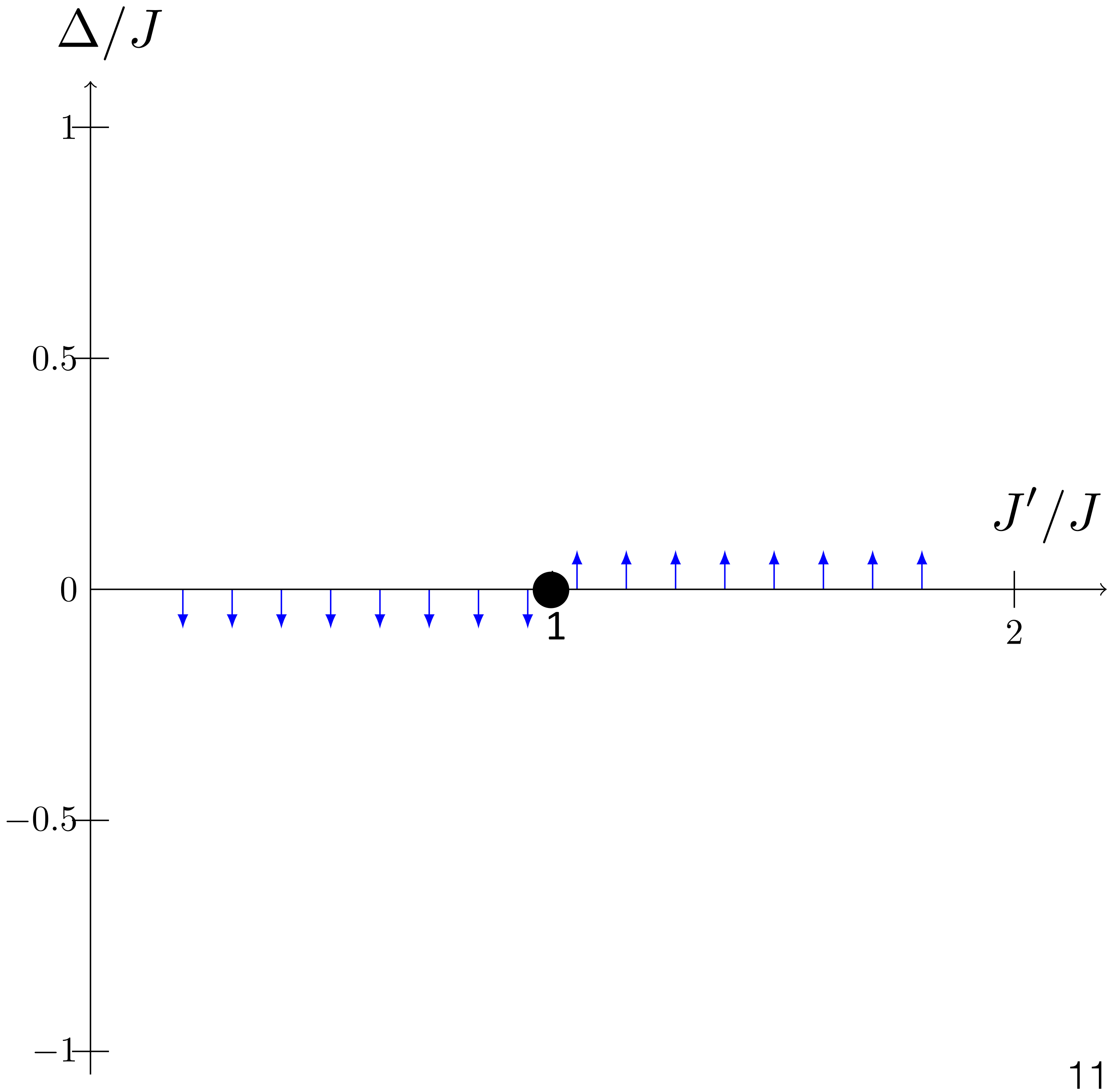
$$= |\boldsymbol{h}| \begin{pmatrix} \sin\theta\ \cos\phi \\ \sin\theta\ \sin\phi \\ \cos\theta \end{pmatrix}$$

Representation of $\Phi_{\text{Zak}}^{(-)}$
by the orientation of a unit vector

↑ : $\Phi_{\text{Zak}}^{(-)} = 0$

← : $\Phi_{\text{Zak}}^{(-)} = \pi/2$

↓ : $\Phi_{\text{Zak}}^{(-)} = \pi$



2.

Optical lattices and superlattices

One-dimension optical lattice

$$k = 2\pi/\lambda$$

Laser standing wave: the intensity is spatially modulated

$$I(x) = I_0 \sin^2(kx)$$ spatial period: $$a = \lambda/2$$

Induced dipole oscillating in time: $\boldsymbol{D} = \alpha \boldsymbol{E}_{\text{light}}$

Dipolar potential: $V(x) = V_0 \ \sin^2(kx)$

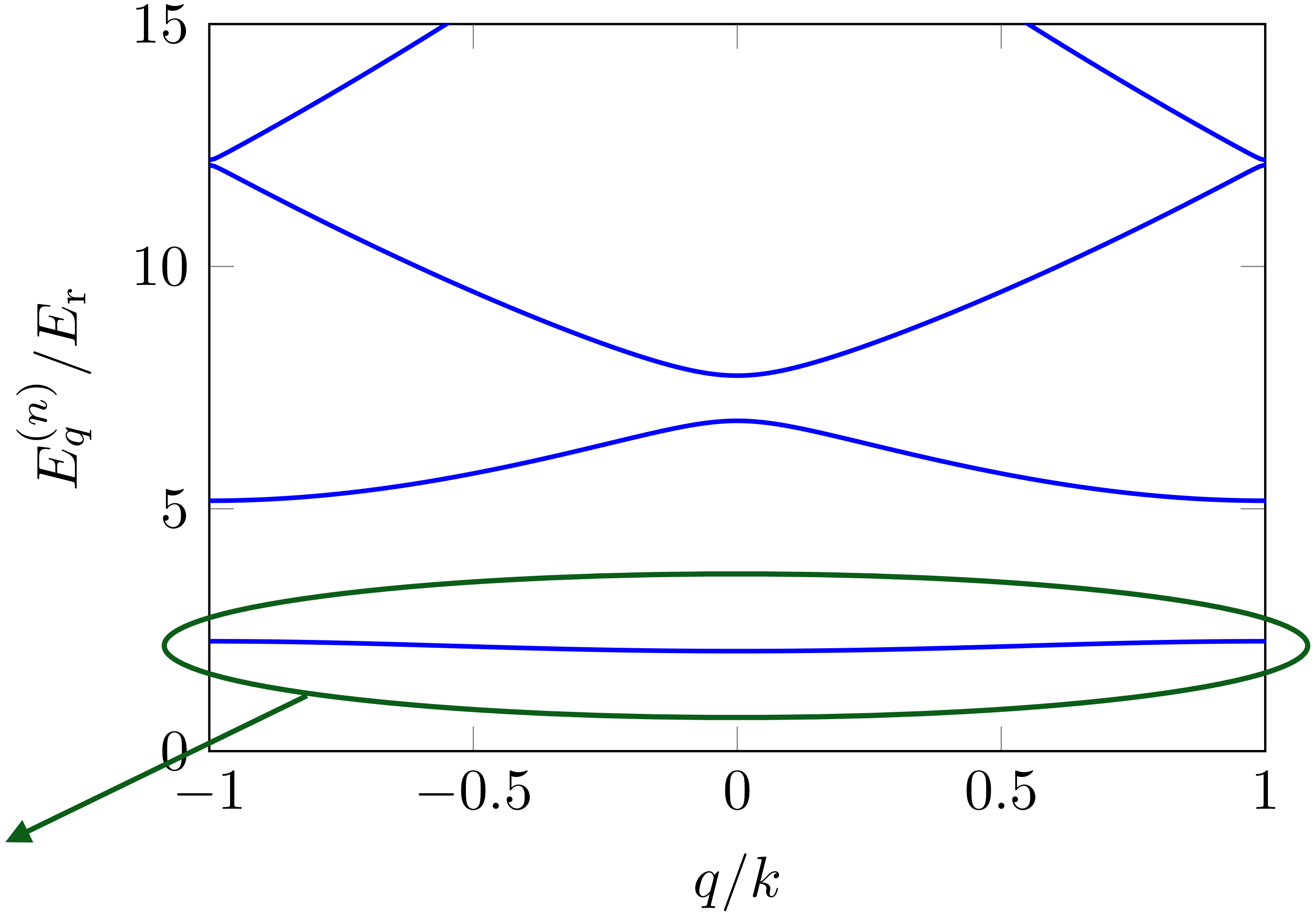
Natural scales for this problem

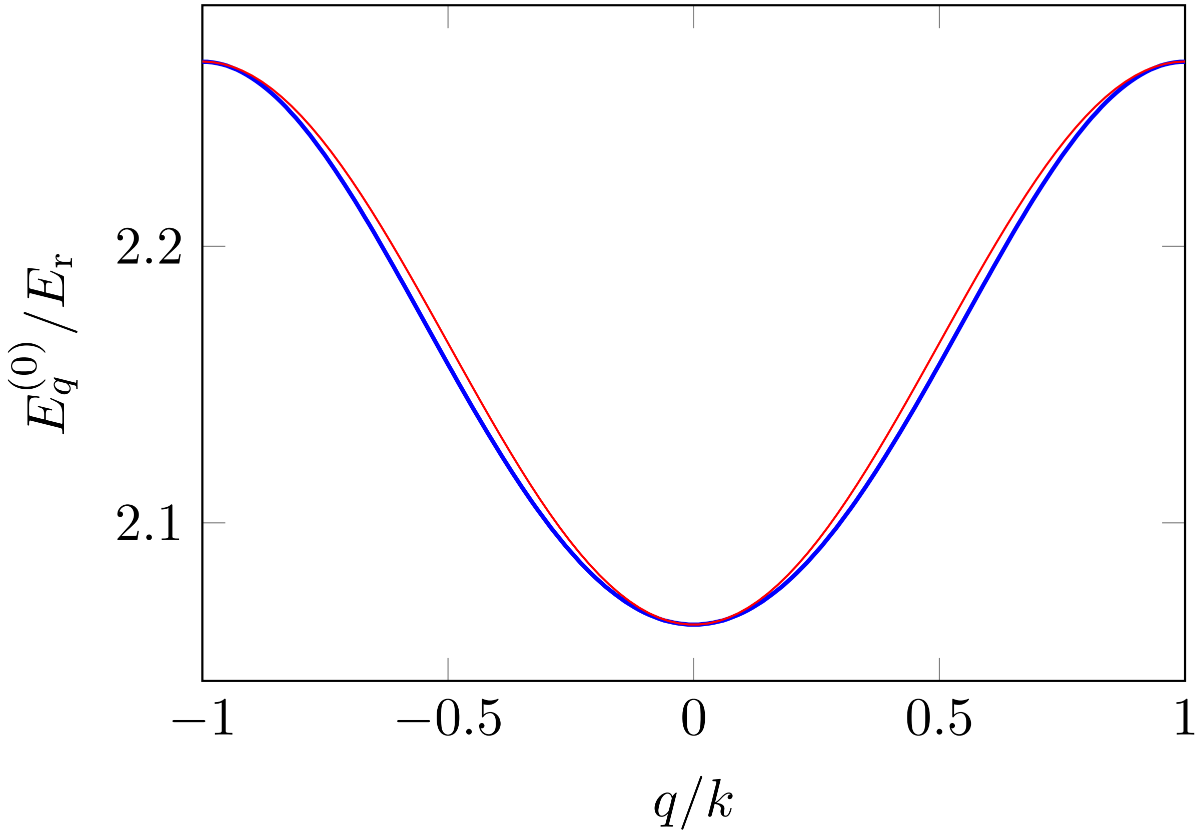
- length : $\lambda = 2\pi/k$ micrometer
- energy : $E_{\text{recoil}} = \dfrac{\hbar^2 k^2}{2m}$ 3-30 kHz

Energy bands for a simple lattice

$$V(x) = V_0\ \sin^2(kx)$$

Plotted for $V_0 = 6E_\mathrm{r}$





Lowest band well described by a Hubbard Hamiltonian:

$$E_q = -2J\cos(qa) + \text{cte.}$$

$$J \approx 0.05\,E_\mathrm{r}$$

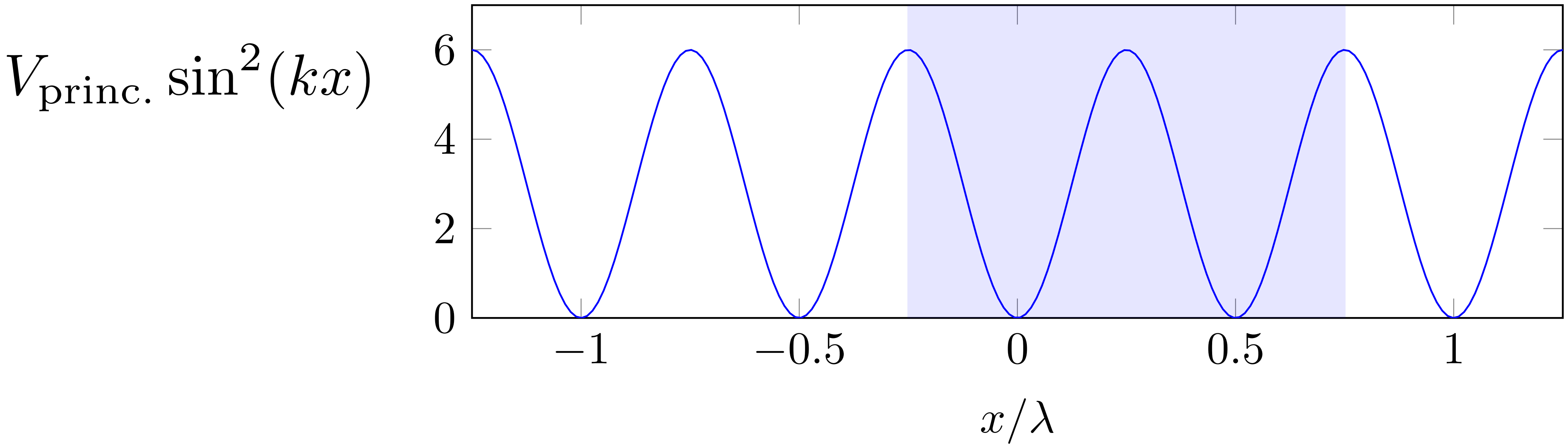
SSH and Rice-Mele: The optical superlattice

Two standing wave at λ and 2λ (typically λ=532 nm)

$$V(x) = V_{\mathrm{princ.}} \sin^2(kx) + V_{\mathrm{sec.}} \sin^2[(kx+\phi)/2]$$

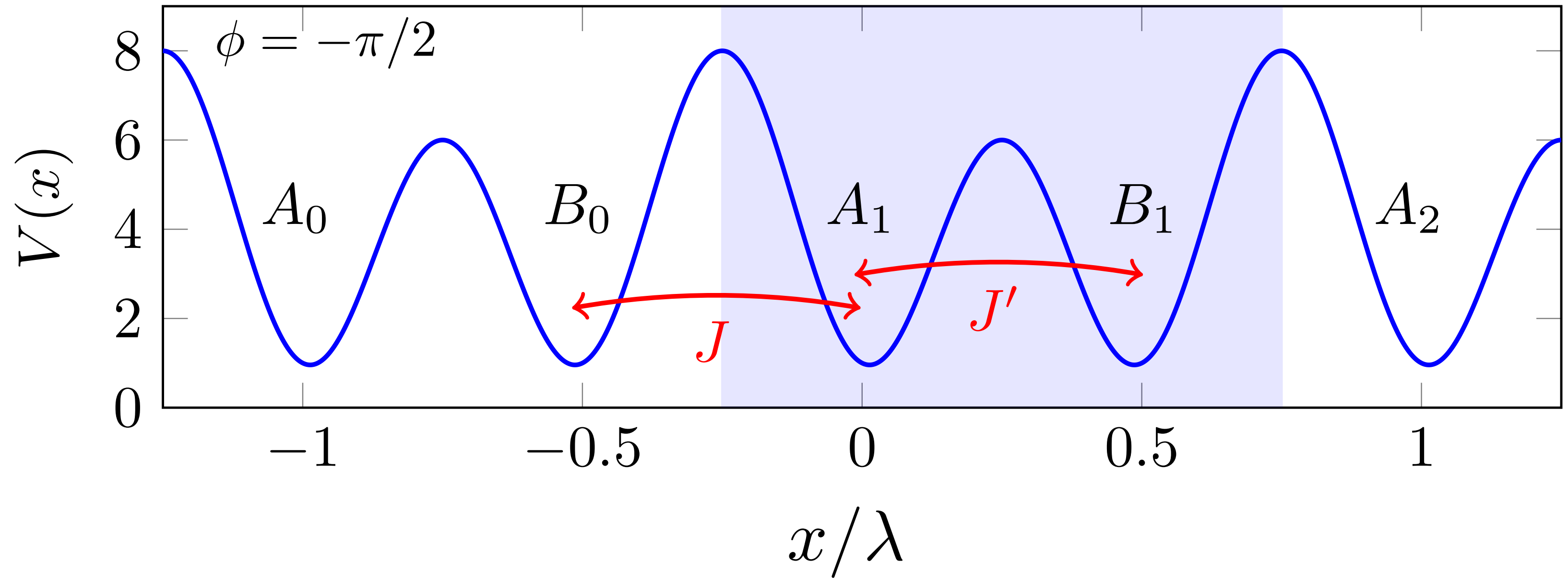
The intensity of the principal lattice (short wavelength) is large compared to that of the secondary lattice (long wavelength)

Potential created by the principal lattice: minima in $x = n\,\lambda/2$



Potential for the superlattice

$$V(x) = V_{\text{princ.}} \sin^2(kx) + V_{\text{sec.}} \sin^2[(kx+\phi)/2]$$



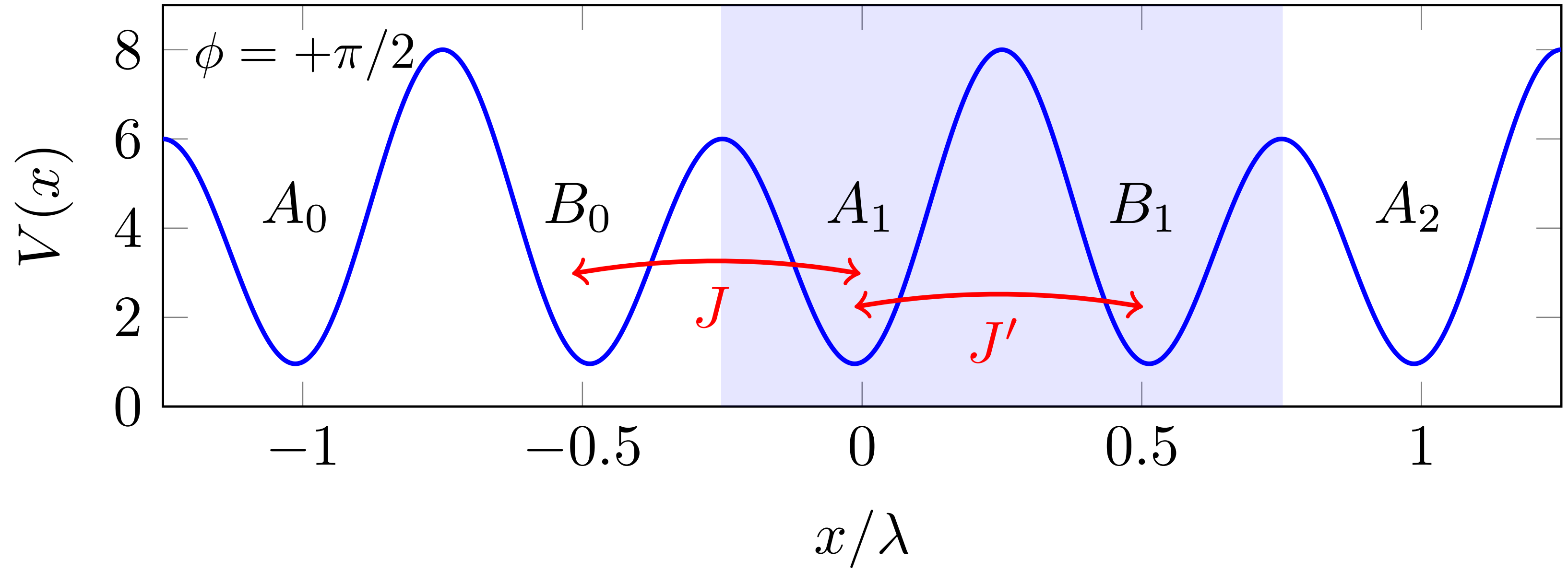
$$\phi = \pm\pi/2$$

$$\downarrow$$

Same value of the secondary potential on all minima of the principal lattice

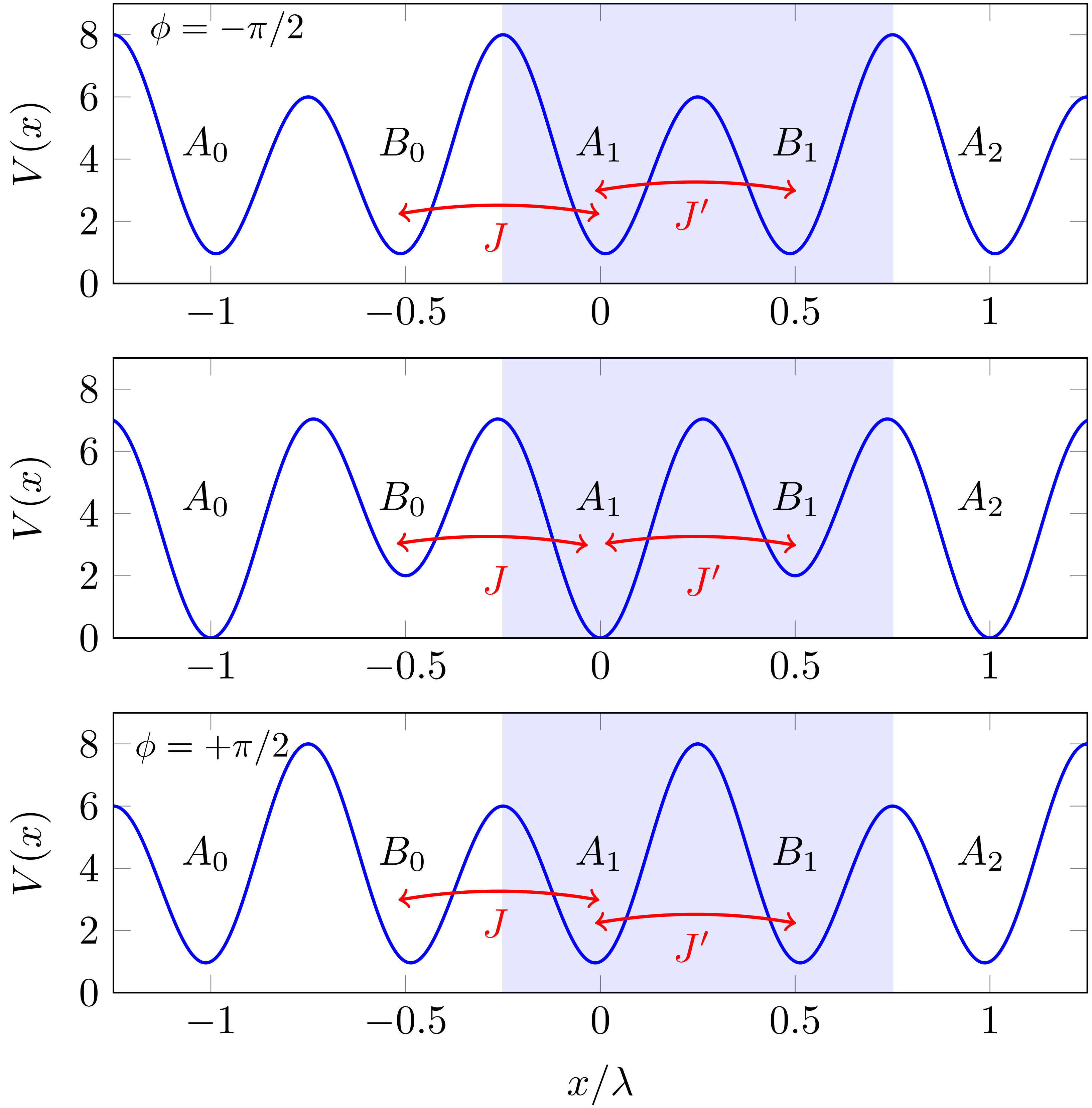
Increase the height of every second barrier

Achieves the two dimerizations of the SSH model



Potential of the superlattice

$$V(x) = V_{\text{princ.}} \sin^2(kx) + V_{\text{sec.}} \sin^2[(kx+\phi)/2]$$



$\phi = 0$

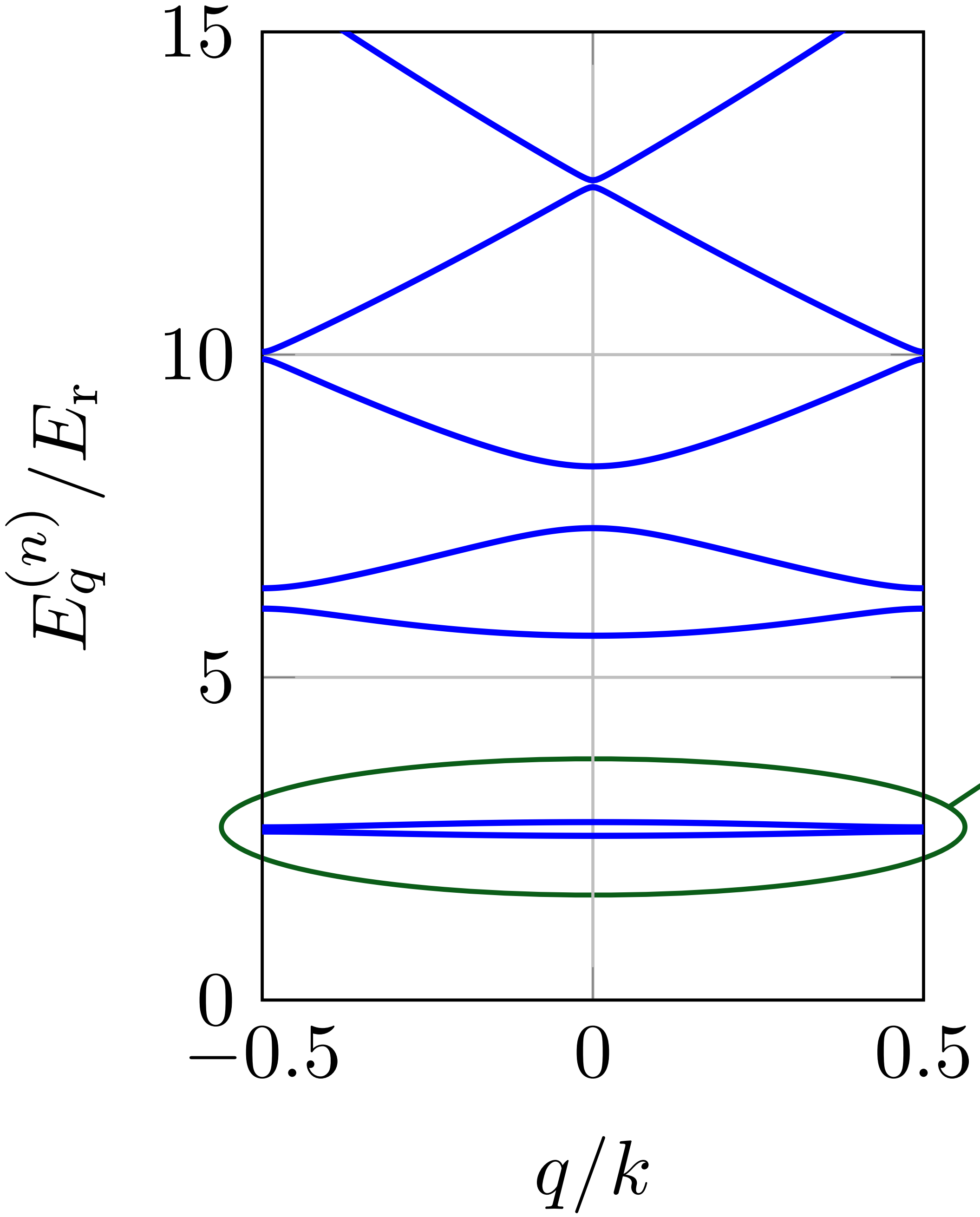
Raises every second minimum, with the same barrier height everywhere

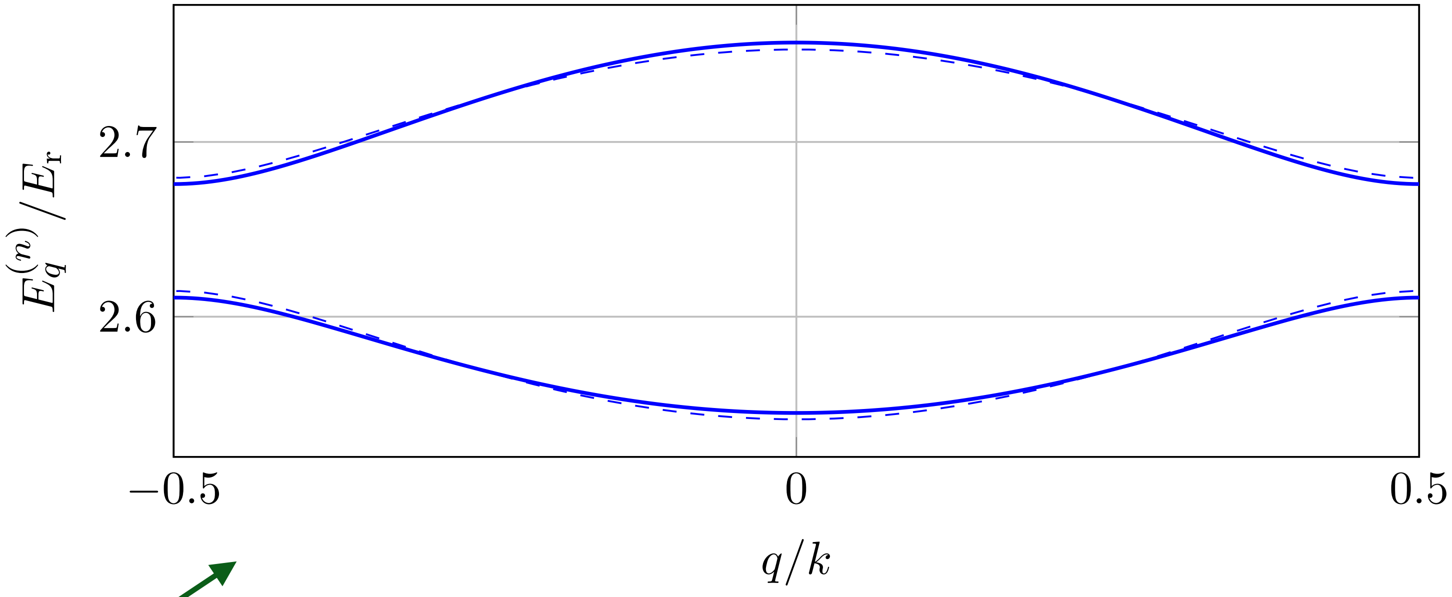
$J = J',\ \Delta > 0$

Energy bands of the superlattice

$$V(x) = V_{\text{princ.}} \sin^2(kx) + V_{\text{sec.}} \sin^2[(kx+\phi)/2]$$

$V_{\text{princ.}} = 6\, E_{\text{r}}$ $\qquad V_{\text{sec.}} = E_{\text{r}}$ $\qquad \phi = \pi/2$





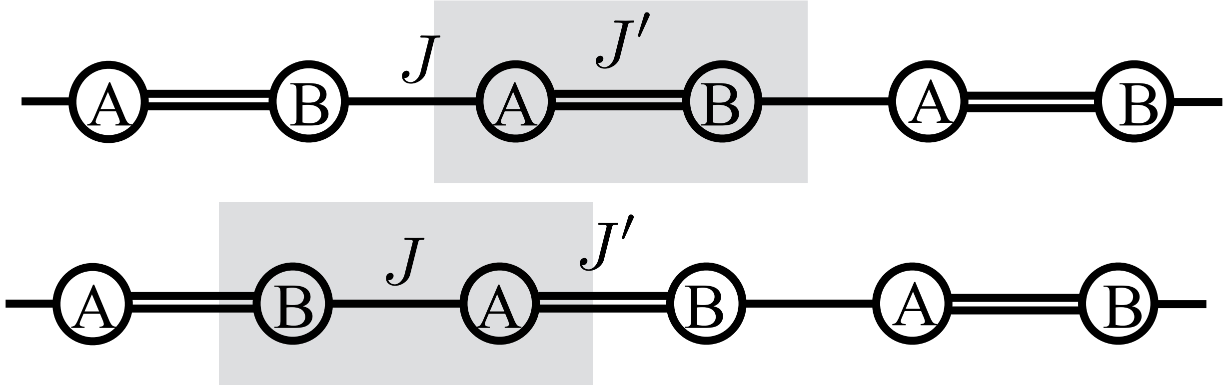
Good agreement with the SSH model (dashed)

$J = 0.07\, E_{\text{r}}$ $\qquad J' = 0.04\, E_{\text{r}}$

Mesure of the Zak phase in a superlattice

M. Atala et al., Nat. Phys. **9**, 795 (2013)

Reminder: for an infinite SSH chain, the Zak phase is not truly a topological invariant, and depends on the choice of parametrization A-B



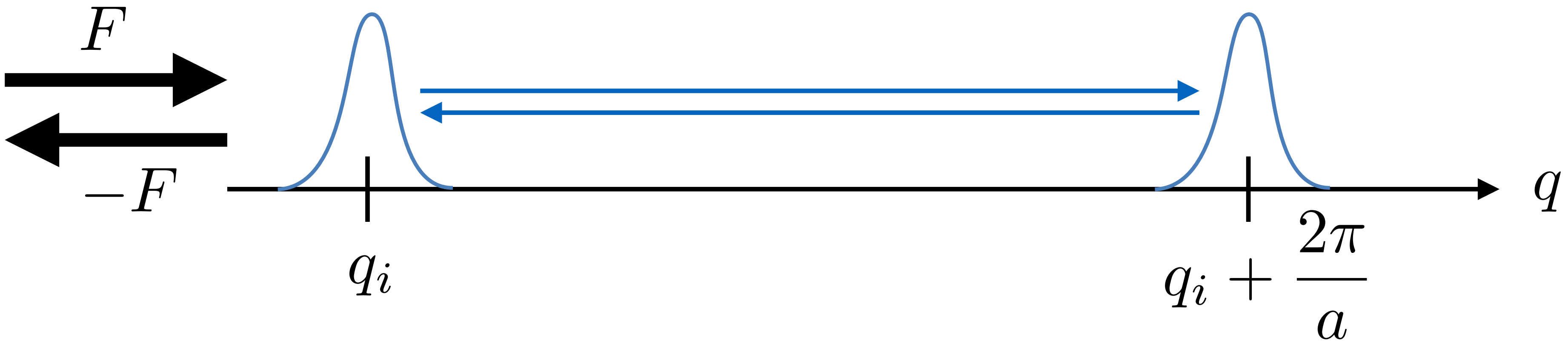
Once this choice is made, one gets:

$$\Delta\Phi \equiv \Phi_{\rm Zak}^{[J'>J]} - \Phi_{\rm Zak}^{[J'<J]} \;=\; \pm\pi$$

Procedure followed by the Munich group:

Mesure the phase difference $\Delta\Phi$ using an interferometric method by switching the values of *J* and *J’* during the experimental sequence

The Munich experiment (simplified)



- Prepare a particle with a wavepacket centered on the quasimomentum ***q**$_i$* in a superlattice SSH with $\phi = -\pi/2, \quad J' > J$

- Apply a uniform force *F* which accelerates the particle (Bloch oscillations)

$$\hbar \frac{\mathrm{d}q}{\mathrm{d}t} = F \quad \longrightarrow \quad q(t) = q_i + Ft/\hbar$$

- When the Bloch momentum has travelled across the Brillouin zone, change the dimerization :

$$\phi = +\pi/2, \quad J' < J$$

  and flip the sign of the force *F*

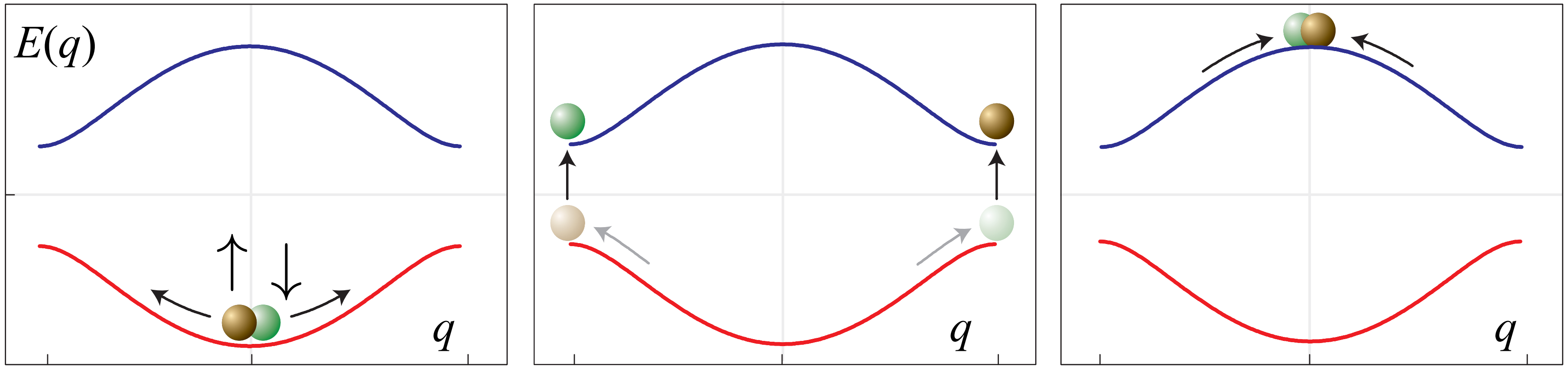
- Measure the accumulated phase when the momentum is back to the initial value ***q**$_i$*

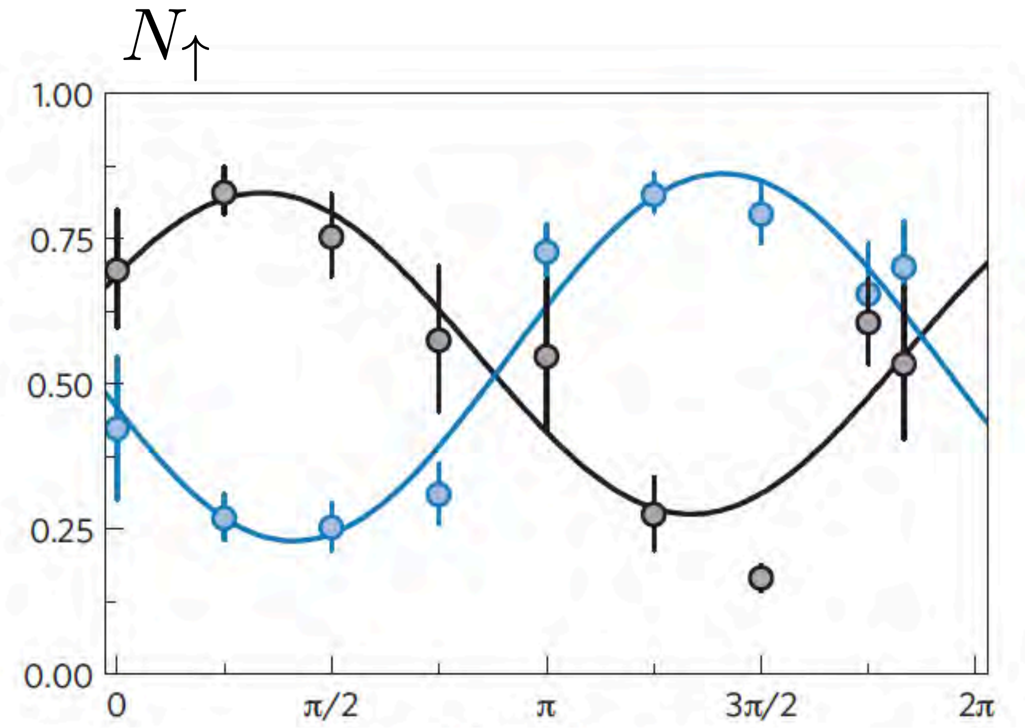
The Munich experiment (more realistic)

The force which induces the Bloch oscillations originates form a magnetic field gradient: one uses a spin echo technique to compensate for the fluctuations of this field

Interferometric measurement of the phase by a series of microwave pulses

Sequence $\frac{\pi}{2} - \pi - \frac{\pi}{2}$





Phase of the final microwave pulse

Blue: with a change of the SSH dimerization

$$\phi = -\pi/2 \quad \longrightarrow \quad \phi = +\pi/2$$

in the middle of the sequence

Black: without change

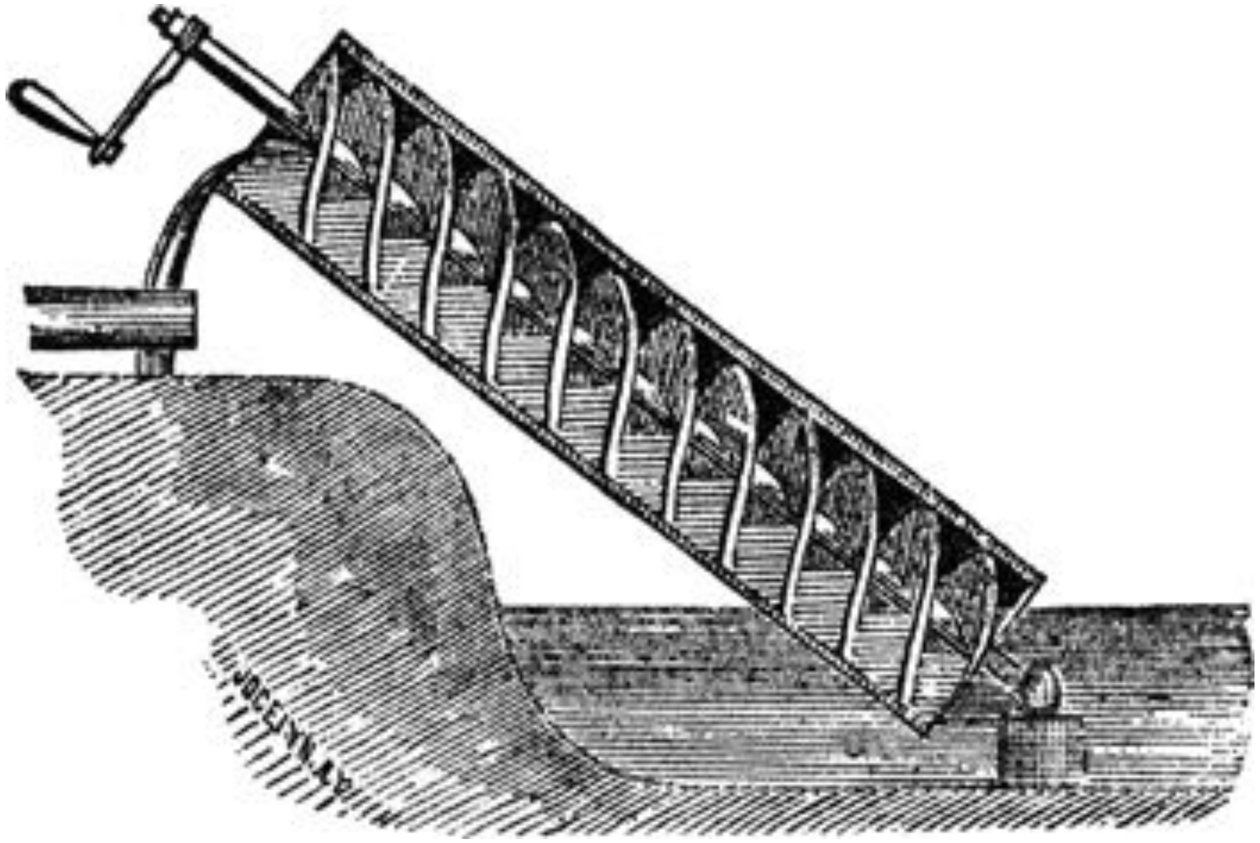
$$\Delta\Phi = 0.97\,(2)\,\pi$$

Mesurement then extended to the Rice-Mele case

3.

Adiabatic pump for the Rice-Mele model

Principle of an adiabatic pump



- Cyclic change of the parameters that control the state of a fluid
- After a cycle, the fluid does not come back to its initial state, but a certain quantity of fluid has been transported
- The amount of matter that is transported does not depend on the cycle duration

Quantum version:

Quantization of the amount that is transported (or of its displacement)

D.J. Thouless, Phys. Rev. B 27, 6083 (1983)

Thouless, Kohmoto, Nightingale, den Nijs, Phys. Rev. Lett. 49, 405 (1982)

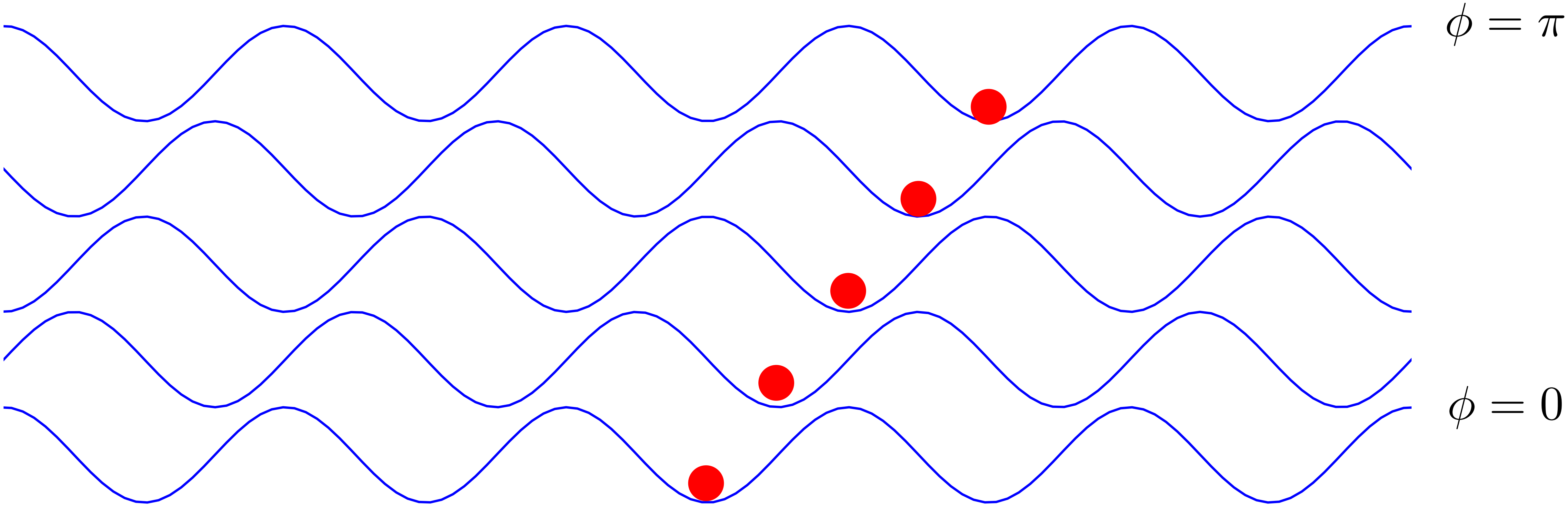
A first example (that looks too simple…)

A translated standing wave $V(x) = V_0 \sin^2(kx - \phi)$

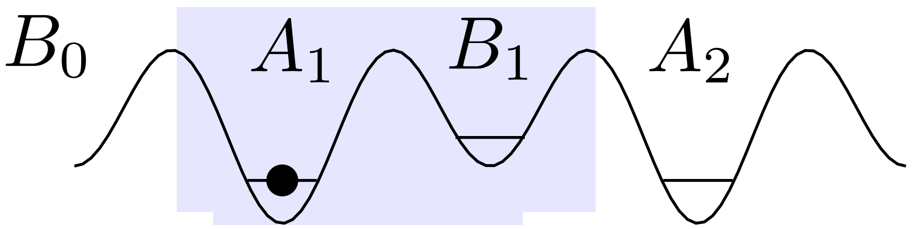
Very deep lattice: no tunnelling

At initial time, $\phi = 0$

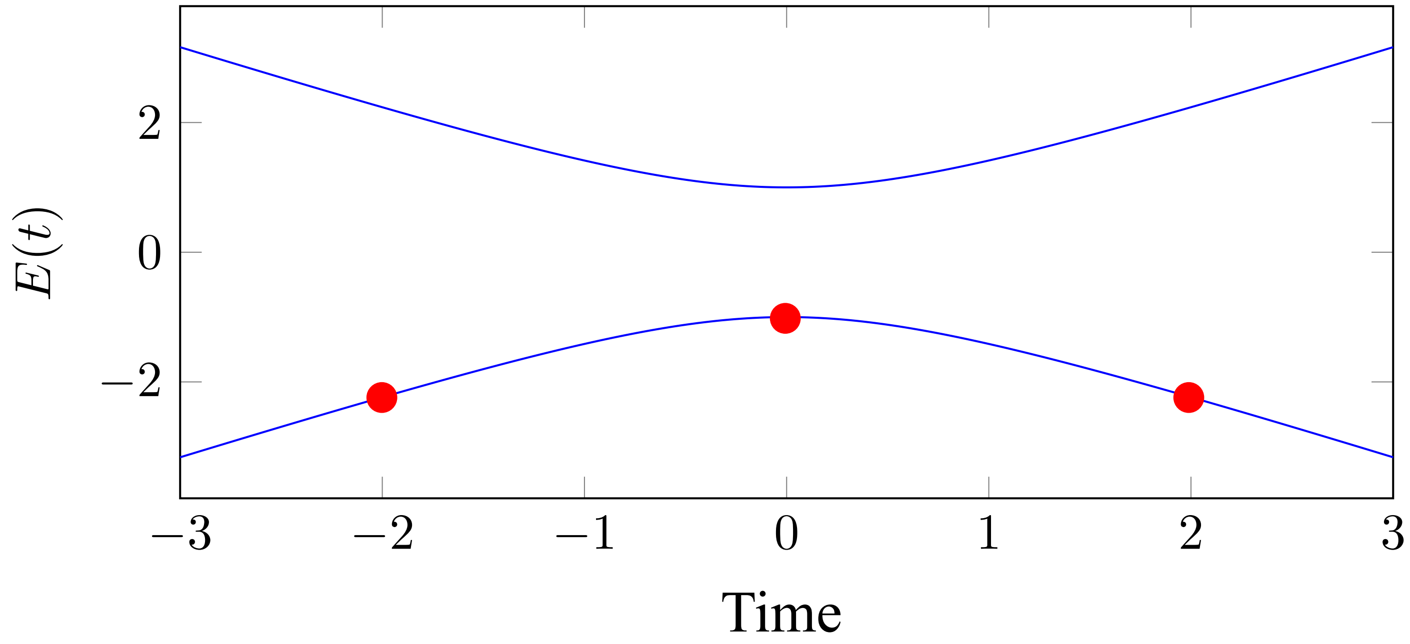
The phase ϕ increase slightly with time



A second example (less simple…)



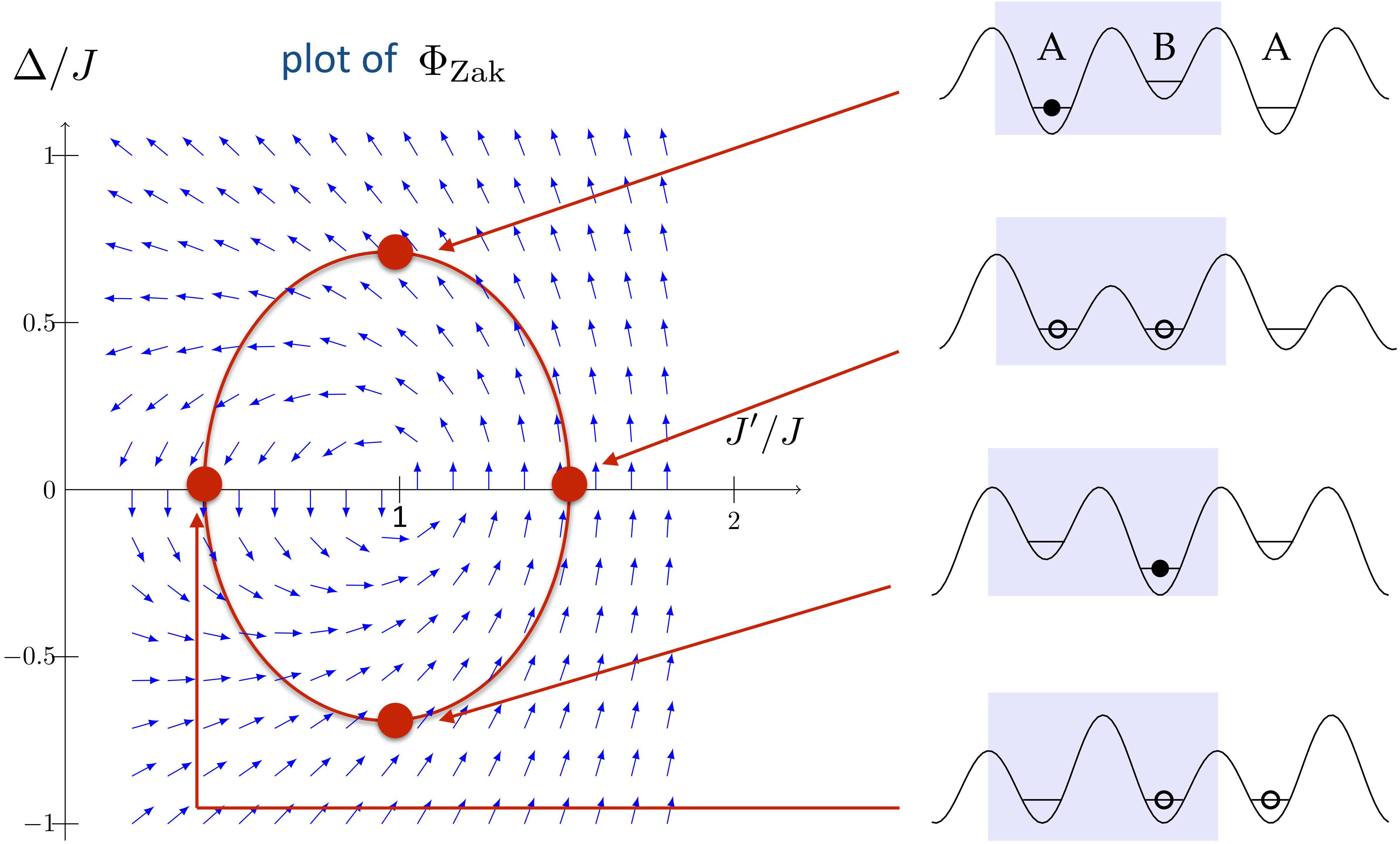
Deep superlattice: no tunnelling across the highest barriers



A particle initially in A_1 ends up in A_2

A particle initially in B_1 ends up in … B_0

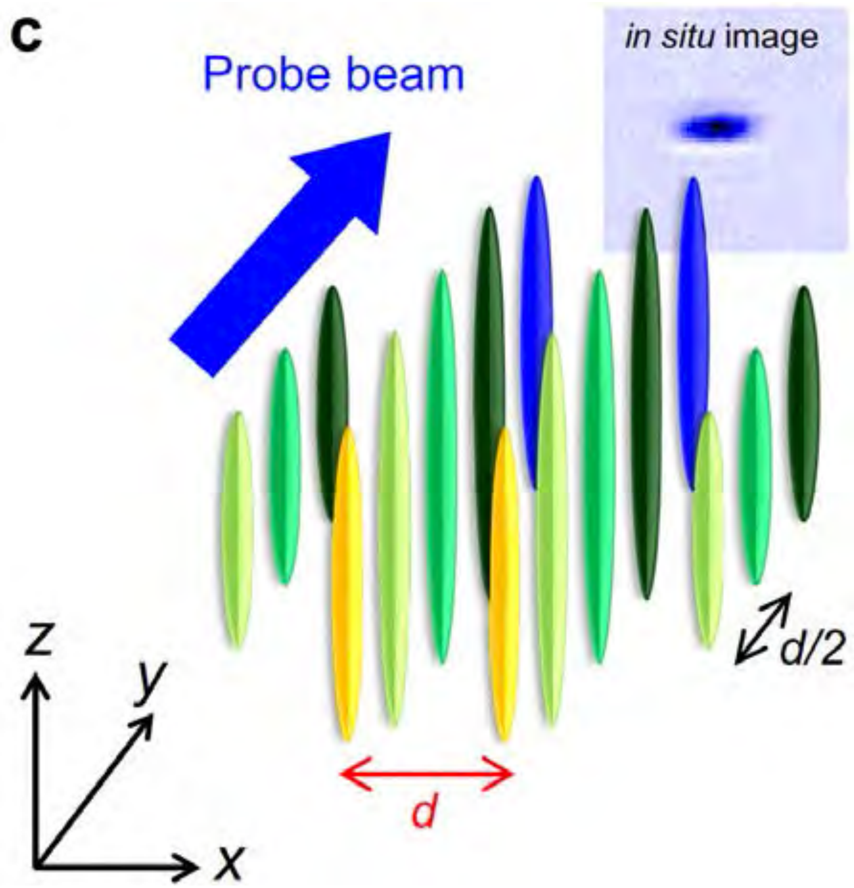
General scheme for an adiabatic pump



The Kyoto experiment

With cold atom experiments, it becomes possible to implement directly Thouless's proposal (1983) : Kyoto, Munich, Maryland

S. Nakajima et al., Nature Phys. **12**, 96 (2016)



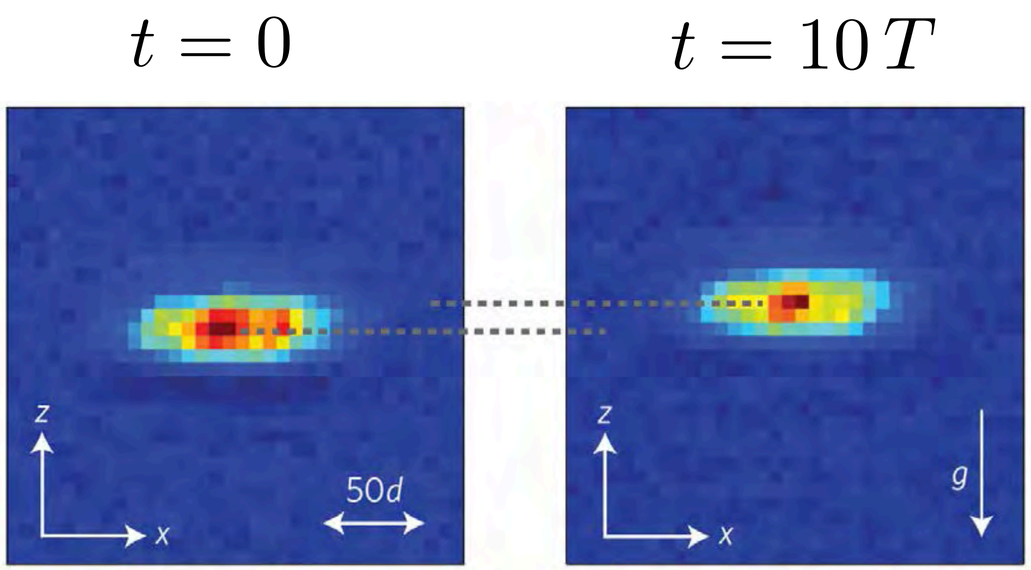
10 000 ^{171}Yb atoms (fermions)

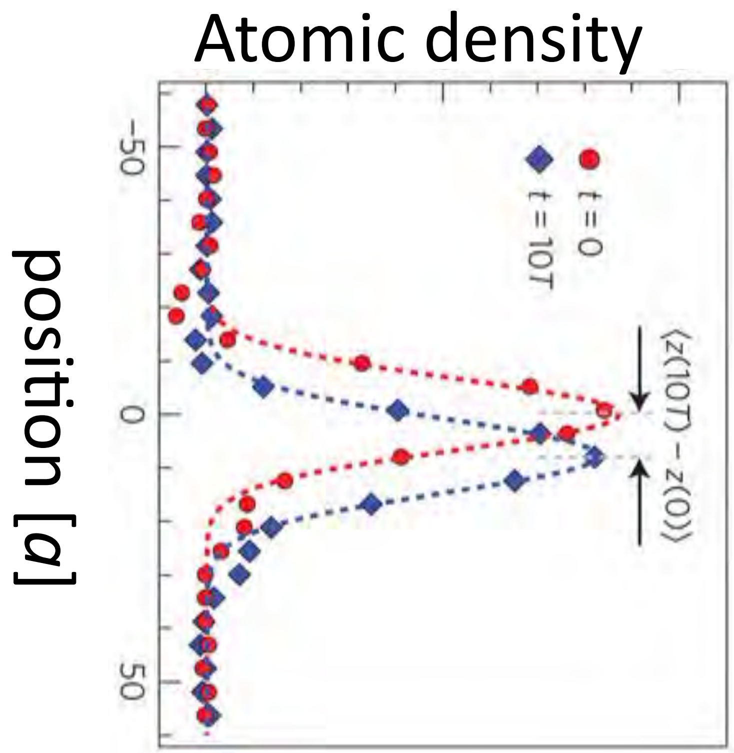
Array of independent vertical tubes

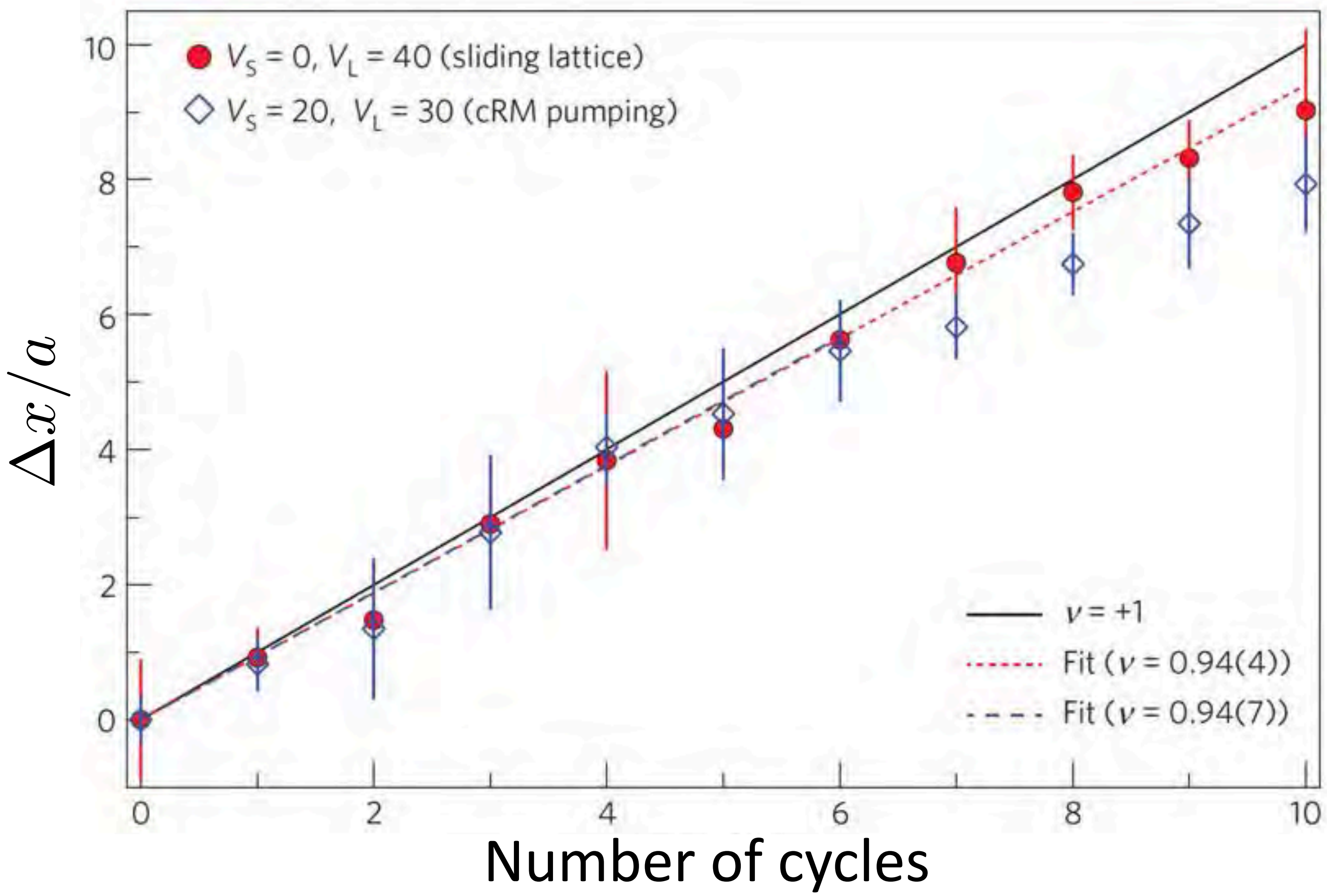
An optical superlattice is placed along each tube (periods 266 et 532 nm)

Filling factor: 0.7 atom/cell

Displacement after a few pump cycles of the superlattice



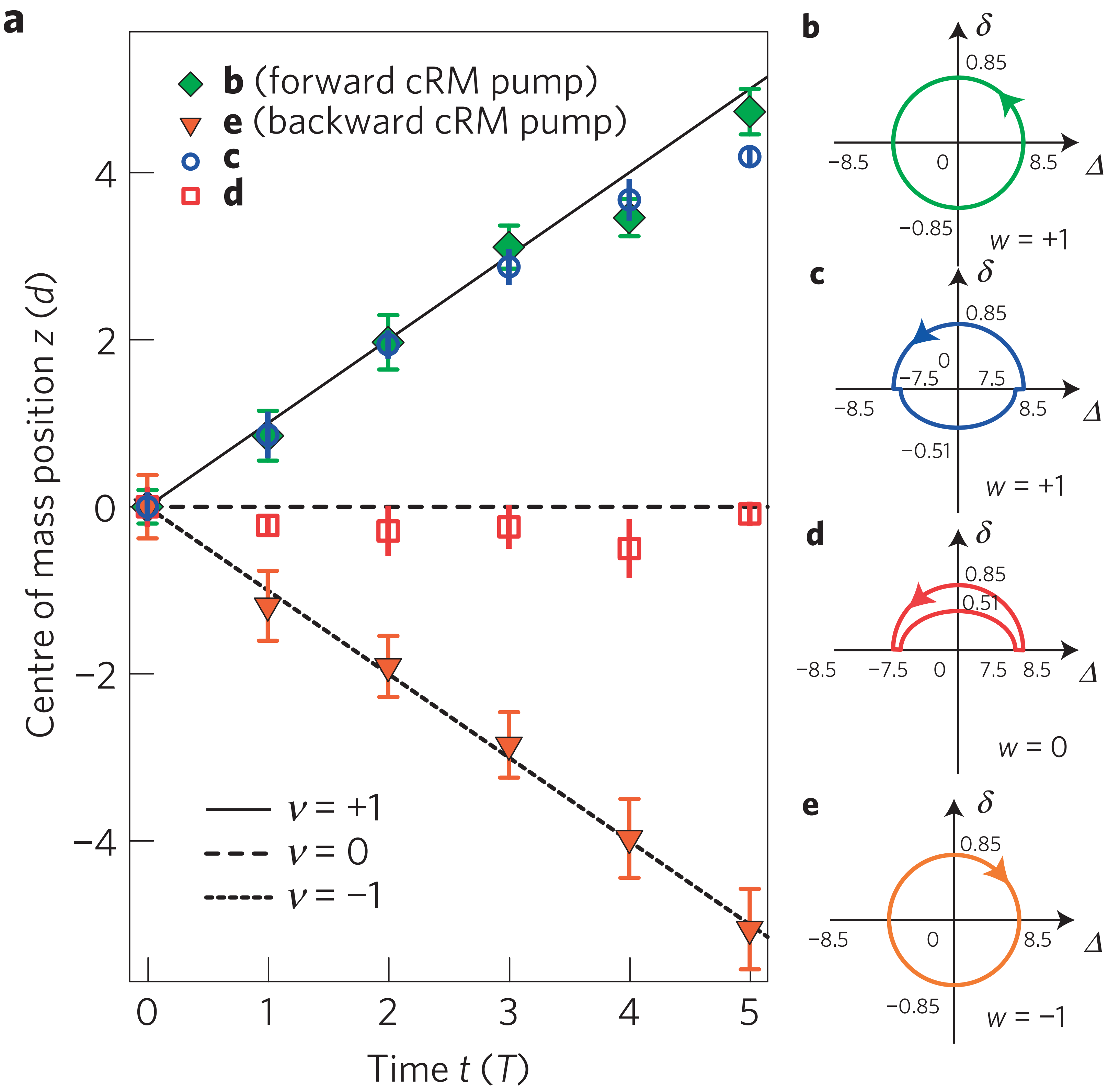




Red: mere translation of the lattice

Blue: loop in the plane (*J'*,Δ)

Topological robustness of the adiabatic pump



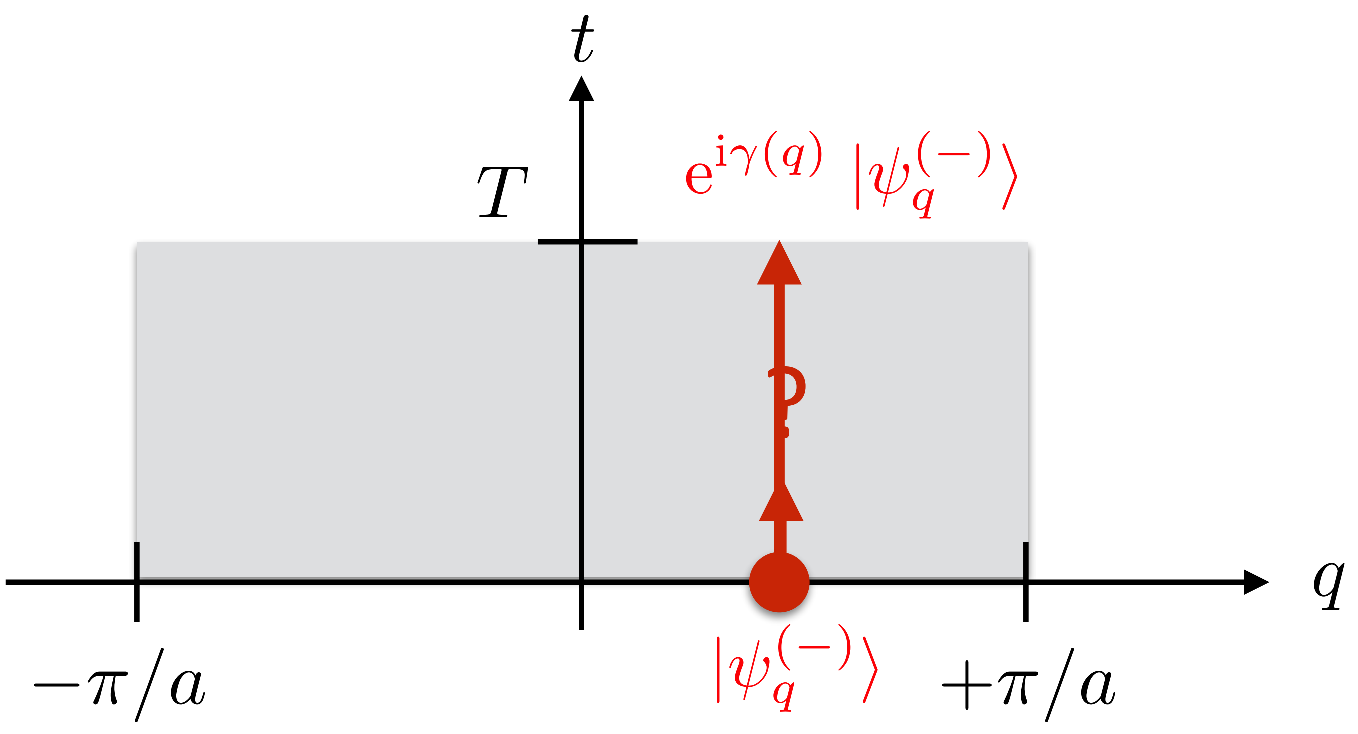
$$2\delta = J' - J$$

S. Nakajima et al.
Nature Phys. **12**, 96 (2016)

4.

Adiabatic pump and Berry phase

Cycling Hamiltonian and Bloch theorem



Start at *t* = 0 from a Bloch state in the lowest band.

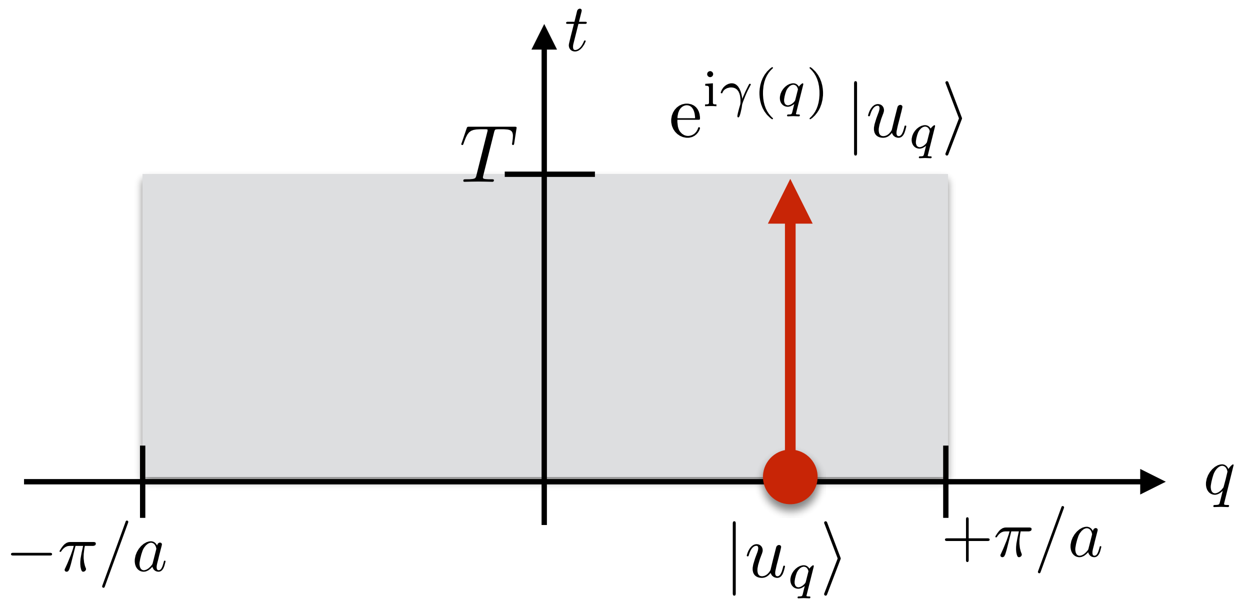
What is the state of the particle after a pump cycle of duration *T*?

- At each time t, the Hamiltonian remains spatially periodic. The state of the particle can thus be written $\mathrm{e}^{\mathrm{i}qx}\ u(x)$

- If the parameters (J, J', Δ) vary slowly in time and if there is no degeneracy (no gap closure), adiabatic following of the state of the lowest band:

$$|\psi_q^{(-)}\rangle \quad \longrightarrow \quad \mathrm{e}^{\mathrm{i}\gamma(q)}\ |\psi_q^{(-)}\rangle$$

Cycling Hamiltonian and geometric phase



$$\gamma(q) = \Phi_{\mathrm{dyn}}(q) + \Phi_{\mathrm{geom}}(q)$$

$$\Phi_{\mathrm{dyn}}(q) = -\frac{1}{\hbar}\int_0^T E_{\boldsymbol{q}}^{(-)}(t)\;\mathrm{d}t$$

$$\Phi_{\mathrm{geom}}(q) = \int_0^T \mathcal{A}_2(q,t)\;\mathrm{d}t$$

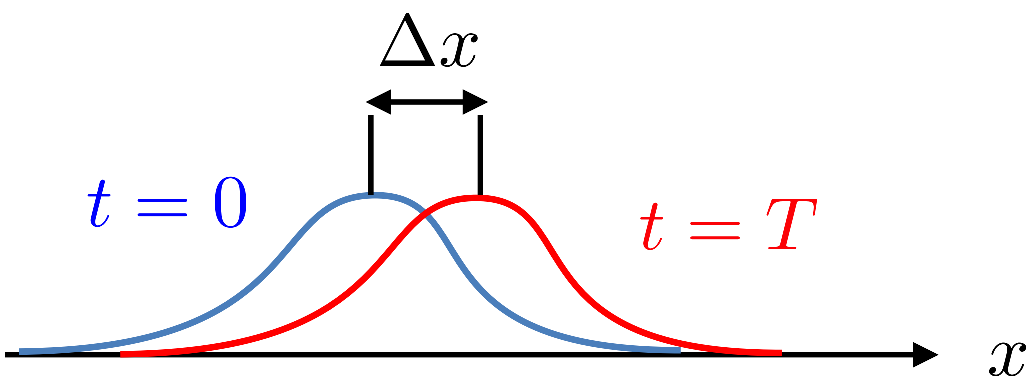
$$\mathcal{A}_2(q,t) = \mathrm{i}\langle u_{q,t}|\partial_t u_{q,t}\rangle$$

“Temporal” Berry connection

Evolution operator over one pump cycle: $\hat{U}(T) = \exp[\mathrm{i}\gamma(\hat{q})]$

where we introduced the operator “Bloch momentum” $\hat{q}$: $\hat{q}|\psi_q^{(-)}\rangle = q\;|\psi_q^{(-)}\rangle$

Displacement of the center of the wave packet



Position operator in the lattice $\hat{x}$,
conjugated with the momentum $\hat{q}$

$$[\hat{x}, \hat{q}] = \mathrm{i}$$

Heisenberg picture:

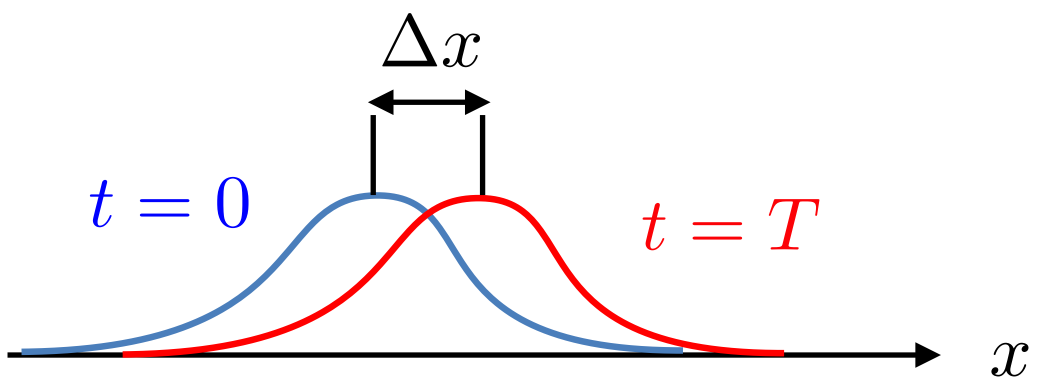
$$\hat{x}(T) = \hat{U}^\dagger(T)\,\hat{x}\,\hat{U}(T) = \hat{x} - \partial_q \gamma(\hat{q}) \qquad \hat{U}(T) = \exp[\mathrm{i}\gamma(\hat{q})]$$

Displacement during a pump cycle, after average over the initial distribution $\Pi(q)$ of the Bloch momentum q:

$$\Delta x = -\int_{-\pi/a}^{+\pi/a} \partial_q \gamma(q)\,\Pi(q)\,\mathrm{d}q$$

i.e., for a uniform initial population of the band: $\Delta x = -\dfrac{a}{2\pi}\displaystyle\int_{-\pi/a}^{+\pi/a} \partial_q \gamma(q)\,\mathrm{d}q$

Displacement of the center of mass and geometrical phase



$$\Delta x = -\frac{a}{2\pi}\int_{-\pi/a}^{+\pi/a} \partial_q \gamma(q)\,\mathrm{d}q$$

$$\gamma(q) = \Phi_{\mathrm{dyn}}(q) + \Phi_{\mathrm{geom}}(q)$$

- The contribution of the dynamical phase vanishes, because of the periodicity of the energy E_q as a function of q over the Brillouin zone

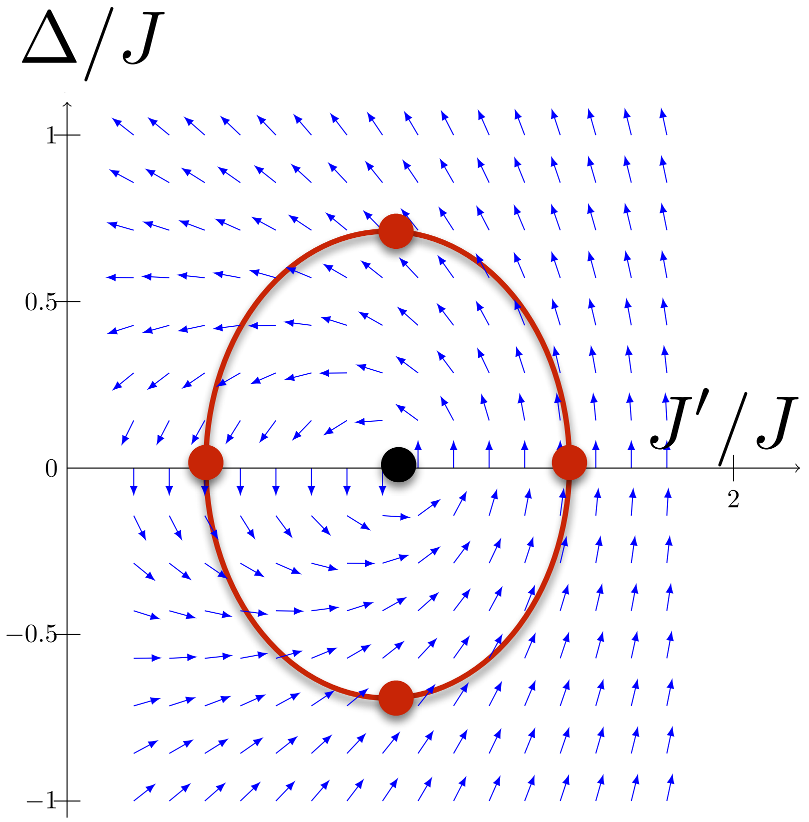
- Contribution of the geometrical phase:

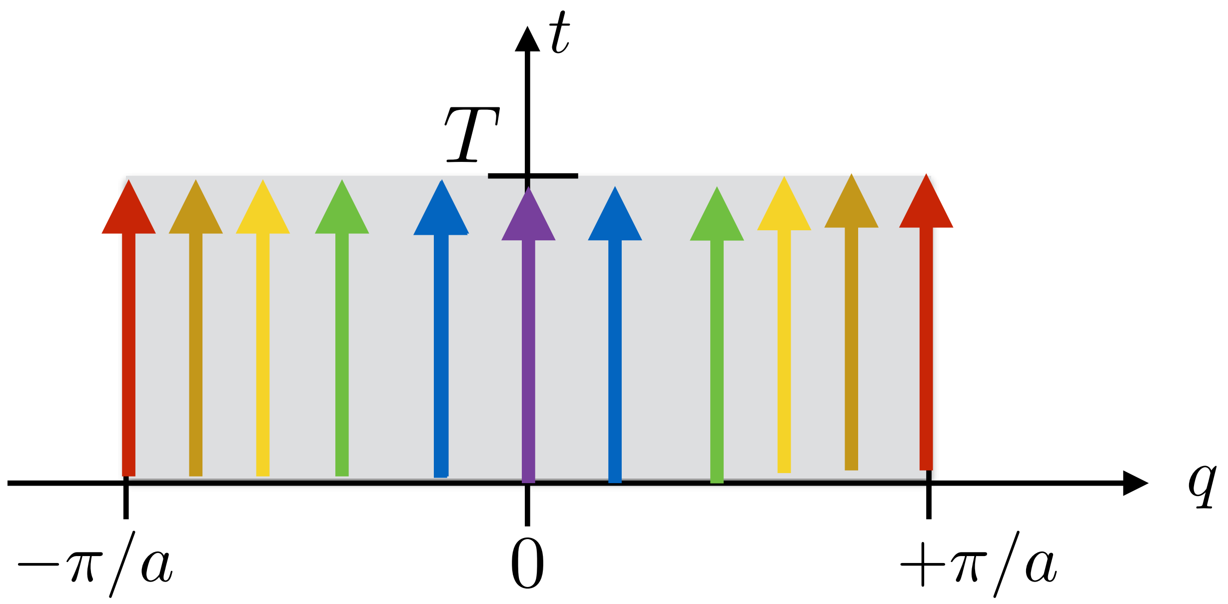
$$\Delta x = -\frac{a}{2\pi}\int_{-\pi/a}^{+\pi/a} \frac{\mathrm{d}\Phi_{\mathrm{geom}}}{\mathrm{d}q}\,\mathrm{d}q \qquad \underset{\uparrow}{=} -\frac{a}{2\pi}\left[\Phi_{\mathrm{geom}}(+\pi/a) - \Phi_{\mathrm{geom}}(-\pi/a)\right]$$

Need to be cautious because of possible mathematical singularities

Here we shall perform a geometrical evaluation of $\Phi_{\mathrm{geom}}(+\pi/a) - \Phi_{\mathrm{geom}}(-\pi/a)$

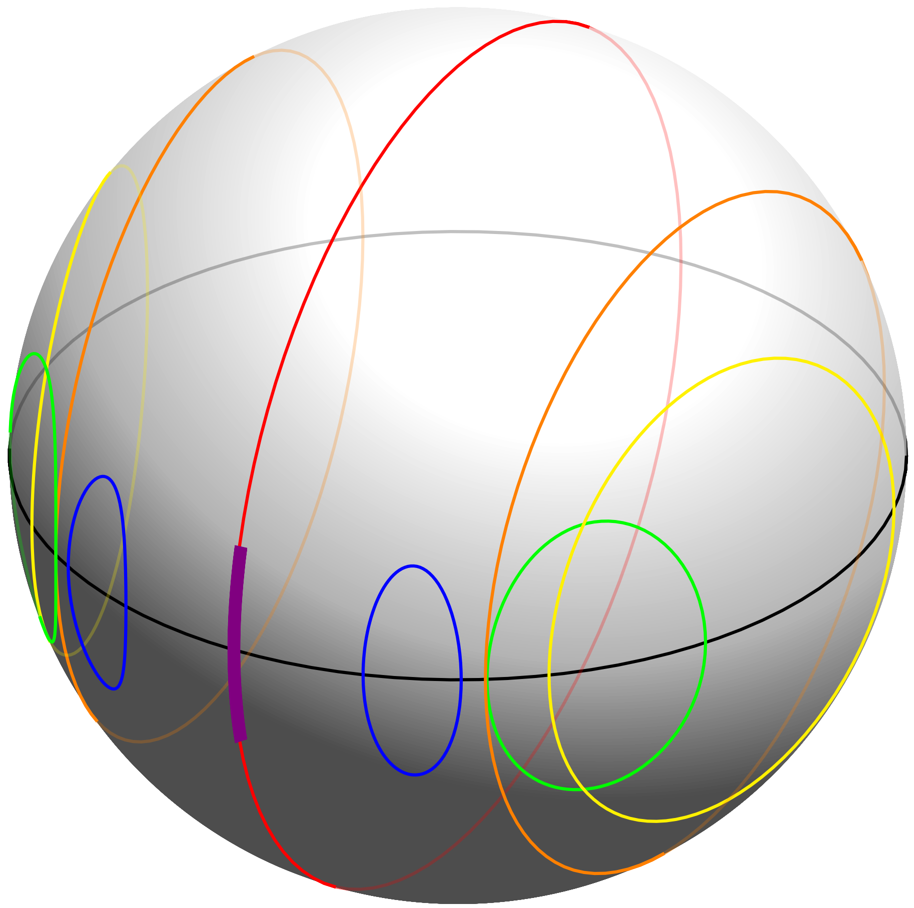
Geometrical phase and Bloch sphere





$$\cos\theta_q = \frac{\Delta}{|\boldsymbol{h}(q)|}$$

$$\mathrm{e}^{\mathrm{i}\phi_q}\ \sin\theta_q = \frac{J' + J\,\mathrm{e}^{\mathrm{i}qa}}{|\boldsymbol{h}(q)|}$$



$qa = 0:\ \ \mathrm{e}^{\mathrm{i}\phi_q}\ \sin\theta_q$ real > 0, poles not reached

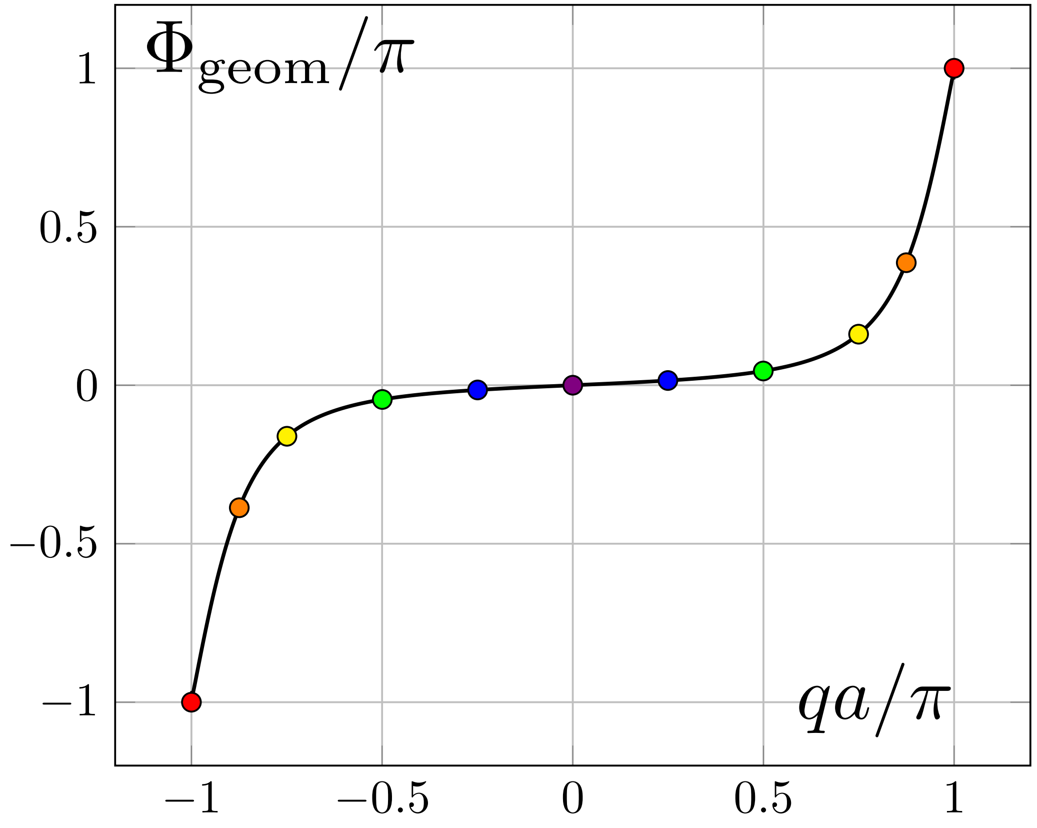
$qa = \pm\pi:\ \ \mathrm{e}^{\mathrm{i}\phi_q}\ \sin\theta_q$ real with a change of sign, poles are reached

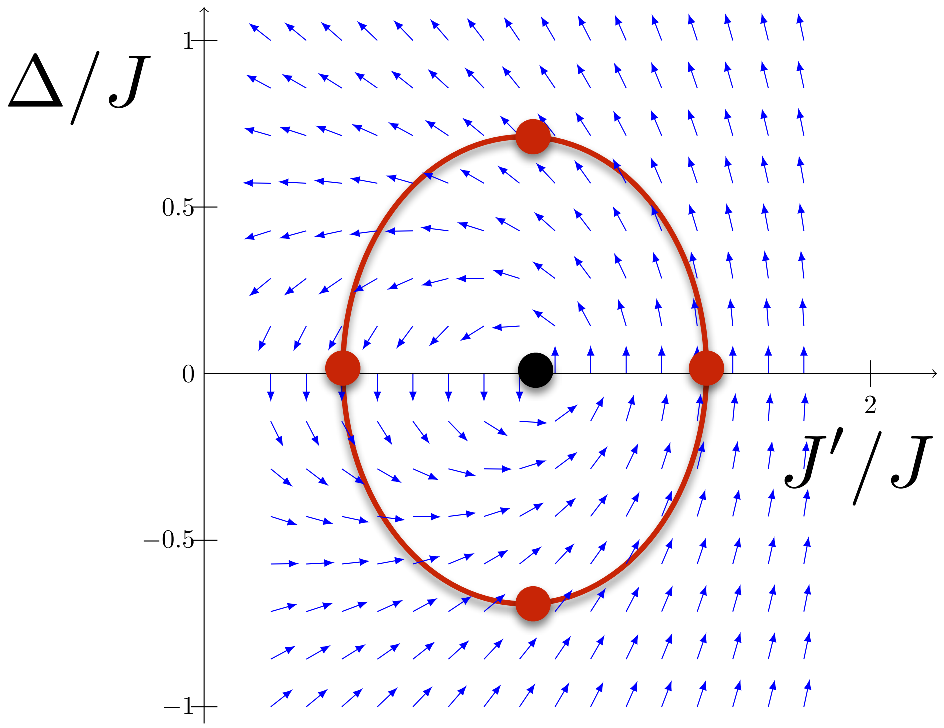
Summary regarding the winding over the Bloch sphere

- By continuity we gave a non-ambiguous meaning to:

$$
\begin{aligned}
\Delta x &= -\frac{a}{2\pi}\left[\Phi_{\text{geom}}(+\pi/a) - \Phi_{\text{geom}}(-\pi/a)\right] \\
&= -\frac{a}{2\pi}\left[(+\pi) - (-\pi)\right] = -a
\end{aligned}
$$

Quantized displacement!





- All points of the Bloch sphere are reached for at least one couple (q,t)

The Bloch sphere is wrapped in a way that cannot be unwrapped

Link with Berry curvature

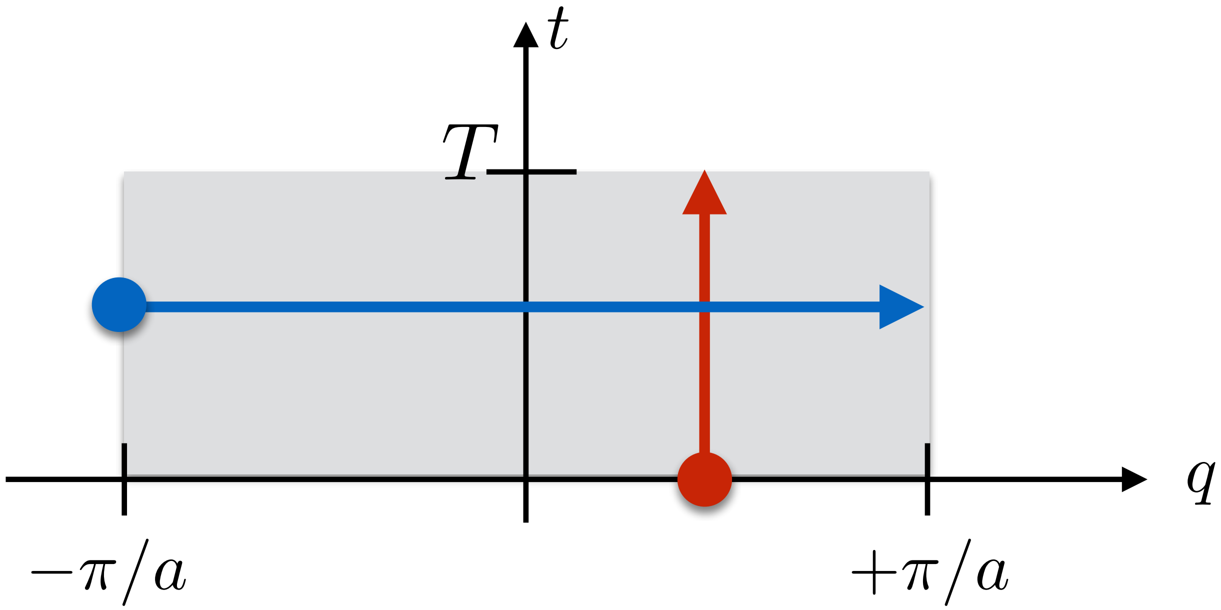
We have introduced two Berry connections

$$\mathcal{A}_1(q,t) = \mathrm{i}\langle u_{q,t}|\partial_q u_{q,t}\rangle$$

Calculation of Zak phase

$$\mathcal{A}_2(q,t) = \mathrm{i}\langle u_{q,t}|\partial_t u_{q,t}\rangle$$

Calculation of the geometrical phase over a pump cycle at fixed ***q***



Berry curvature for this effective two-dimensional problem:

$$\Omega(q,t) = \begin{pmatrix}\partial_q \\ \partial_t\end{pmatrix} \times \begin{pmatrix}\mathcal{A}_1 \\ \mathcal{A}_2\end{pmatrix} \quad = \mathrm{i}\left(\langle \partial_q u_{q,t}|\partial_t u_{q,t}\rangle - \langle \partial_t u_{q,t}|\partial_q u_{q,t}\rangle\right)$$

real

Integration by parts

$$\Delta x = -\frac{a}{2\pi}\int_{-\pi/a}^{+\pi/a} \frac{\mathrm{d}\Phi_{\mathrm{geom}}}{\mathrm{d}q}\,\mathrm{d}q \quad \longrightarrow \quad \Delta x = -\frac{a}{2\pi}\int_{-\pi/a}^{+\pi/a}\int_0^T \Omega(q,t)\,\mathrm{d}q\,\mathrm{d}t$$

ROBUST!

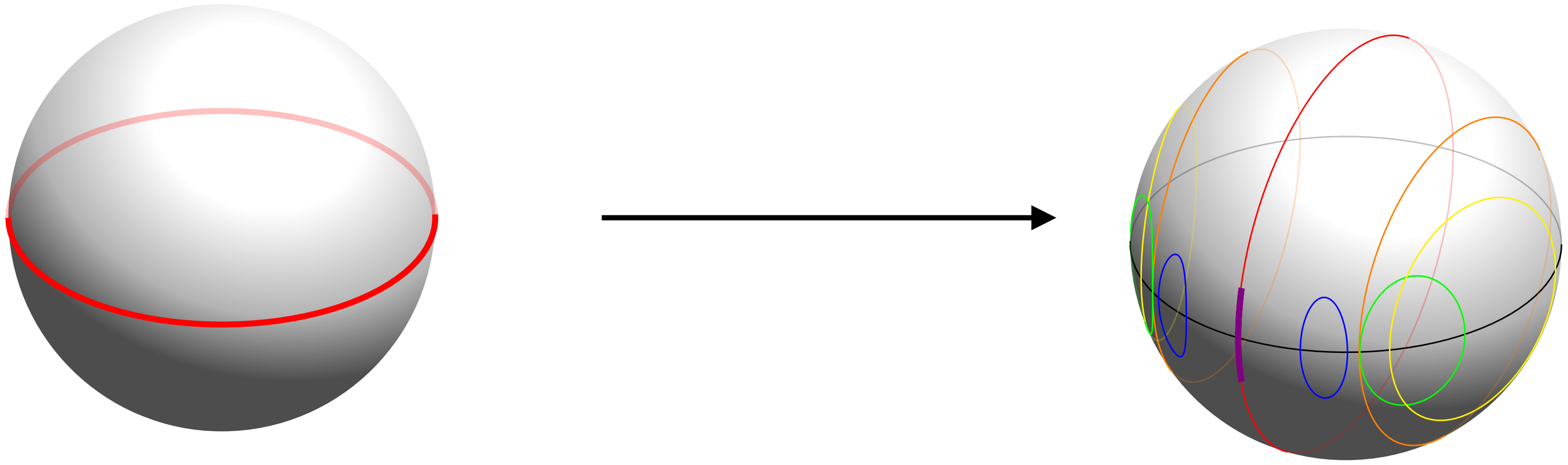
Conclusions

Adiabatic pump: first step towards two-dimensional problems:

$$q \longrightarrow q, t \longrightarrow q_x, q_y$$

Quantization of transport in a pump cycle [0,*T*]

New topological invariant: how to wrap Bloch sphere



Emergence of Berry curvature to calculate the quantized quantity:

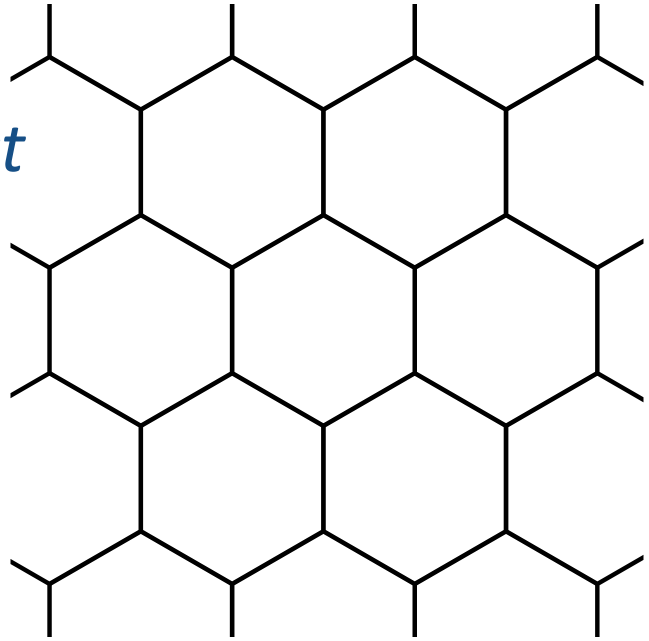
$$\frac{\Delta x}{a} = \frac{1}{2\pi}\iint \Omega(q,t)\,\mathrm{d}q\,\mathrm{d}t$$

Integral over “1D Brillouin Zone” x [0,*T*]

Topology and Berry curvature in a two-dimensional lattice

Goal for this part

Start the study of two-dimensional periodic lattices and characterize their topological properties



Problem that originates from the Quantum Hall effect

Emergence of robust quantum numbers:
Chern indices

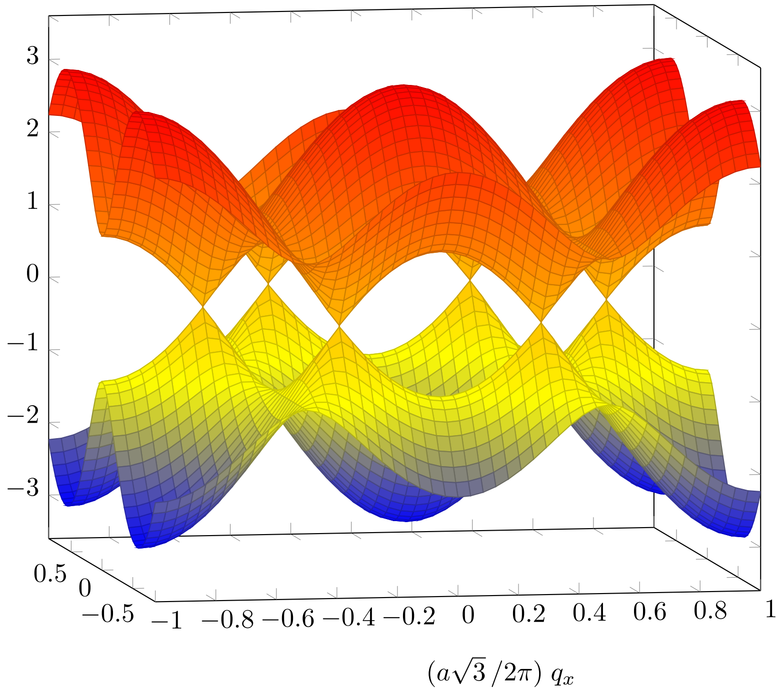
Unconventional statistics:
any-ons

These numbers appear in an equivalent manner from different points of view

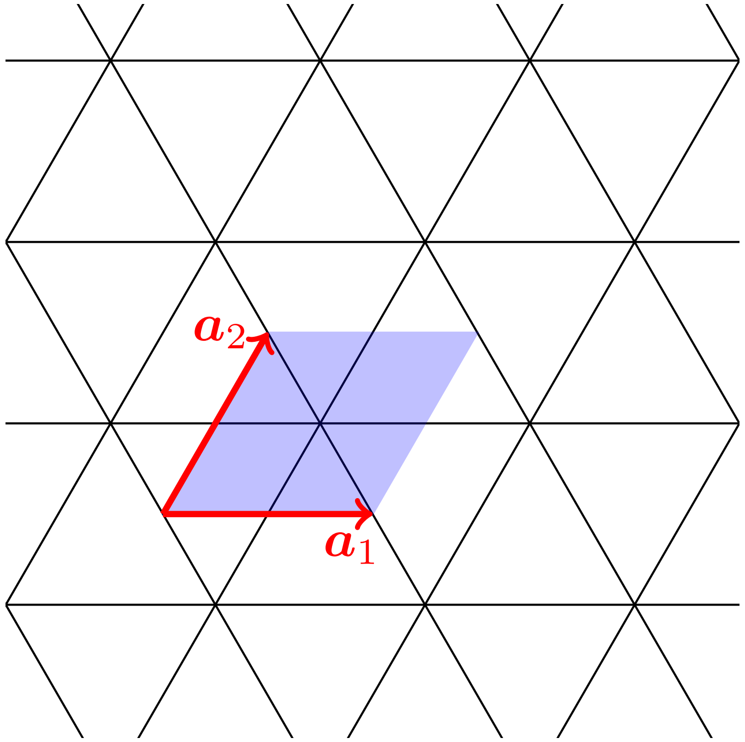
- Geometrical: wrapping the Bloch sphere
- Physical, with the study of transport and the quantization of conductivity
- Physical, with the existence of edge states

1.

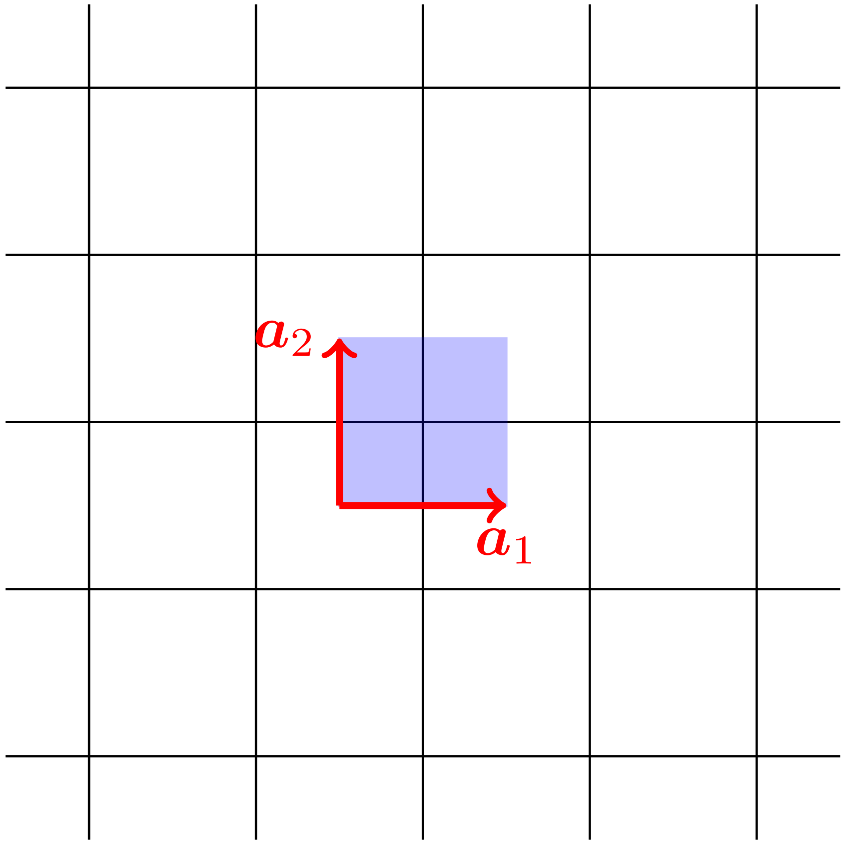
Bi-partite lattices and Dirac points



Triangular and square lattices



Bravais lattices,
one site per unit cell



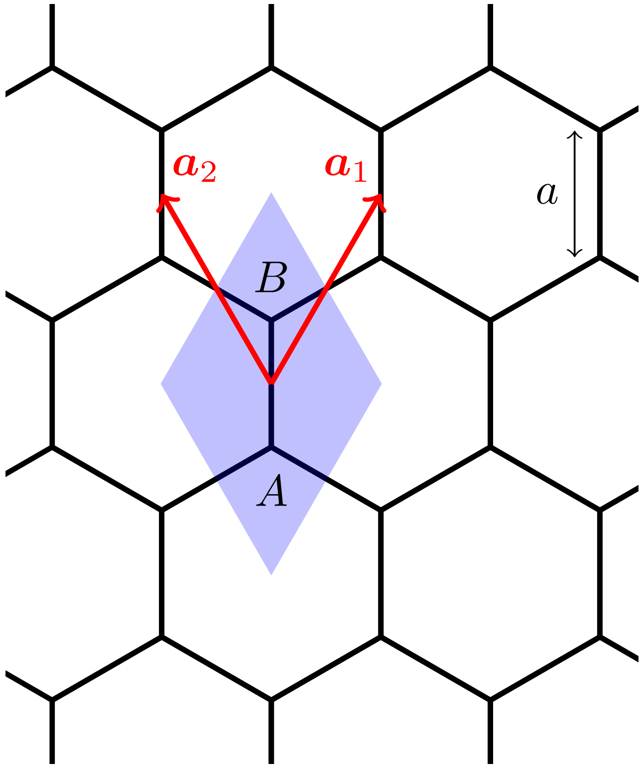
General Bloch theorem in 2D : $\psi_{\boldsymbol{q}}(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} u_{\boldsymbol{q}}(\boldsymbol{r})$ with $u_{\boldsymbol{q}}(\boldsymbol{r})$ periodic

In the tight-binding limit, only one periodic function

$$|u_{\boldsymbol{q}}\rangle = \sum_{\boldsymbol{j}} |A_{\boldsymbol{j}}\rangle$$

Real and with no ***q*** variation: no topological properties expected

The hexagonal (graphene) lattice



Two sites A and B per unit cell

Lattice generated by the translation of $\boldsymbol{a}_{1,2} = \frac{\sqrt{3}}{2}a\begin{pmatrix}\pm 1\\ \sqrt{3}\end{pmatrix}$

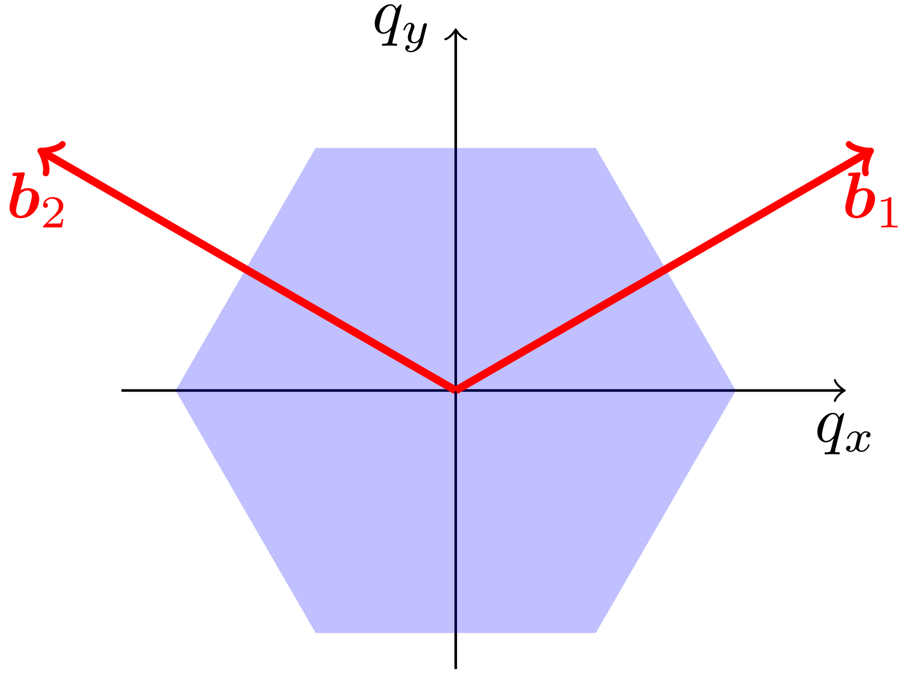
In the tight-binding regime, the functions that are periodic over the lattice read:

$$|u_{\boldsymbol{q}}\rangle = \alpha_{\boldsymbol{q}}\left(\sum_{\boldsymbol{j}}|A_{\boldsymbol{j}}\rangle\right) + \beta_{\boldsymbol{q}}\left(\sum_{\boldsymbol{j}}|B_{\boldsymbol{j}}\rangle\right)$$

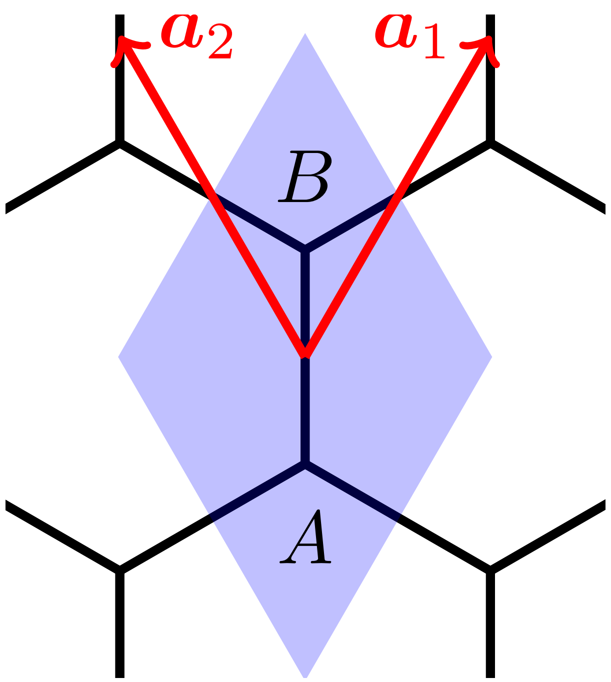
spin 1/2

Brillouin zone

$$\boldsymbol{b}_{1,2} = \frac{2\pi}{3a}\begin{pmatrix}\pm\sqrt{3}\\ 1\end{pmatrix}$$



The periodic Hamiltonian for graphene



Same energy for A and B : $E_A = E_B = 0$

Nearest coupling only:

The A site is coupled to three B sites

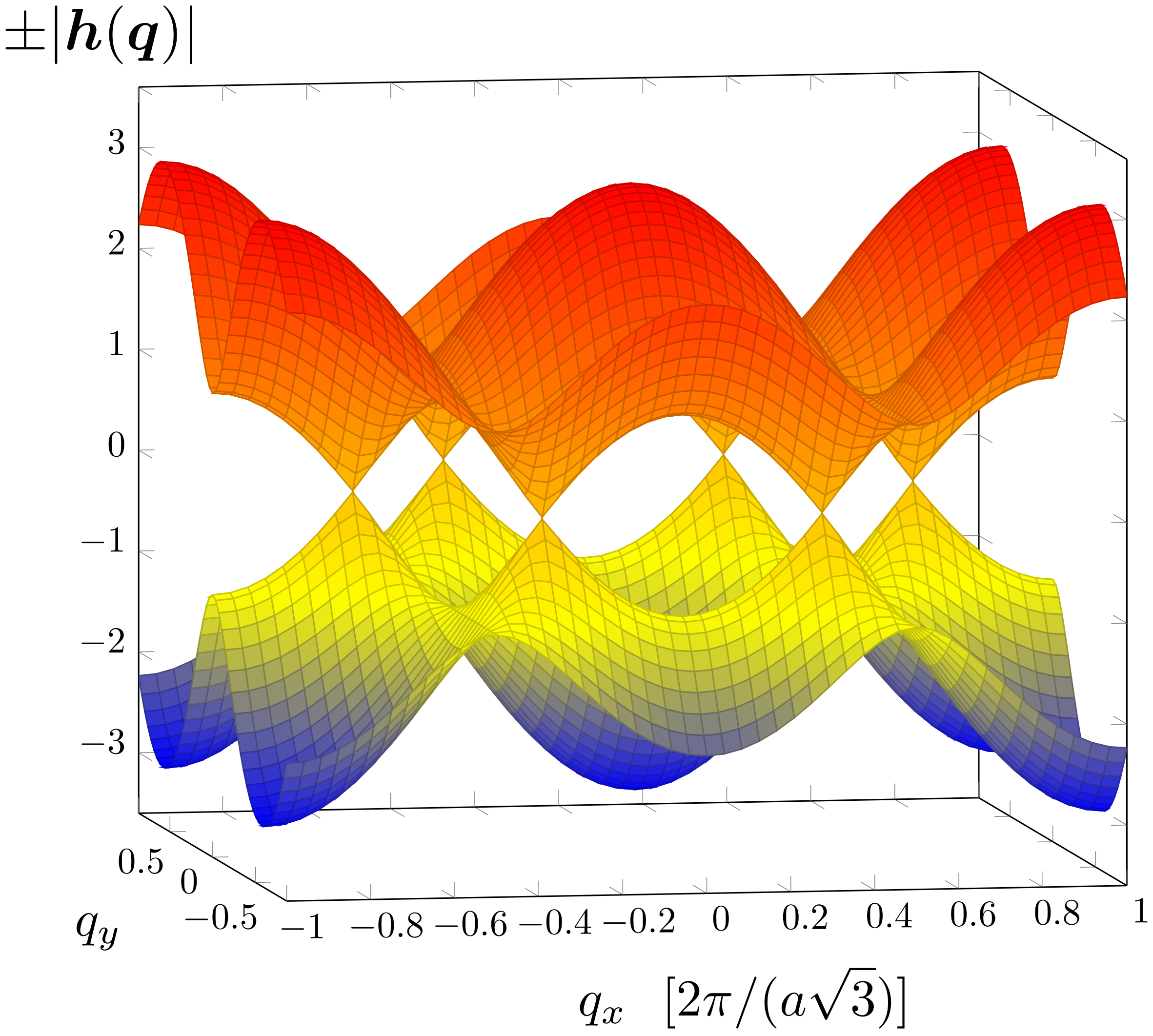
The B site is coupled to three A sites

Hamiltonian for the periodic part $|u_{\boldsymbol{q}}\rangle$

$$\hat{H}_{\boldsymbol{q}} = -J \begin{pmatrix} 0 & 1 + \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_1} + \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_2} \\ 1 + \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_1} + \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_2} & 0 \end{pmatrix} = -\boldsymbol{h}(\boldsymbol{q}) \cdot \hat{\boldsymbol{\sigma}}$$

with: $\boldsymbol{h}(\boldsymbol{q}) = \begin{pmatrix} 1 + \cos(\boldsymbol{q}\cdot\boldsymbol{a}_1) + \cos(\boldsymbol{q}\cdot\boldsymbol{a}_2) \\ \sin(\boldsymbol{q}\cdot\boldsymbol{a}_1) + \sin(\boldsymbol{q}\cdot\boldsymbol{a}_2) \\ 0 \end{pmatrix}$ Energies : $\pm|\boldsymbol{h}(\boldsymbol{q})|$

The Dirac points



Energies : $\pm|\boldsymbol{h}(\boldsymbol{q})|$

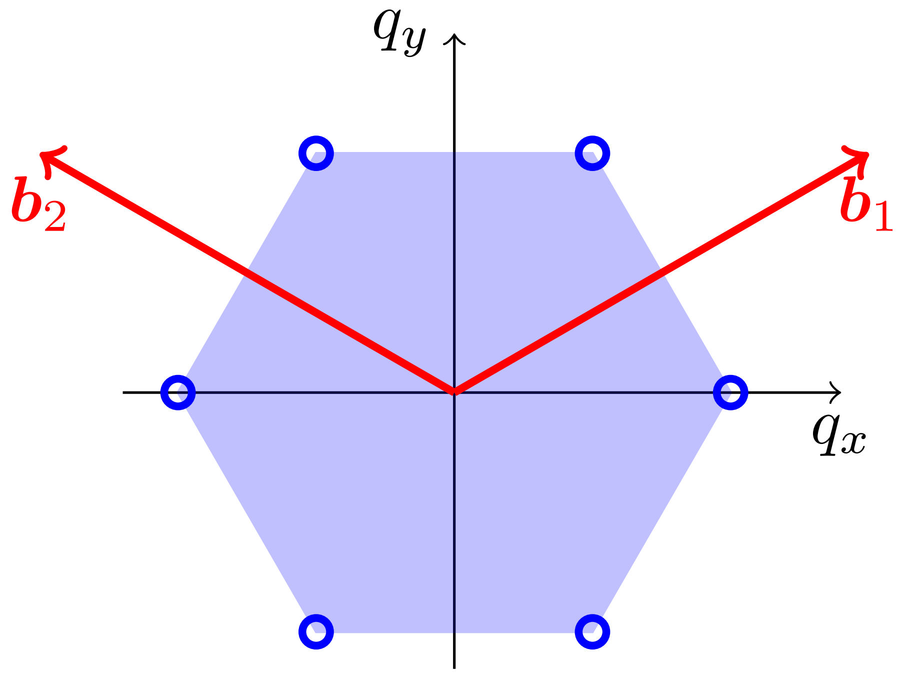
$$\boldsymbol{h}(\boldsymbol{q}) = \begin{pmatrix} 1 + \cos(\boldsymbol{q}\cdot\boldsymbol{a}_1) + \cos(\boldsymbol{q}\cdot\boldsymbol{a}_1) \\ \sin(\boldsymbol{q}\cdot\boldsymbol{a}_1) + \sin(\boldsymbol{q}\cdot\boldsymbol{a}_1) \\ 0 \end{pmatrix}$$

Contact between the bands where

$$|\boldsymbol{h}(\boldsymbol{q})| = 0$$

i.e. :

$$h_x(q_x, q_y) = 0$$
$$h_y(q_x, q_y) = 0$$



Contact between bands: marginal situation with respect to topology

The Dirac points (continued)

Linear dispersion relation $E_{\boldsymbol{q}}$ near these points: relativistic physics

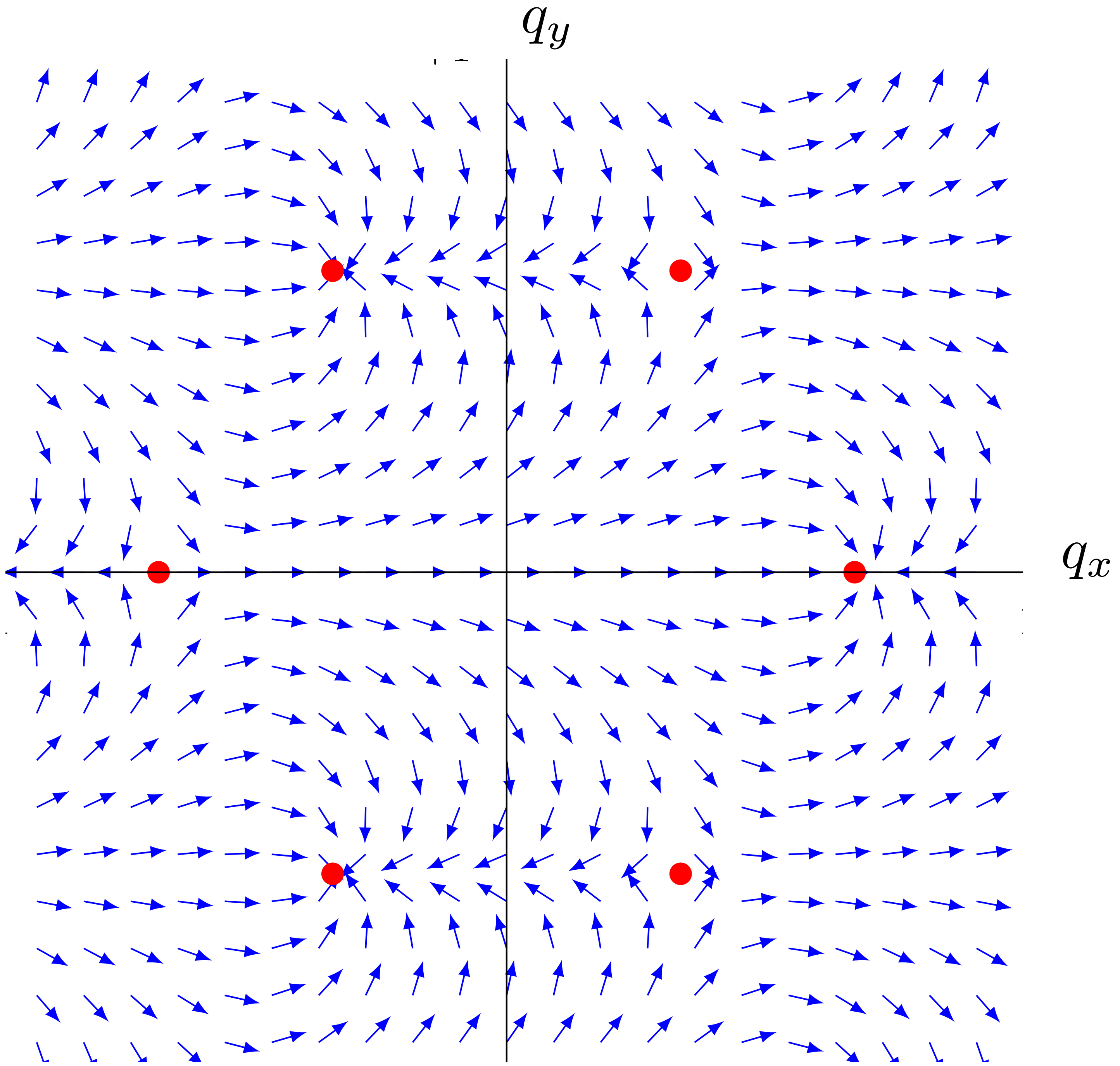
Winding of the vector $\boldsymbol{h}(\boldsymbol{q})$ around these points

$$\boldsymbol{h}(\boldsymbol{q}) = \begin{pmatrix} 1 + \cos(\boldsymbol{q}\cdot\boldsymbol{a}_1) + \cos(\boldsymbol{q}\cdot\boldsymbol{a}_1) \\ \sin(\boldsymbol{q}\cdot\boldsymbol{a}_1) + \sin(\boldsymbol{q}\cdot\boldsymbol{a}_1) \\ 0 \end{pmatrix}$$

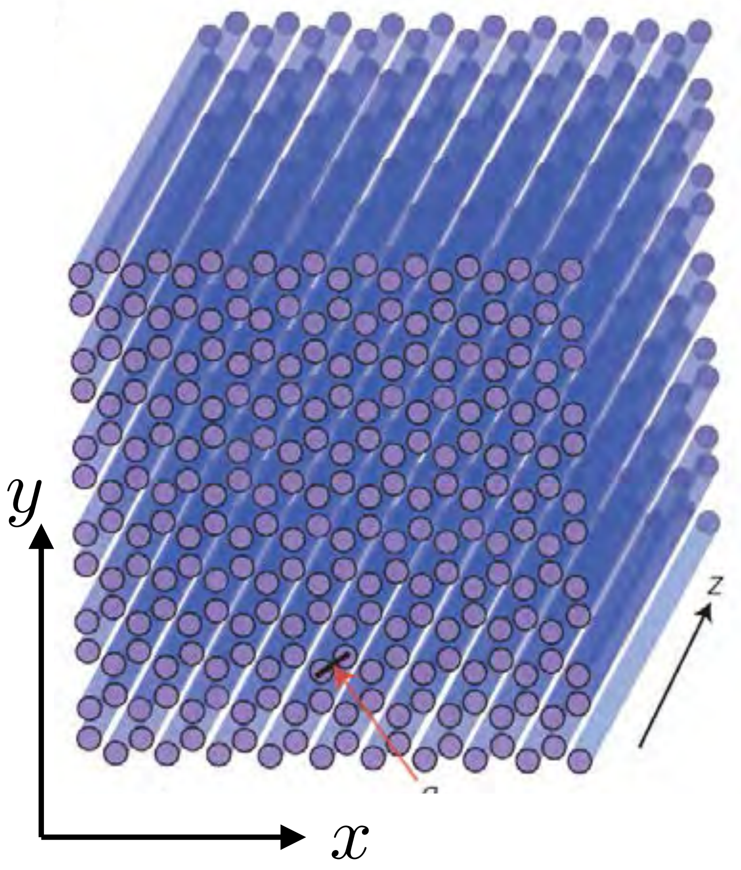
Plot of the vector field

$$\boldsymbol{n} = \frac{\boldsymbol{h}}{|\boldsymbol{h}|}$$

in the plane (q_x, q_y)

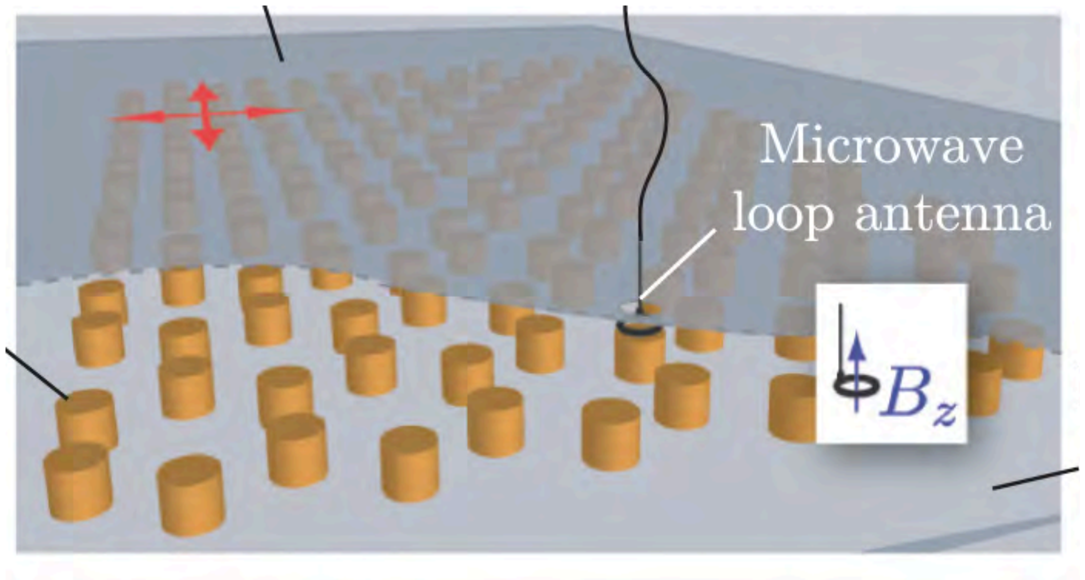


Hexagonal lattices outside condensed matter physics

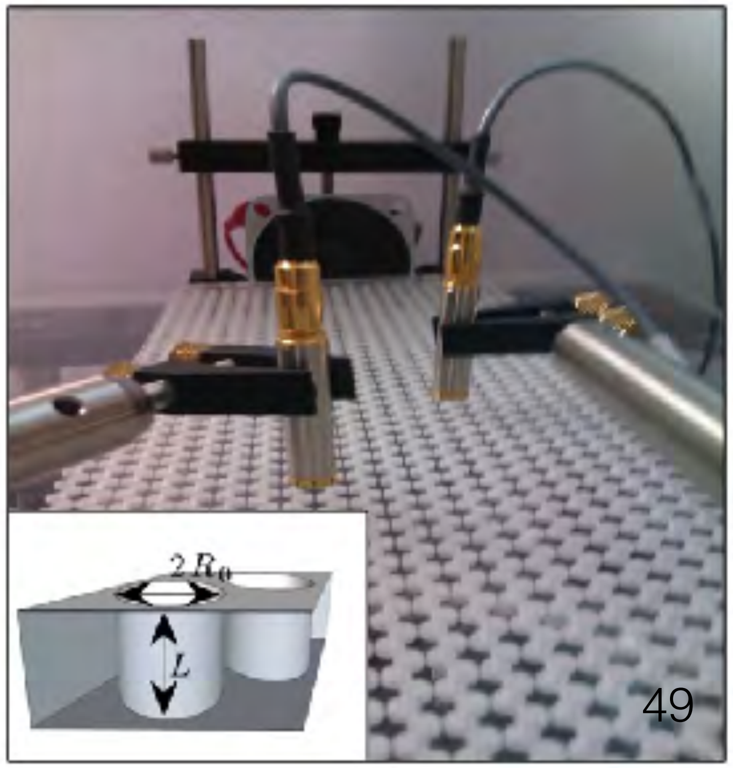


Rechtsman, Zeuner et al.,2013,
Lattice of optical waveguides

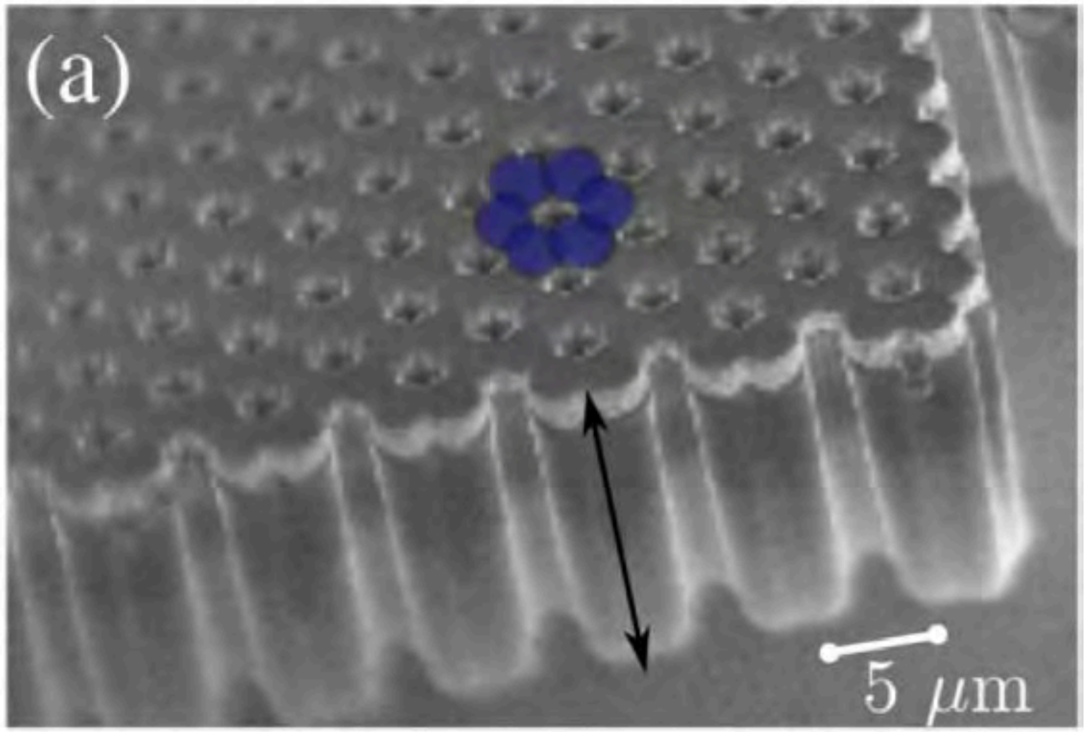
Bellec, Kuhl et al., 2013,
Microwave resonators



Torrent & Sanchez-Dehesa, 2012,
Acoustic domain

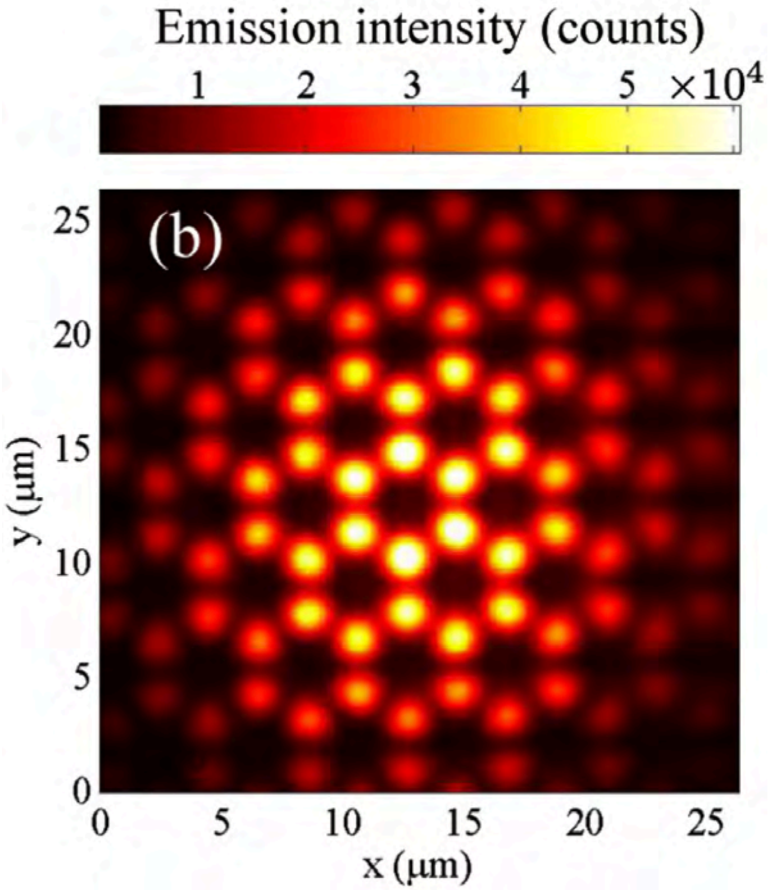


Graphene lattice with polaritons

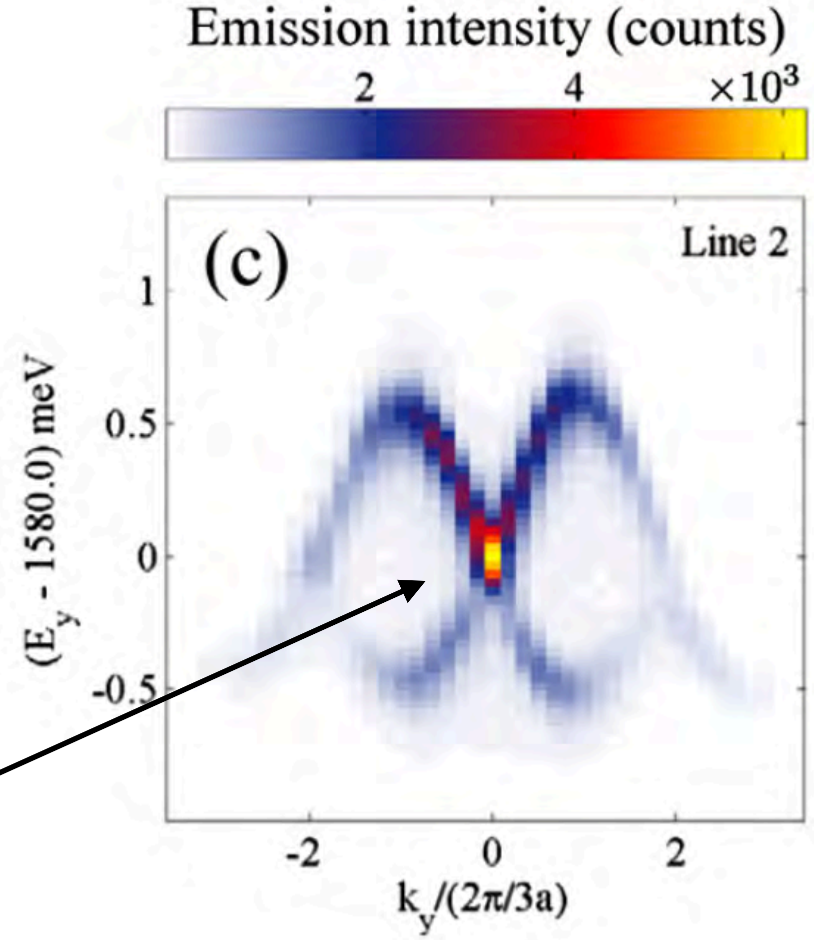


Jacqmin, Carusotto et al.,
Phys. Rev. Lett. 112, 116402 (2014)

Microstructure of AlGaAs quantum wells, pumped with non-resonant light



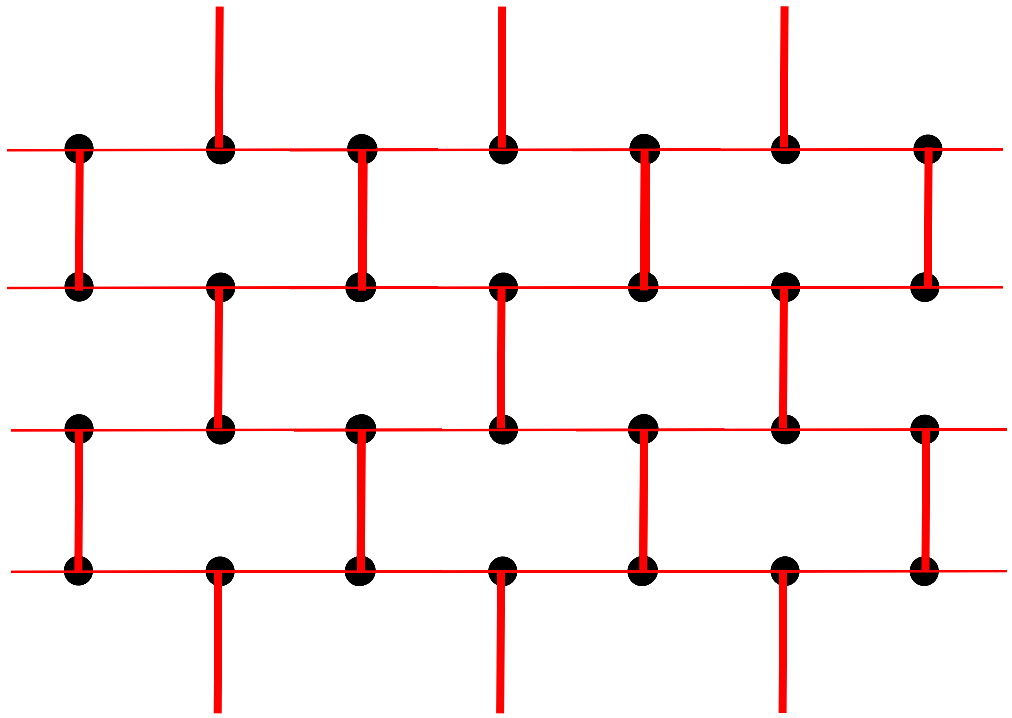
Imaging in position space or in momentum space



Dirac cones!

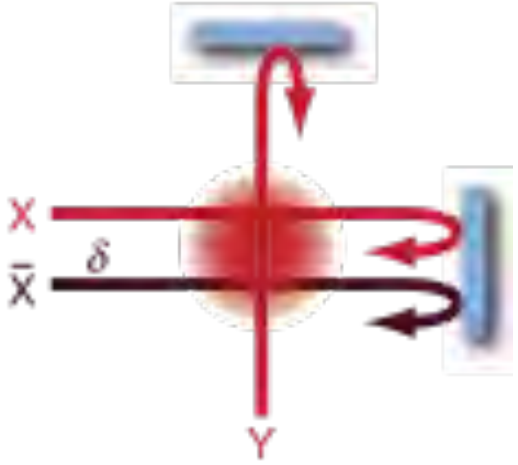
A graphene-like structure :
A brick-wall lattice for cold atoms

Leticia Tarruell, Daniel Greif, Thomas Uehlinger, Gregor Jotzu & Tilman Esslinger,
Creating, moving and merging Dirac points with a Fermi gas in a tunable honeycomb lattice
Nature 483, 302 (2012).



The brick-wall lattice with light

Superimpose several laser standing wave along the axes x and y



→ An intense standing wave along x

$$V_1(\vec{r}) = -V_{\bar{X}} \sin^2(kx)$$

→ A weak pair of phase-locked waves

$$V_2(\vec{r}) = -V_Y \cos^2(ky) \; - \; 2\sqrt{V_X V_Y} \; \cos(kx) \; \cos(ky) \; - \; \xcancel{V_X \cos^2(kx)}$$

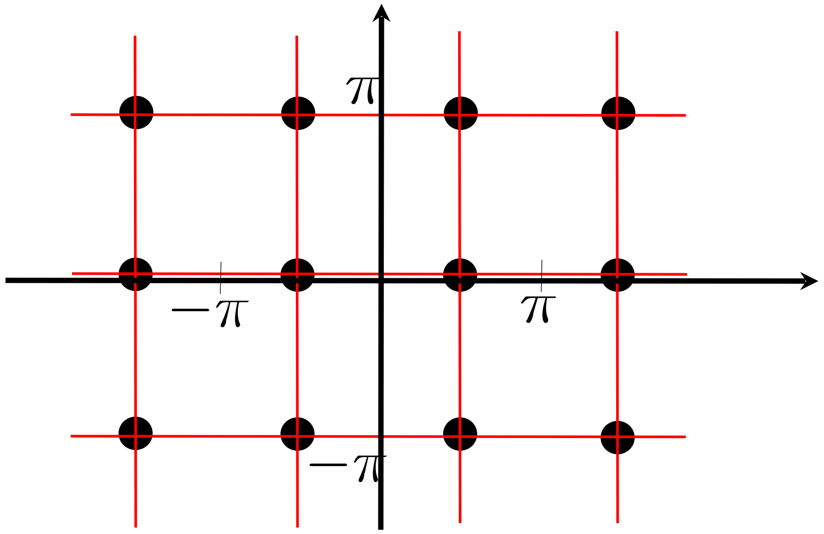
Choose the intensities such that $V_X \ll \sqrt{V_X V_Y} \ll V_Y < V_{\bar{X}}$

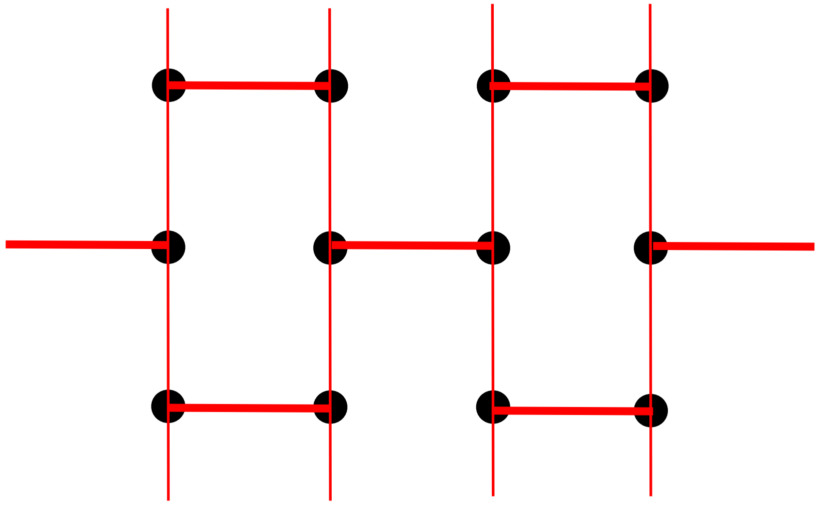
If we keep only the two dominant terms, square lattice:

$$-V_{\bar{X}} \sin^2(kx) - V_Y \cos^2(ky)$$

The brick-wall lattice with light

$-V_{\bar{X}} \sin^2(kx) - V_Y \cos^2(ky)$ $\quad$ $V_{\bar{X}}, V_Y > 0$





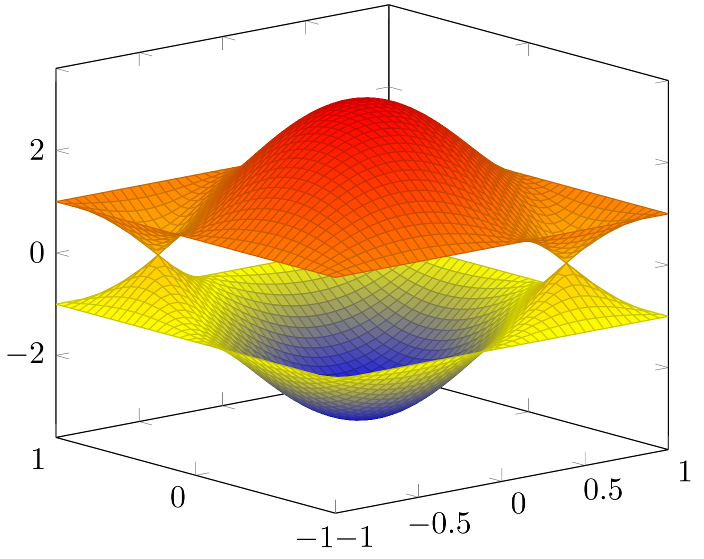
Now take into account $- 2\sqrt{V_X V_Y}\,\cos(kx)\,\cos(ky)$

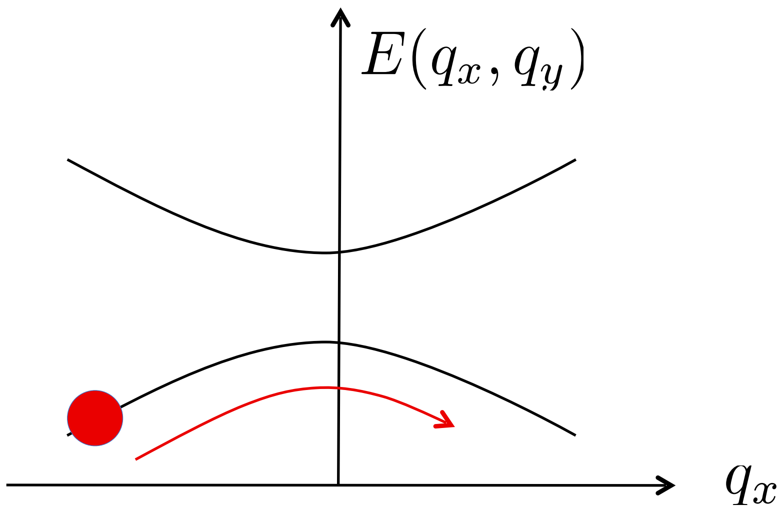
Link centered in $\cos(kx)\,\cos(ky) = +1$: tunnelling is increased

Link centered in $\cos(kx)\,\cos(ky) = 0$: tunnelling is unchanged

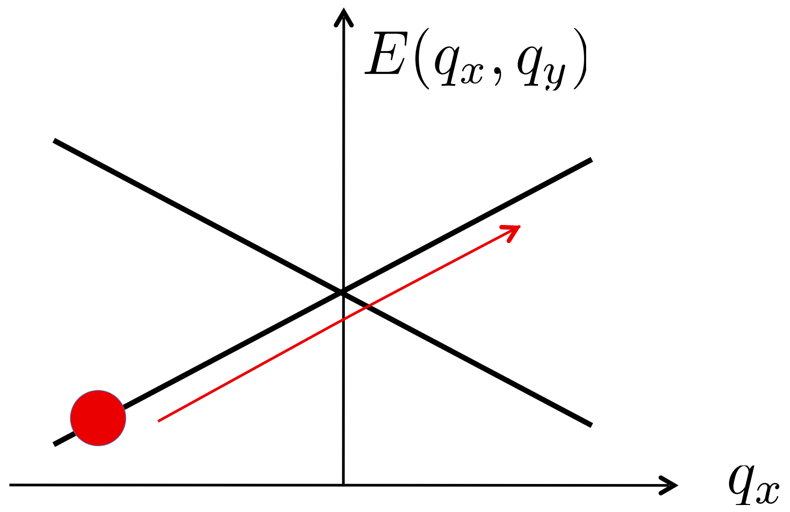
Link centered in $\cos(kx)\,\cos(ky) = -1$: tunnelling is decreased

Dirac points and Bloch oscillations



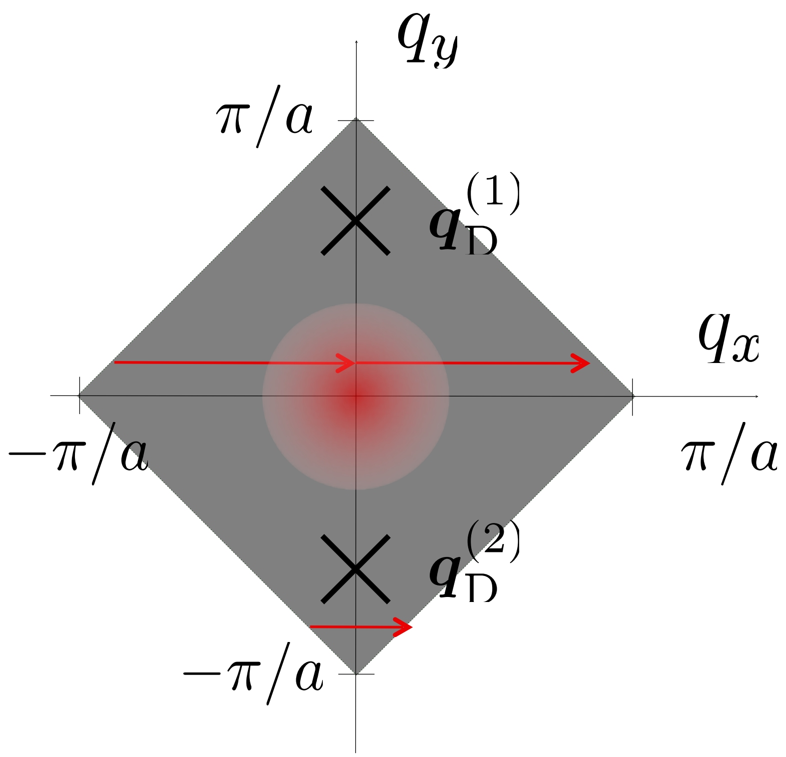


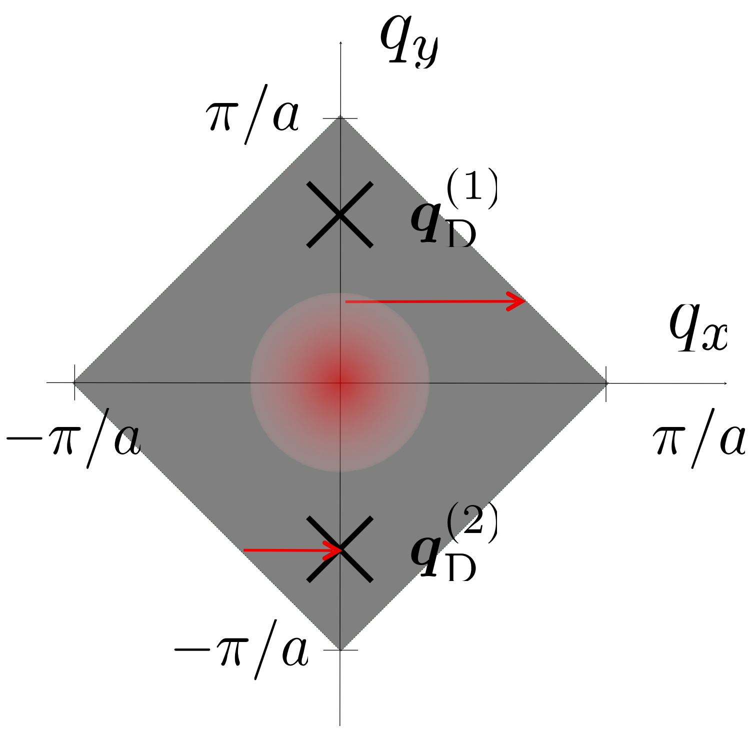
avoided crossing
adiabatic following is possible

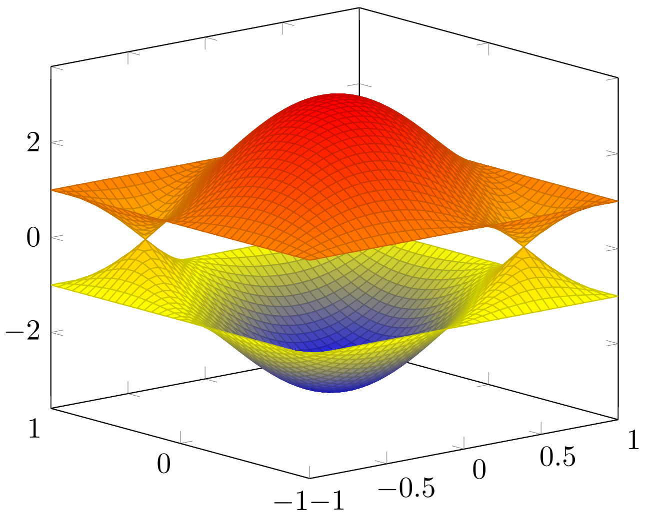


Dirac points:
adiabatic following is impossible

Bloch oscillations induced by a force created by a magnetic gradient



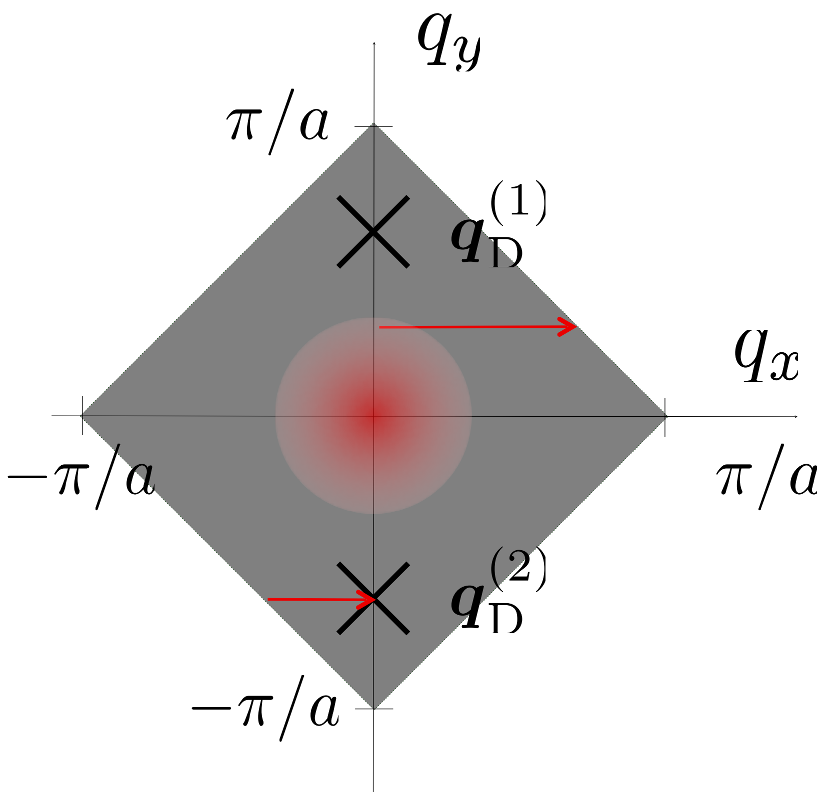


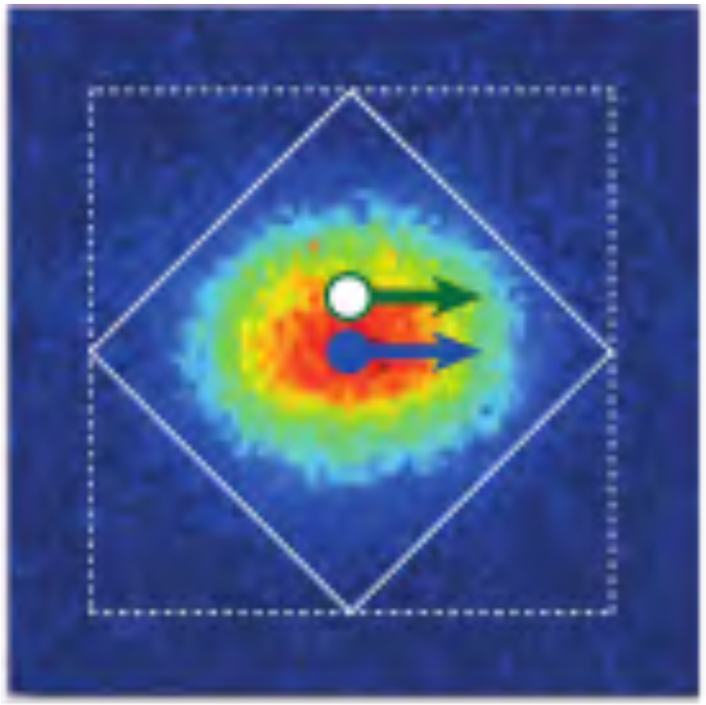


Dirac points and Bloch oscillations

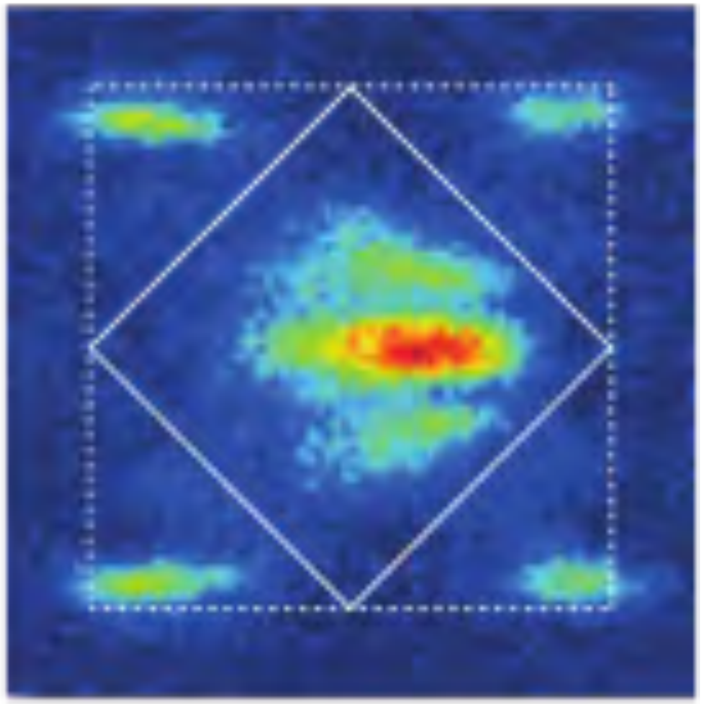
Leticia Tarruell et al., Nature 483, 302 (2012)

^{40}K atoms (polarized fermions, no interaction)





Initial time

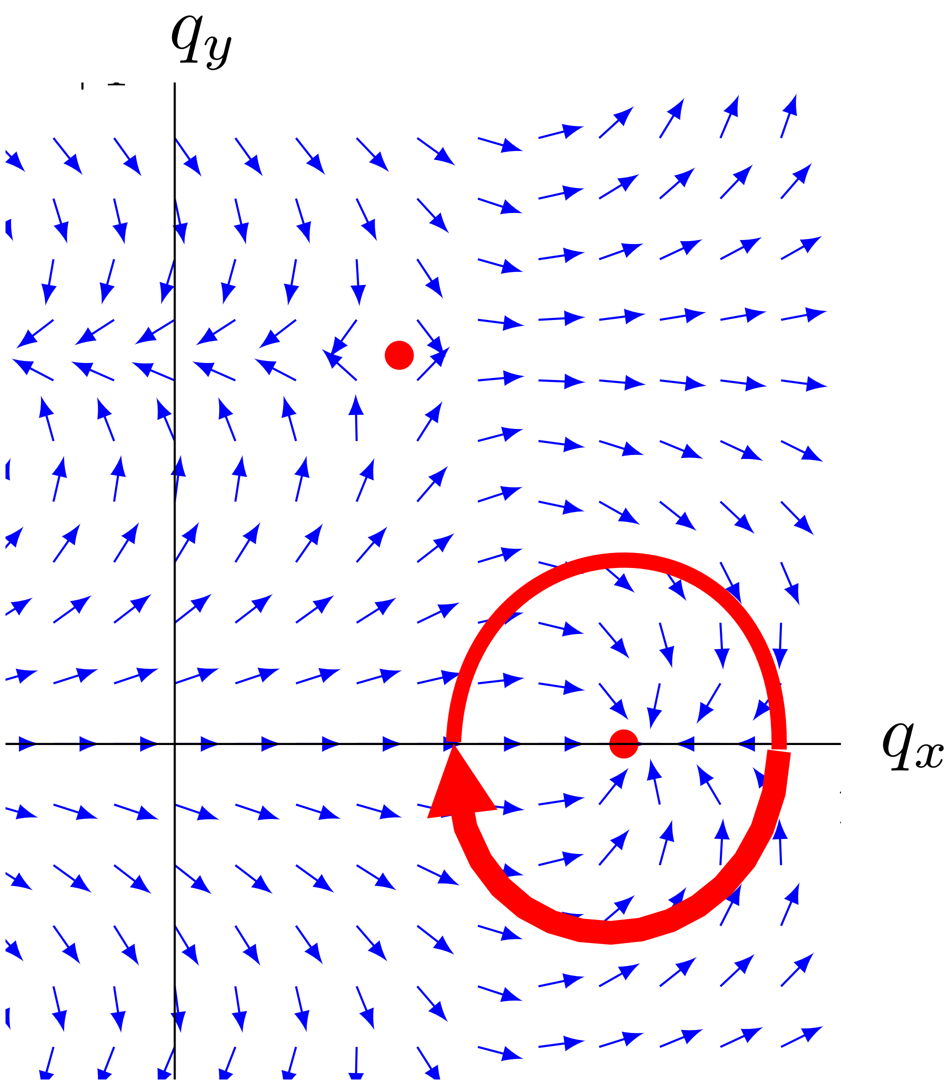


After a Bloch period

Pictures obtained after time-of-flight:
Band mapping technique, where the various in-situ bands end up at various places in space

Phase winding around a Dirac point

What is the geometrical phase accumulated by a particle that follows a closed contour in momentum space, which encircles a Dirac point?



The vector $\boldsymbol{n} = \dfrac{\boldsymbol{h}}{|\boldsymbol{h}|}$ remains on the equator

of the Bloch sphere and makes a full turn

Solide angle 2π irrespective of the contour shape

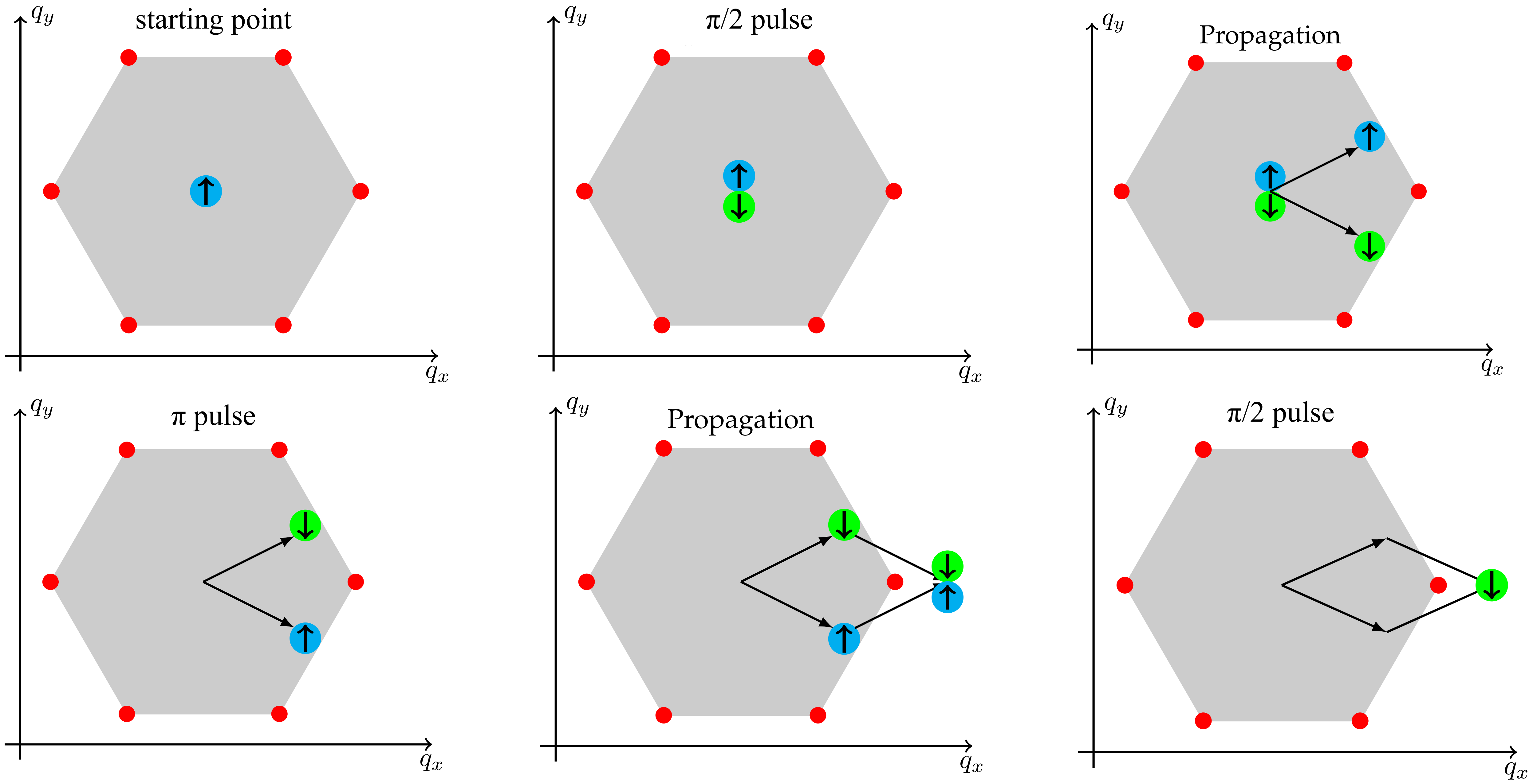
Geometric phase: $\dfrac{1}{2}\,2\pi \;=\; \pi$

The Munich experiment

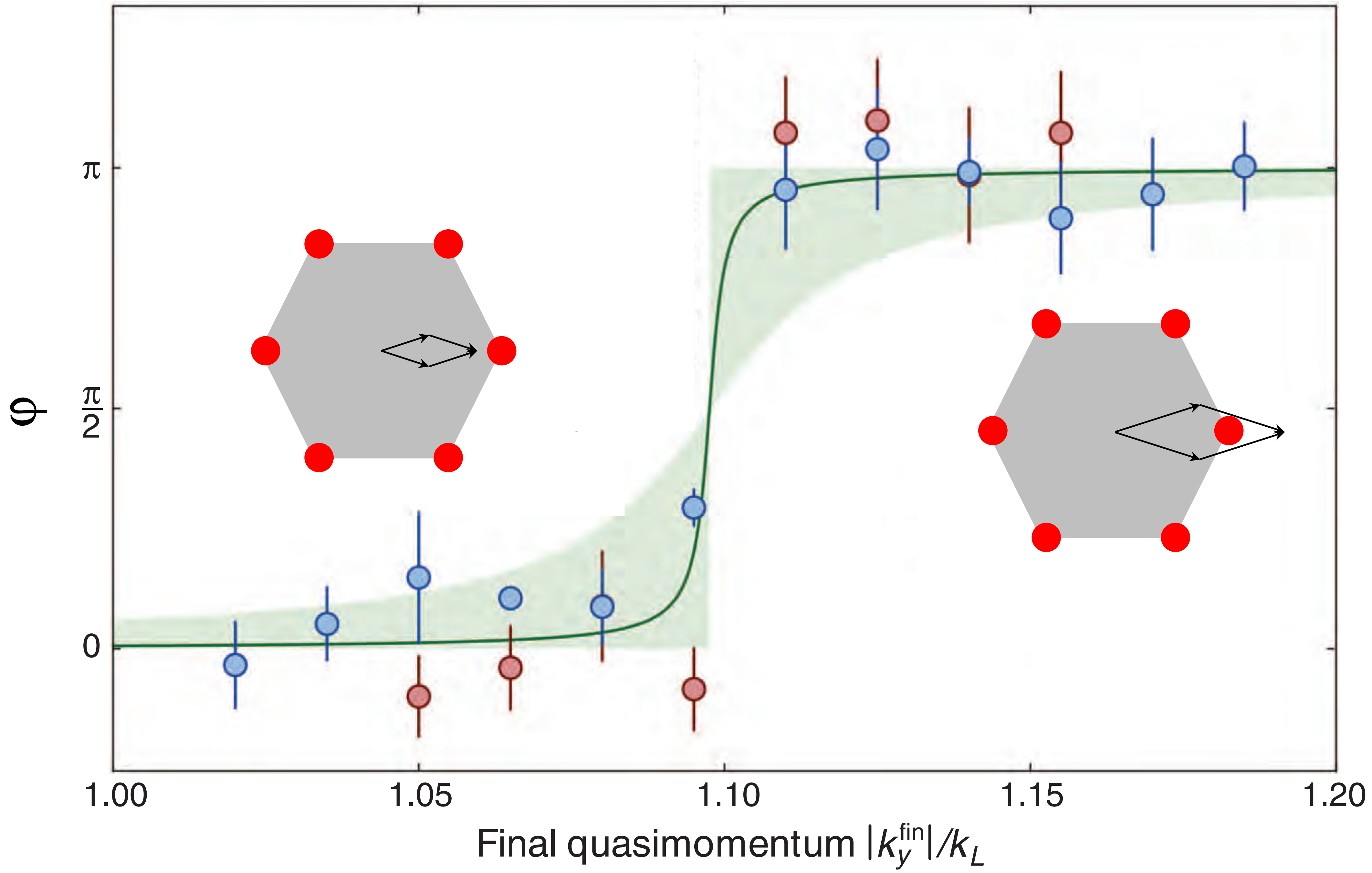
Duca et al., Science 347, 288 (2015) : An Aharonov-Bohm interferometer for determining Bloch band topology

Optical lattice for ^{87}Rb formed by 3 laser beams at 120°

Interferometric measurement of the geometric phase: $\frac{\pi}{2} - \pi - \frac{\pi}{2}$ scheme

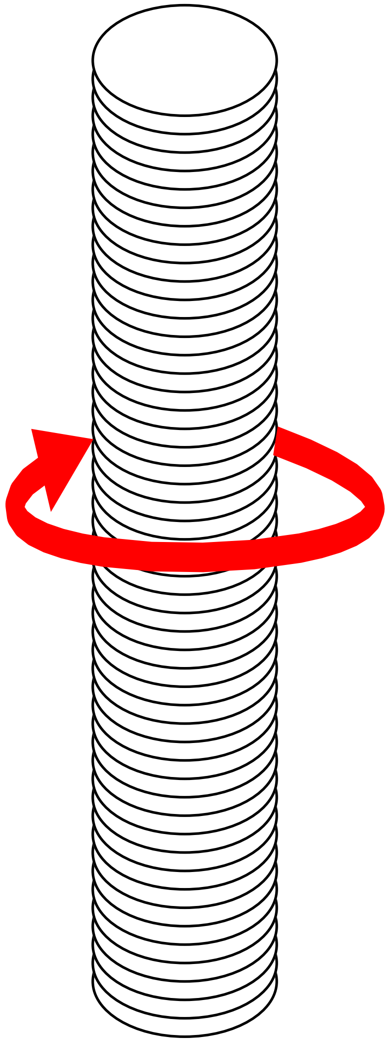


The Munich experiment



Zero geometrical phase as long as the Dirac point is outside the zone delimited by the interferometer, phase equal to π otherwise

Analogy with the Aharonov-Bohm effect



Infinite solenoid: the field is confined inside the solenoid

What is the phase accumulated by a particle
on the contour which encircles the solenoid?

$$\Phi_{\mathrm{AB}} = \frac{e}{2\pi\hbar} \oint_{\mathcal{C}} \boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r}$$

$$\oint_{\mathcal{C}} \boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = \iint \boldsymbol{B}(\boldsymbol{r})\, \mathrm{d}^2 r$$

2.

Topological bands in two dimensions: Geometrical characterization

Brillouin zone and Bloch sphere

Periodic Hamiltonian for a two-site cell, tight-binding limit

A-A coupling **A-B coupling**

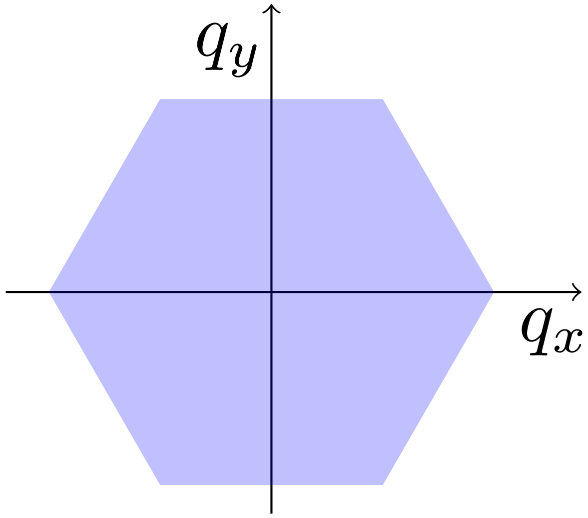
$$\hat{H}_{\boldsymbol{q}} = E_0(\boldsymbol{q})\,\hat{1} - \boldsymbol{h}(\boldsymbol{q})\cdot\hat{\boldsymbol{\sigma}} \quad = \begin{pmatrix} E_0 - h_z & -h_x + \mathrm{i}h_y \\ -h_x - \mathrm{i}h_y & E_0 + h_z \end{pmatrix}$$

B-B coupling

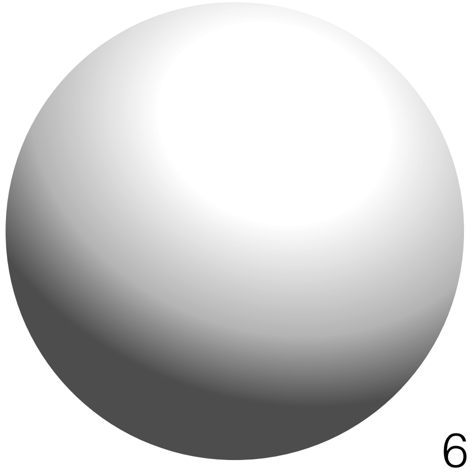
Energies : $E_0 \pm |\boldsymbol{h}|$

Eigenstates determined using $\boldsymbol{n} = \dfrac{\boldsymbol{h}}{|\boldsymbol{h}|}$, characterized by the angles $\theta_{\boldsymbol{q}}, \phi_{\boldsymbol{q}}$

Characterization of $\hat{H}_{\boldsymbol{q}}$ by the mapping:

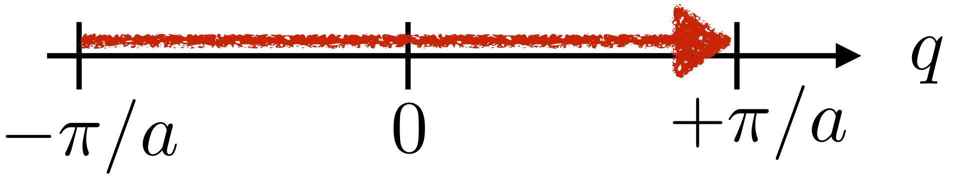


$$\boldsymbol{q} = (q_x, q_y) \longrightarrow \boldsymbol{n}_{\boldsymbol{q}} = \begin{pmatrix} \cos\phi_q \ \sin\theta_q \\ \sin\phi_q \ \sin\theta_q \\ \cos\theta_q \end{pmatrix}$$

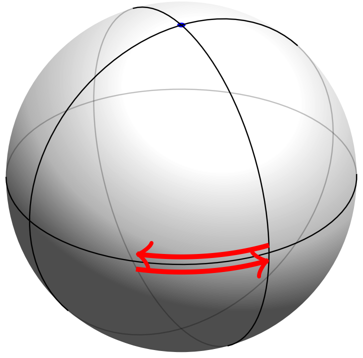


Wrapping of the Bloch sphere

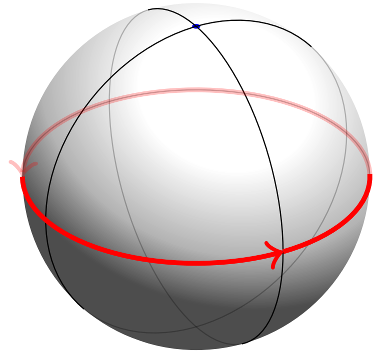
In one dimension (for instance SSH) :



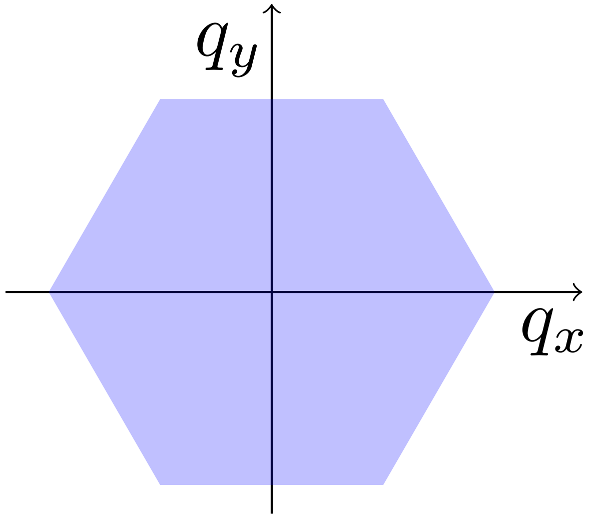
leads to



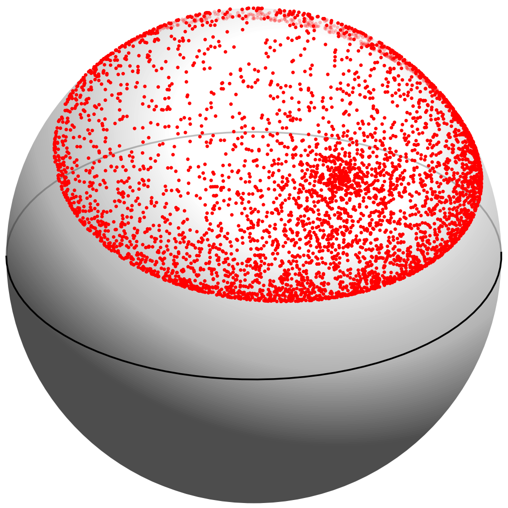
or



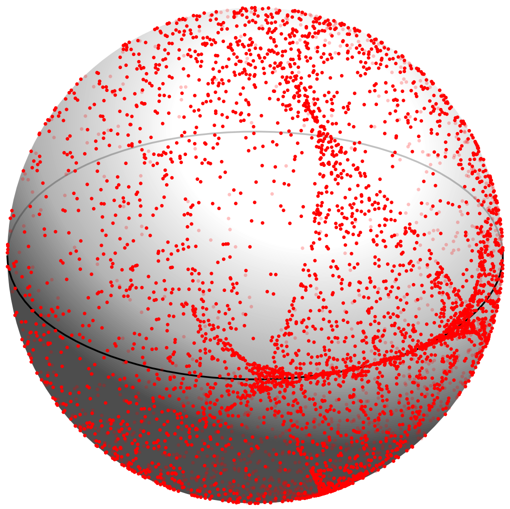
In two dimensions :



may lead to



or

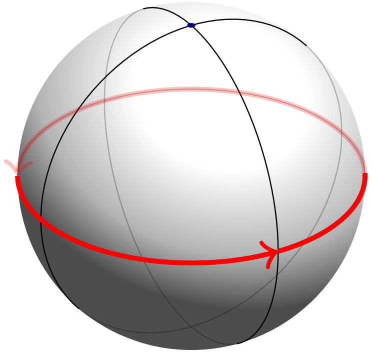


Total wrapping of the Bloch sphere, which cannot be unwrapped

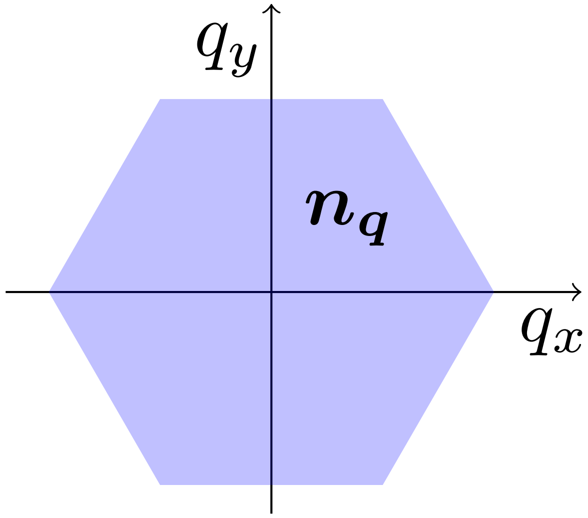
A result from geometry: the wrapping number

Winding number in one dimension:

$$\mathcal{N} = \frac{1}{2\pi} \int_{\mathrm{ZB}} \frac{\mathrm{d}\phi}{\mathrm{d}q}\, \mathrm{d}q$$

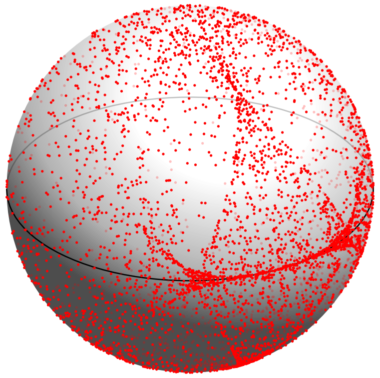


Wrapping number in two dimensions:

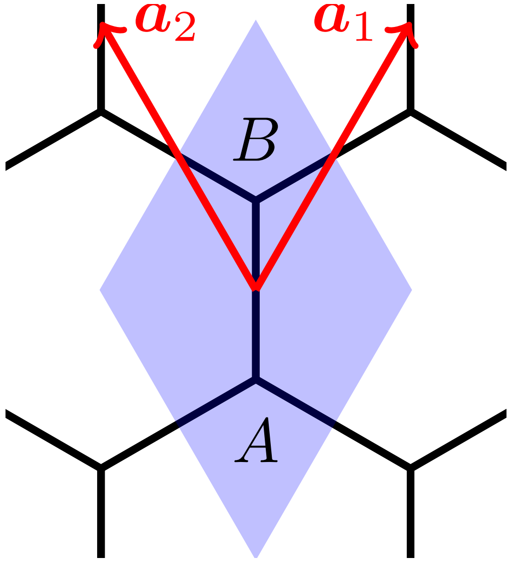


$$\mathcal{C} = -\frac{1}{4\pi} \iint_{\mathrm{ZB}} \boldsymbol{n} \cdot \left[(\partial_{q_x} \boldsymbol{n}) \times (\partial_{q_y} \boldsymbol{n}) \right]\, \mathrm{d}q_x\, \mathrm{d}q_y$$

Integer number that is non-zero if and only if the sphere is fully wrapped (*cf.* result for adiabatic pumps)



The graphene case

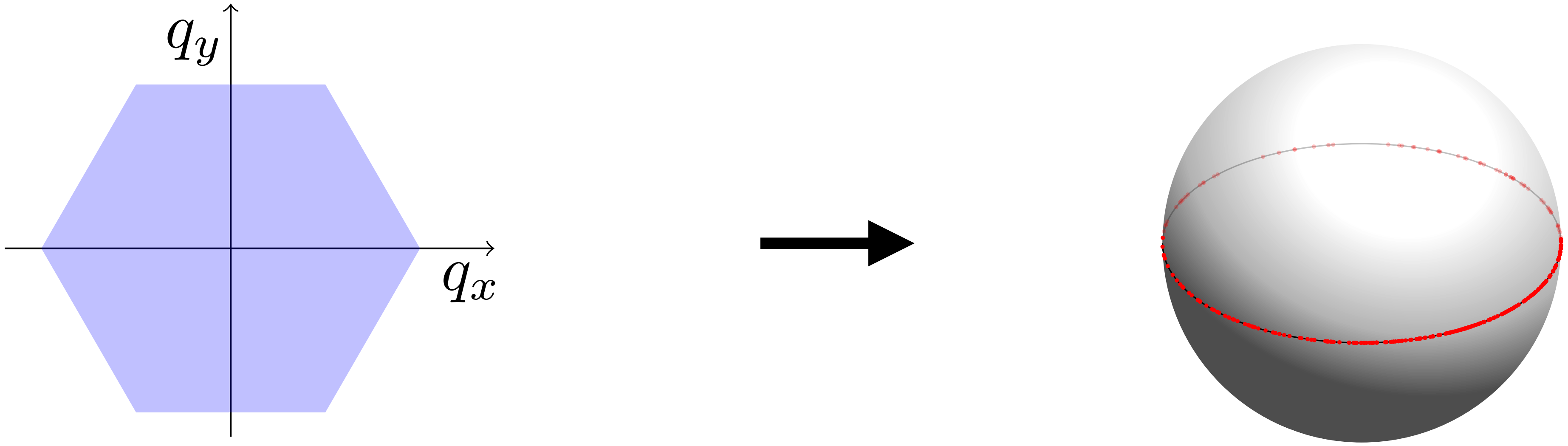


$E_A = E_B$ and no coupling to second neighbors:

$\longrightarrow$ $\hat{H}_{\boldsymbol{q}}$ has no diagonal element

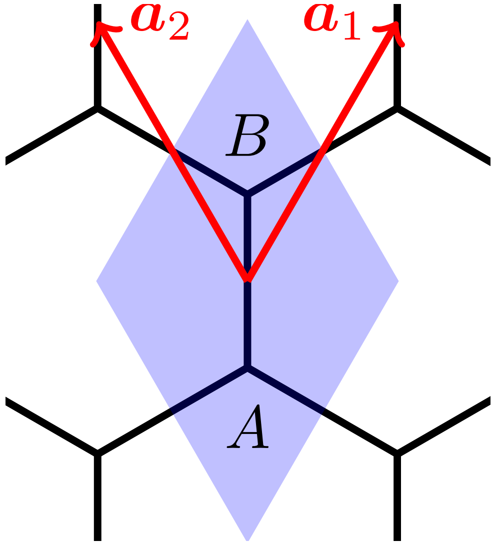
$$\hat{H}_{\boldsymbol{q}} = -J \begin{pmatrix} 0 & 1 + \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_1} + \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_2} \\ 1 + \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_1} + \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_2} & 0 \end{pmatrix}$$

The vector $\boldsymbol{h}_{\boldsymbol{q}}$ remains along the equator of the Bloch sphere, which therefor cannot be wrapped



Marginal situation

Partial or total coverage?



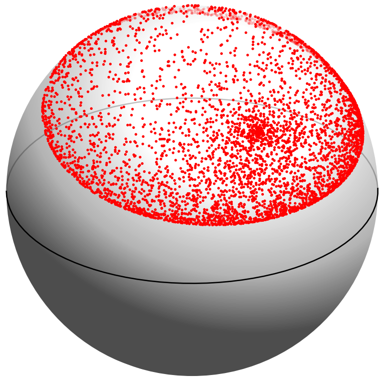
Let us make the A et B sites different by an energy splitting (cf. passage from SSH to Rice-Mele)

$$\hat{H}_{\boldsymbol{q}} = -\begin{pmatrix} \Delta & h_x(\boldsymbol{q}) - \mathrm{i}h_y(\boldsymbol{q}) \\ h_x(\boldsymbol{q}) + \mathrm{i}h_y(\boldsymbol{q}) & -\Delta \end{pmatrix}$$

$$E_A = -\Delta \qquad\qquad E_B = +\Delta$$

$$h_z(\boldsymbol{q}) = \Delta$$

The sign of $h_z(\boldsymbol{q})$ is constant over the full Brillouin zone: we can cover at most one hemisphere of the Bloch sphere



To wrap completely the Bloch sphere, we need to go beyond nearest-neighbor couplings: Haldane model (next week)

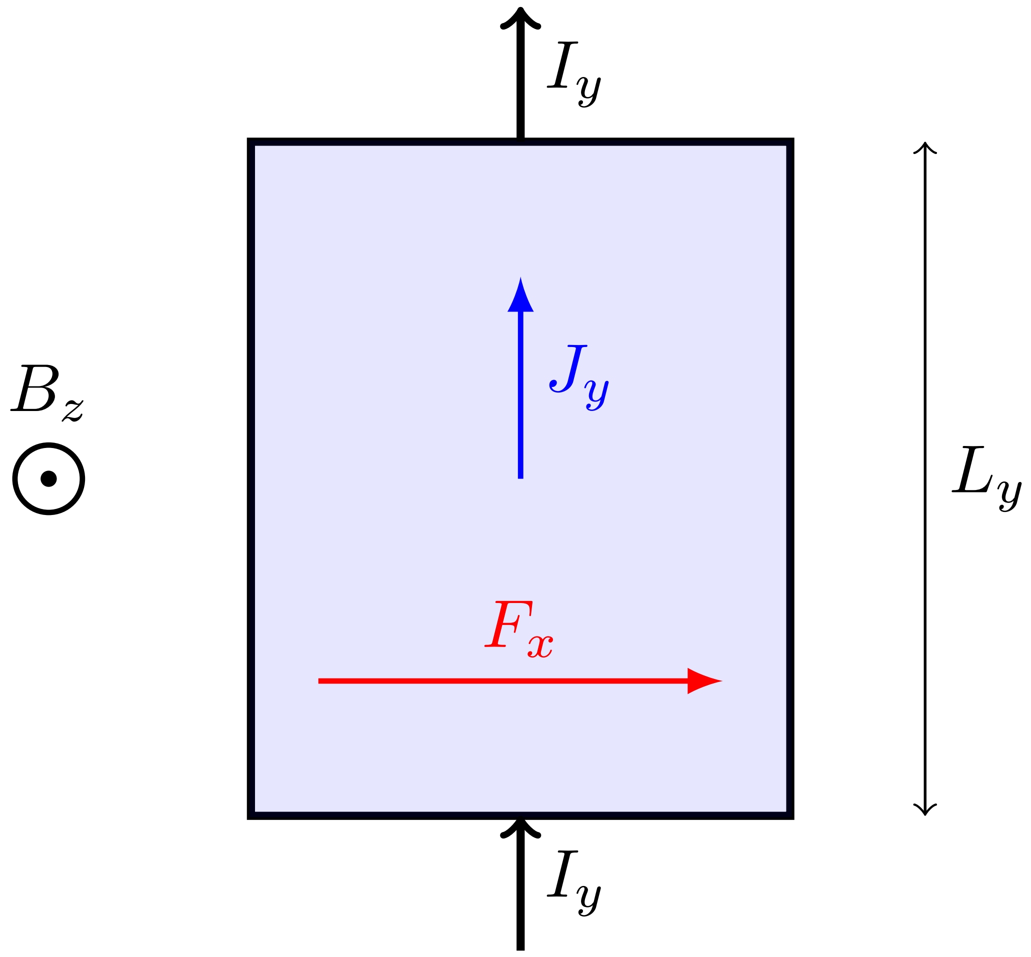
3.

Topological bands in two dimensions: Physical characterization

The quantum Hall effect

2D electron gas confined in a quantum well
in the presence of a large magnetic field

$$[0, L_x] \times [0, L_y]$$



Voltage $V_x = \mathcal{E}_x L_x$

Force on a charge e: $F_x = e\mathcal{E}_x$

Current along the direction ***y*** : I_y

Current density : $J_y = I_y / L_x$

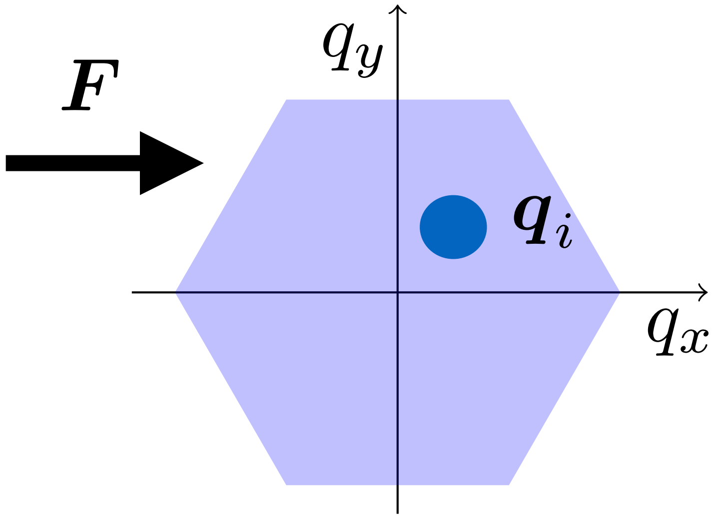
Hall conductance: $I_y = \sigma_{yx} V_x$ or $J_y = \sigma_{yx} \mathcal{E}_x$

Quantized conductance! $\sigma_{yx} = \dfrac{e^2}{h} n$ n integer

Origin: Topological nature of energy bands (Landau levels)

Equations of motion in an energy band

Let us restrict to the lowest band $|u_{\boldsymbol{q}}^{(0)}\rangle$ to simplify the discussion



Wave packet initially centered in $\boldsymbol{q}_i$

Apply a uniform force $\boldsymbol{F}$

$$\hbar \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \boldsymbol{F}$$ *Bloch oscillations*

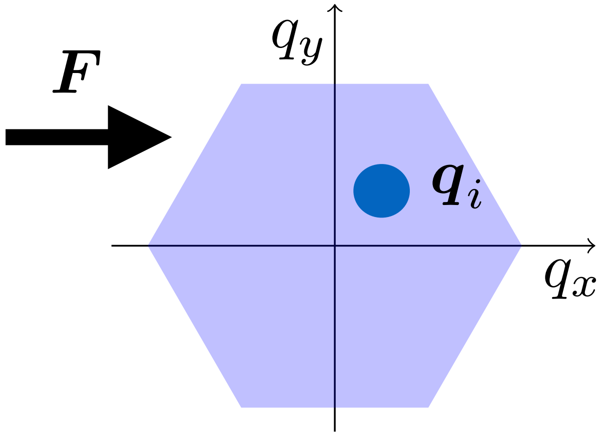
$$\hbar \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \hbar \boldsymbol{v} = \underbrace{\boldsymbol{\nabla}_{\boldsymbol{q}} E_{\boldsymbol{q}}^{(0)}}_{\text{Group velocity}} + \underbrace{\boldsymbol{\Omega}_{\boldsymbol{q}} \times \boldsymbol{F}}_{\text{Anomalous velocity}}$$

$\boldsymbol{\Omega}_{\boldsymbol{q}}$: Berry curvature for the lowest band

$\boldsymbol{\mathcal{A}}_{\boldsymbol{q}} = \mathrm{i}\langle u_{\boldsymbol{q}}^{(0)}|\boldsymbol{\nabla}_{\boldsymbol{q}} u_{\boldsymbol{q}}^{(0)}\rangle$: Berry connection

$\boldsymbol{\Omega}_{\boldsymbol{q}} = \boldsymbol{\nabla}_{\boldsymbol{q}} \times \boldsymbol{\mathcal{A}}_{\boldsymbol{q}}$ oriented along *z* $\quad \Omega_{\boldsymbol{q}} = \mathrm{i}\ \langle \partial_{q_x} u_{\boldsymbol{q}}^{(0)}|\partial_{q_y} u_{\boldsymbol{q}}^{(0)}\rangle\ +\ \mathrm{c.c.}$

Equation 1: Evolution of the momentum



Hamiltonian in the presence of a uniform external force

$$\hat{H}_t = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\hat{\boldsymbol{r}}) - \boldsymbol{F}_t \cdot \hat{\boldsymbol{r}}$$

This hamiltonian is not spatially periodic anymore: Do we loose Bloch theorem?

Not really, thanks to the unitary transform:

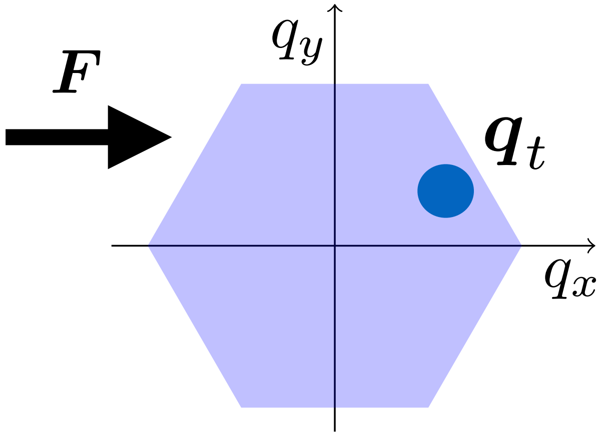
$$\hat{U}_t = \exp\left[-\mathrm{i}\,\boldsymbol{A}_t \cdot \hat{\boldsymbol{r}}\right] \qquad \text{with} \qquad \boldsymbol{A}_t = \frac{1}{\hbar}\int_0^t \boldsymbol{F}_{t'}\,\mathrm{d}t' \qquad\qquad \tilde{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{A}$$

Hamiltonian after transformation : $\hat{\tilde{H}}_t = \dfrac{(\hat{\boldsymbol{p}} + \hbar\boldsymbol{A}_t)^2}{2m} + V(\hat{\boldsymbol{r}})$

- The Bloch form is preserved for the “transformed” states: $\tilde{\boldsymbol{q}}(t) = \tilde{\boldsymbol{q}}(0)$
- If the force $\boldsymbol{F}$ is weak enough, the particle stays in the lowest band

Back to initial states : $\boldsymbol{q}(t) = \boldsymbol{q}(0) + \boldsymbol{A}(t) \longrightarrow \dfrac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \dfrac{1}{\hbar}\boldsymbol{F}$

Equation 2: The anomalous velocity



Adiabatic approximation at order 1 in the perturbation

$$|u_t\rangle \;=\; |u^{(0)}_{\boldsymbol{q}_t}\rangle \;+\; \mathrm{i}\hbar \sum_{n\neq 0} |u^{(n)}_{\boldsymbol{q}_t}\rangle \frac{\langle u^{(n)}_{\boldsymbol{q}_t}|\partial_t u^{(0)}_{\boldsymbol{q}_t}\rangle}{E^{(n)}_{\boldsymbol{q}} - E^{(0)}_{\boldsymbol{q}}} \;+\; \dots$$

order 0: stays in the lowest band

order 1: coupling to excited bands, linear in $\boldsymbol{F}$

Average velocity of a wave packet centered in $\boldsymbol{q}_t$

$$\boldsymbol{v} = \left(\langle u_t|\mathrm{e}^{-\mathrm{i}\boldsymbol{q}_t\cdot\boldsymbol{r}}\right) \frac{\hat{\boldsymbol{p}}}{m} \left(\mathrm{e}^{\mathrm{i}\boldsymbol{q}_t\cdot\boldsymbol{r}}|u_t\rangle\right) \qquad \hat{\boldsymbol{p}} = -\mathrm{i}\hbar\,\boldsymbol{\nabla}_{\boldsymbol{r}}$$

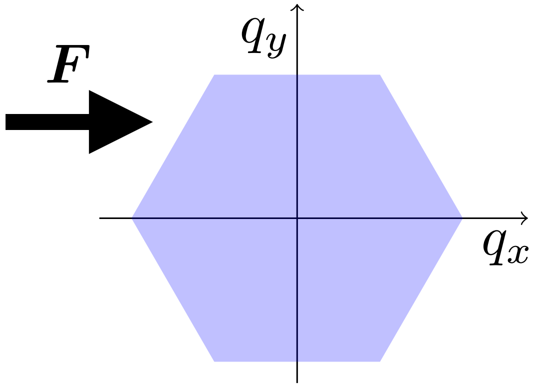
Order 0 in $\boldsymbol{F}$: $\boldsymbol{v}_0 = \frac{1}{\hbar}\boldsymbol{\nabla}_{\boldsymbol{q}} E^{(0)}_{\boldsymbol{q}_t}$ ***group velocity***

Order 1 in $\boldsymbol{F}$: $\boldsymbol{v}_{1,t} = \frac{1}{\hbar}\boldsymbol{\Omega}_{\boldsymbol{q}_t} \times \boldsymbol{F}_t$ ***anomalous velocity***

Karplus & Luttinger 1954
Adams & Blount 1959

$$\Omega_{\boldsymbol{q}} = \mathrm{i}\;\langle \partial_{q_x} u^{(0)}_{\boldsymbol{q}}|\partial_{q_y} u^{(0)}_{\boldsymbol{q}}\rangle \;+\; \mathrm{c.c.}$$ *: Berry curvature*

Conductance of a filled band



Band insulator

1 particle per unit cell:
filled lowest band, all excited bands are empty

We apply a weak force $\boldsymbol{F}$; What is the particle current?

$$\boldsymbol{J} = \rho^{(2D)} \ \langle \boldsymbol{v} \rangle \qquad\qquad \langle \boldsymbol{v} \rangle = \frac{1}{A_{\mathrm{ZB}}} \iint_{\mathrm{ZB}} \boldsymbol{v}_{\boldsymbol{q}} \ \mathrm{d}^2 q$$

We use: $\hbar \boldsymbol{v}_{\boldsymbol{q}} \ = \ \boldsymbol{\nabla}_{\boldsymbol{q}} E_{\boldsymbol{q}}^{(0)} + \boldsymbol{\Omega}_{\boldsymbol{q}} \times \boldsymbol{F}$

- Zero contribution for the group velocity: an insulator does not conduct electricity!
- Contribution of the anomalous velocity: current orthogonal to $\boldsymbol{F}$

For a force oriented along ***x***, current along ***y*** : $J_y = \sigma_{yx} \ F_x$

$$\sigma_{yx} = \frac{1}{h} \, \mathcal{C} \qquad\qquad \mathcal{C} = \frac{1}{2\pi} \iint_{\mathrm{ZB}} \Omega_{\boldsymbol{q}} \ \mathrm{d}^2 q$$

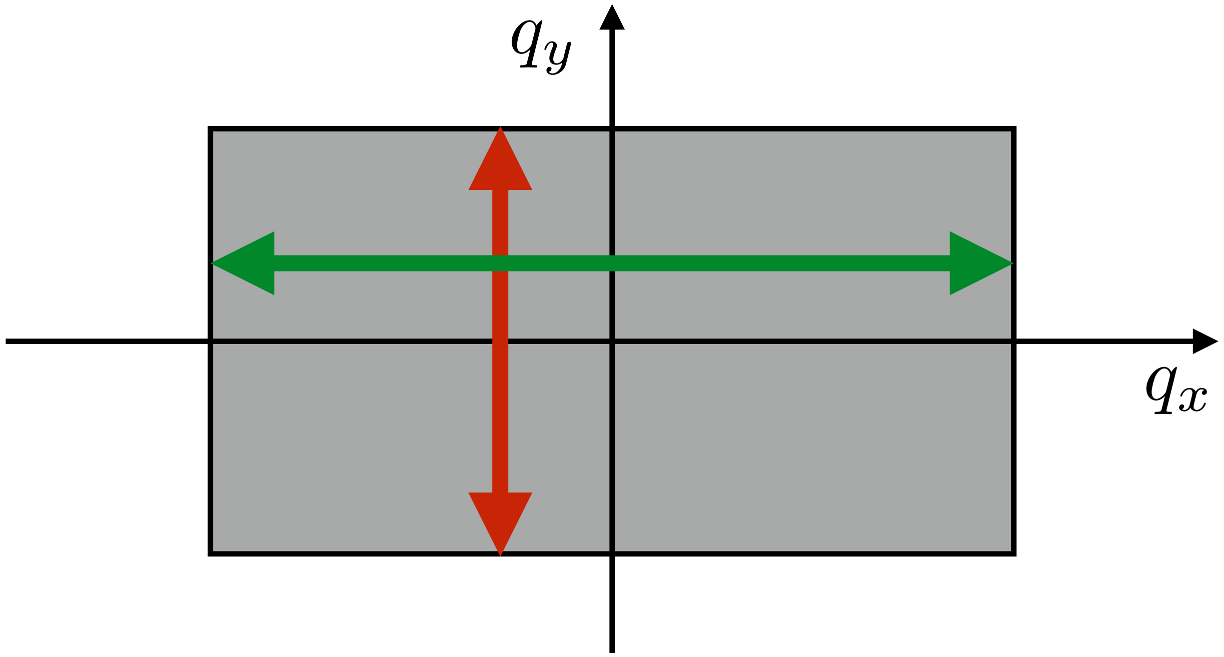
Hall conductivity

Chern number

Why the Chern number maybe non-zero

$$\mathcal{C} = \frac{1}{2\pi} \iint_{\mathrm{ZB}} \Omega_{\boldsymbol{q}} \, \mathrm{d}^2 q \qquad\qquad \boldsymbol{\Omega}_{\boldsymbol{q}} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}_{\boldsymbol{q}} \quad = \Omega_{\boldsymbol{q}} \, \boldsymbol{u}_z$$

The Brillouin zone has by construction periodic boundary conditions



cf. adiabatic pumps

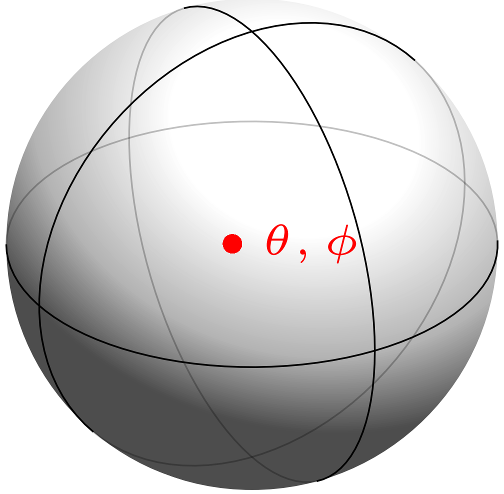
If the Berry connection $\boldsymbol{\mathcal{A}}_{\boldsymbol{q}}$ is regular over the whole Brillouin zone, Stokes theorem leads to

$$\frac{1}{2\pi} \iint_{\mathrm{ZB}} \Omega_{\boldsymbol{q}} \, \mathrm{d}^2 q \quad = \quad \frac{1}{2\pi} \oint_{\mathrm{ZB}} \boldsymbol{\mathcal{A}}_{\boldsymbol{q}} \cdot \mathrm{d}\boldsymbol{q} \qquad = 0 \text{ !!!}$$

but Berry connection is not always regular…

The singularities of Berry connection $\boldsymbol{\mathcal{A}_q}$

Unit cell with two sites



Gauge choice: $|u\rangle = \begin{pmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{pmatrix}$

The south pole problem: for $\theta = \pi$, we obtain for this gauge choice

$$|u\rangle = \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix}$$

infinite gradient in that point

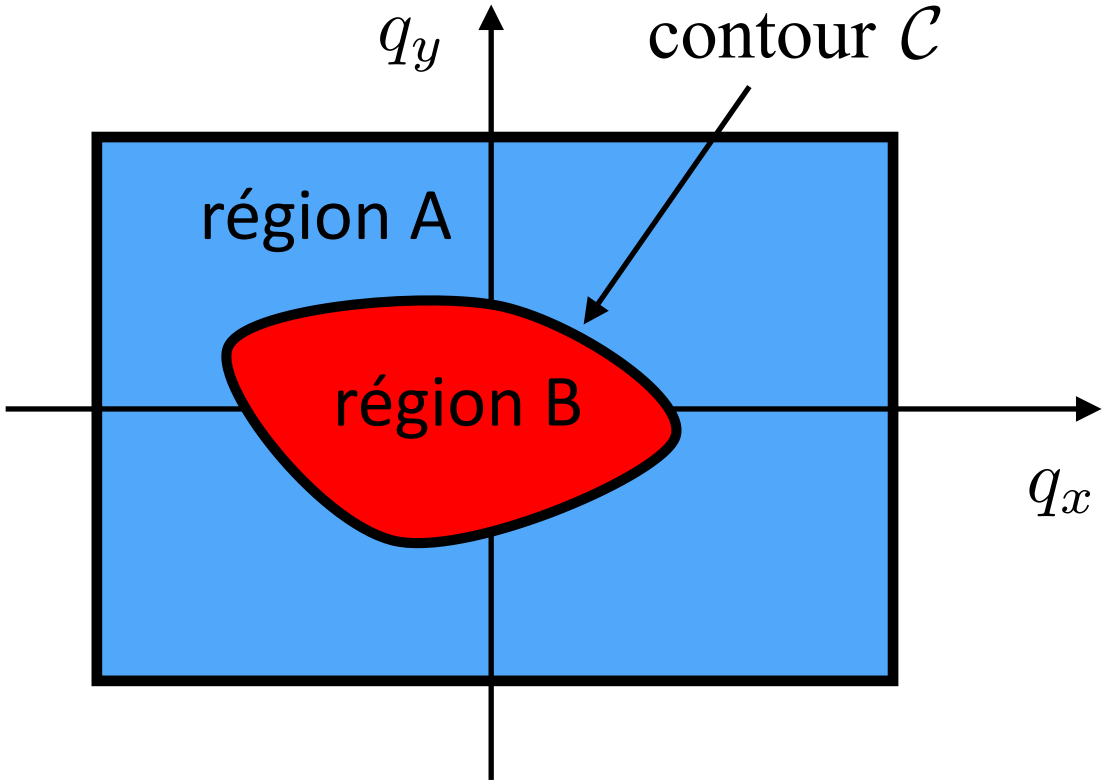
What about another gauge, for instance $\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi}\cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}$?

The south pole problem has disappeared : $|u\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

… but a problem appears now at the north pole : $\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi} \\ 0 \end{pmatrix}$

If the Bloch sphere is completely wrapped, no "good" gauge choice

The Chern is not always zero, but it must be an integer



Separate the Brillouin zone in two regions A et B :

- Gauge choice *(I)* non singular over A
- Gauge choice *(II)* non singular over B

$$|u_{\boldsymbol{q}}^{(II)}\rangle = \mathrm{e}^{-\mathrm{i}\chi_{\boldsymbol{q}}}\,|u_{\boldsymbol{q}}^{(I)}\rangle$$

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{q}}^{(II)} = \boldsymbol{\mathcal{A}}_{\boldsymbol{q}}^{(I)} + \boldsymbol{\nabla}_{\boldsymbol{q}}\chi_{\boldsymbol{q}}$$

Surface integral of Berry curvature and Stokes theorem:

$$\iint_{\mathrm{ZB}} \Omega_{\boldsymbol{q}}\,\mathrm{d}^2q = \iint_{A} \Omega_{\boldsymbol{q}}\,\mathrm{d}^2q + \iint_{B} \Omega_{\boldsymbol{q}}\,\mathrm{d}^2q \;=\; \left(\oint_{\mathrm{ZB}} - \oint_{\mathcal{C}}\right) \boldsymbol{\mathcal{A}}_{\boldsymbol{q}}^{(I)}\cdot\mathrm{d}\boldsymbol{q} + \oint_{\mathcal{C}} \boldsymbol{\mathcal{A}}_{\boldsymbol{q}}^{(II)}\cdot\mathrm{d}\boldsymbol{q}$$

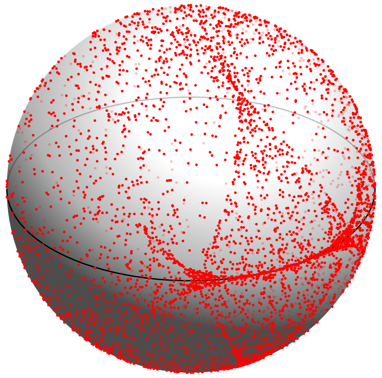
Periodicity of the BZ: $\oint_{\mathrm{ZB}} \boldsymbol{\mathcal{A}}_{\boldsymbol{q}}^{(I)}\cdot\mathrm{d}\boldsymbol{q} = 0$

We are left with: $\iint_{\mathrm{ZB}} \Omega_{\boldsymbol{q}}\,\mathrm{d}^2q = \oint_{\mathcal{C}} \boldsymbol{\nabla}_{\boldsymbol{q}}\chi_{\boldsymbol{q}}\cdot\mathrm{d}\boldsymbol{q}$ = multiple of 2π Q.E.D.

Link between our two approaches

- Geometrical approach for a two-site unit cell

Wrapping of the Bloch sphere when q spans the Brillouin zone



$$-\frac{1}{4\pi}\iint_{\mathrm{ZB}} \boldsymbol{n}\cdot\left[(\partial_{q_x}\boldsymbol{n})\times(\partial_{q_y}\boldsymbol{n})\right]\ \mathrm{d}^2q$$ non zero integer

- Physical approach: quantized Hall conductance

$$\frac{1}{2\pi}\iint_{\mathrm{ZB}} \Omega_{\boldsymbol{q}}\ \mathrm{d}^2q$$ non-zero integer $$\Omega_{\boldsymbol{q}} = \mathrm{i}\ \langle\partial_{q_x}u_{\boldsymbol{q}}^{(-)}|\partial_{q_y}u_{\boldsymbol{q}}^{(-)}\rangle\ +\ \mathrm{c.c.}$$

These are directly related quantities: $|u_{\boldsymbol{q}}^{(-)}\rangle = \begin{pmatrix}\cos(\theta/2)\\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2)\end{pmatrix}$ and $\boldsymbol{n} = \begin{pmatrix}\sin\theta\ \cos\phi\\ \sin\theta\ \sin\phi\\ \cos\theta\end{pmatrix}$

$$\longrightarrow\qquad \Omega_{\boldsymbol{q}} = -\frac{1}{2}\,\boldsymbol{n}\cdot\left[(\partial_{q_x}\boldsymbol{n})\times(\partial_{q_y}\boldsymbol{n})\right]$$ ***: The two criteria are equivalent***

Chern number and symmetries

Inversion symmetry: $\hat{S}_0\,\psi(\boldsymbol{r}) = \psi(-\boldsymbol{r})$

If $[\hat{S}_0, \hat{H}] = 0$ then : $\Omega_{\boldsymbol{q}} = \Omega_{-\boldsymbol{q}}$

Time-reversal symmetry: $\boldsymbol{r} \longrightarrow \boldsymbol{r}$ $\qquad \boldsymbol{p} \longrightarrow -\boldsymbol{p}$

For a (spinless) wavefunction : $\hat{K}_0\,\psi(\boldsymbol{r}) = \psi^*(\boldsymbol{r})$

$$\hat{K}_0\,\left(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right) = \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

If $[\hat{K}_0, \hat{H}] = 0$ then : $\Omega_{\boldsymbol{q}} = -\Omega_{-\boldsymbol{q}}$ $\longrightarrow$ $\mathcal{C} = \frac{1}{2\pi}\iint \Omega_{\boldsymbol{q}}\,\mathrm{d}^2 q = 0$

Non topological band

If the two symmetries are simultaneously present: $\Omega_{\boldsymbol{q}} = 0$

The anomalous velocity is zero at any point of the Brillouin zone

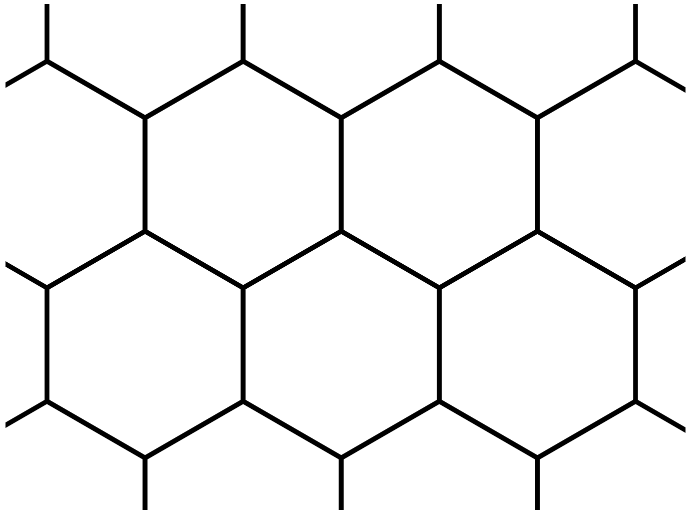
4.

Measurement of Berry curvature in a 2D optical lattice

Fläschner, Rem, et al., Science 352, 1091 (2016)   [Hamburg]
Experimental reconstruction of the Berry curvature in a Floquet Bloch band

Local measurement of Berry curvature

Hexagonal lattice that is modulated in time, with parameters such that the tight-binding two-band model is a good description



Measure of $\Omega_{\boldsymbol{q}}$ $\longleftrightarrow$ Measure of $\boldsymbol{n_q}$: $\theta_{\boldsymbol{q}}, \phi_{\boldsymbol{q}}$

Measurement of the momentum distribution of atoms in the lowest band

Hauke, Lewenstein, Eckardt (2014)

Sudden switch off of the lattice and ballistic expansion:

$$|u_{\boldsymbol{q}}^{(-)}\rangle = \begin{pmatrix} \cos(\theta_{\boldsymbol{q}}/2) \\ \mathrm{e}^{\mathrm{i}\phi_{\boldsymbol{q}}} \sin(\theta_{\boldsymbol{q}}/2) \end{pmatrix} \longrightarrow \mathcal{N}(\boldsymbol{q}) = f(\boldsymbol{q}) \ \left|\cos(\theta_{\boldsymbol{q}}/2) + \mathrm{e}^{\mathrm{i}\phi_{\boldsymbol{q}}} \sin(\theta_{\boldsymbol{q}}/2)\right|^2$$

Enveloppe : Wannier function on sites A or B

Local measurement of Berry curvature (2)

$$|u_{\boldsymbol{q}}^{(-)}\rangle = \begin{pmatrix} \cos(\theta_{\boldsymbol{q}}/2) \\ \mathrm{e}^{\mathrm{i}\phi_{\boldsymbol{q}}} \sin(\theta_{\boldsymbol{q}}/2) \end{pmatrix}$$

Signal after a ballistic expansion:

$$\mathcal{N}(\boldsymbol{q}) = f(\boldsymbol{q}) \ \left|\cos(\theta_{\boldsymbol{q}}/2) + \mathrm{e}^{\mathrm{i}\phi_{\boldsymbol{q}}} \sin(\theta_{\boldsymbol{q}}/2)\right|^2 \qquad = f(\boldsymbol{q}) \left[1 - \sin\theta_{\boldsymbol{q}} \cos\phi_{\boldsymbol{q}}\right]$$

In order to measure separately θ_q and ϕ_q , multiple step procedure:

- Preparation in the lattice
- Sudden quench: $\hat{H}'_{\boldsymbol{q}} = (\hbar\omega_0/2)\ \hat{\sigma}_z$ for a duration ***t***
- Ballistic expansion

$$\mathcal{N}(\boldsymbol{q}, t) = f(\boldsymbol{q}) \left[1 - \sin\theta_{\boldsymbol{q}} \cos(\phi_{\boldsymbol{q}} + \omega_0 t)\right]$$

The measurement of the time evolution of $\mathcal{N}(\boldsymbol{q}, t)$ for a large number of points q in the Brillouin zone gives access to θ_q and ϕ_q and thus Ω_q

Results from the Hamburg experiment

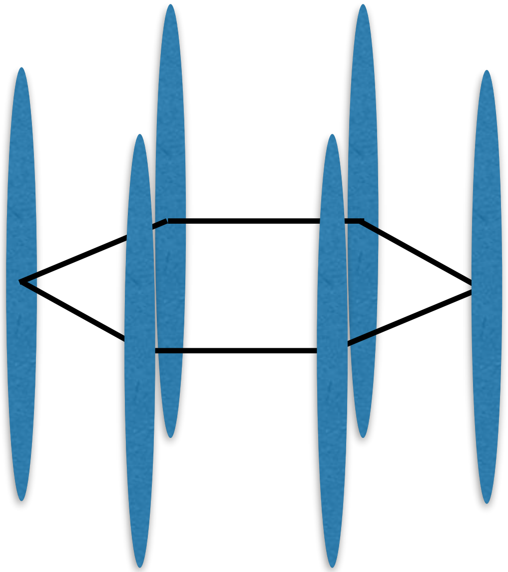
Fläschner, Rem, et al.,
Science (2016)

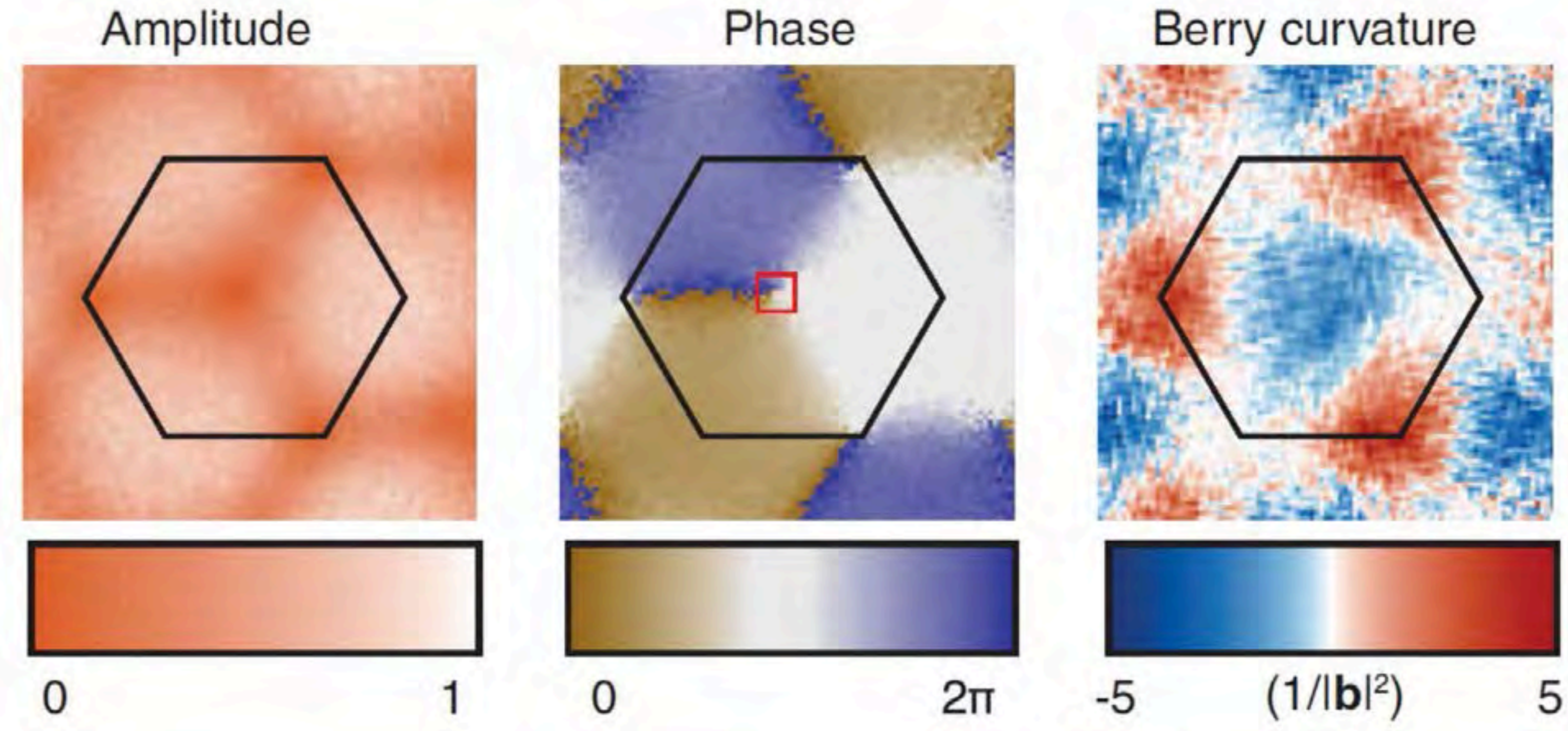
Hexagonal lattice of tubes

^{40}K atoms

$$\mathcal{N}(\boldsymbol{q},t) = f(\boldsymbol{q})\,[1 - \sin\theta_{\boldsymbol{q}}\cos(\phi_{\boldsymbol{q}} + \omega_0 t)]$$

Amplitude : $\sin\theta_{\boldsymbol{q}}$ Phase : $\phi_{\boldsymbol{q}}$





Reconstructed curvature: $\Omega_{\boldsymbol{q}} = \mathrm{i}\,\langle\partial_{q_x}u_{\boldsymbol{q}}^{(-)}|\partial_{q_y}u_{\boldsymbol{q}}^{(-)}\rangle + \text{c.c.} = \frac{1}{2}\sin\theta\,(\boldsymbol{\nabla}_{\boldsymbol{q}}\phi\times\boldsymbol{\nabla}_{\boldsymbol{q}}\theta)\cdot\boldsymbol{u}_z$

For this particular lattice: $\iint \Omega_{\boldsymbol{q}}\,\mathrm{d}^2q = 0$ ***Non topological band***

Conclusion

We now have a physical criterion to characterize the topology of an energy band in a two-dimensional lattice

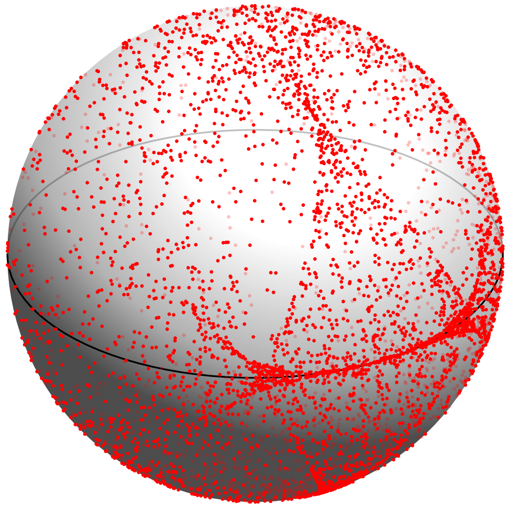
Quantized Hall conductance for a uniformly filled band

$$\sigma_{yx} = \frac{1}{h}\mathcal{C} \qquad \text{with} \qquad \mathcal{C} = \frac{1}{2\pi}\iint_{\mathrm{ZB}} \Omega_{\boldsymbol{q}}\, \mathrm{d}^2 q$$

integer

Central role of Berry curvature

For a two-site unit cell, the criterion $\mathcal{C} \neq 0$ coincides with the condition of full wrapping of the Bloch sphere



What are the physical models leading to such a wrapping?