Quantum field theory
Exercises 11. Solutions
2006-04-03

• Exercise 11.1.

Let us rewrite T-product via theta functions and differetiate it using the rules $\partial_0\theta(x^0) = \delta(x^0)$, $\partial_0\theta(-x^0) = -\delta(x^0)$ (also we can set $y = 0$ without loss of generality, just to shorten the expression)

$$\begin{aligned}
&(\partial^\mu\partial_\mu + m^2)D(x) = (\partial^\mu\partial_\mu + m^2)\langle 0|T(\phi(x)\phi(0))|0\rangle = \\
&= (\partial_0\partial_0 - \partial_i\partial_i + m^2)\langle 0|\theta(x^0)\phi(x)\phi(0) + \theta(-x^0)\phi(0)\phi(x)|0\rangle \\
&= \langle 0|\delta'(x^0)\phi(x)\phi(0) - \delta'(x^0)\phi(0)\phi(x) + 2\delta(x^0)\partial_0\phi(x)\phi(0) - 2\delta(x^0)\phi(0)\partial_0\phi(x) \\
&\quad +\theta(x^0)\partial_0\partial_0\phi(x)\phi(0) + \theta(-x^0)\phi(0)\partial_0\partial_0\phi(x) \\
&\quad +(-\partial_i\partial_i + m^2)T(\phi(x)\phi(0))|0\rangle\ .
\end{aligned}$$

Derivative of the delta-function is defined via integration by parts: $\delta'(t)f(t) = -\delta(t)f'(t)$. So, we get

$$\begin{aligned}
&\langle 0|\delta(x^0)(\partial_0\phi(x)\phi(0) - \phi(0)\partial_0\phi(x)) + (\partial_0\partial_0 - \partial_i\partial_i + m^2)T(\phi(x)\phi(0))|0\rangle \\
= &\langle 0|\delta(x^0)[\partial_0\phi(x),\phi(0)] + (\partial_0\partial_0 - \partial_i\partial_i + m^2)T(\phi(x)\phi(0))|0\rangle\ .
\end{aligned}$$

The commutator in the first term is taken at one moment of time because of the delta-function ($x^0 = 0$), so one can use the expression $[\partial_0\phi(\mathbf{x}),\phi(0)]|_{x^0=0} = -i\delta^{(3)}(\mathbf{x})$. The second term is just the Klein–Gordon equation for the field $\phi(x)$, so it is zero. Finally,

$$\begin{aligned}
&(\partial^\mu\partial_\mu + m^2)D(x) = \\
&= \langle 0|\delta(x^0)(-i\delta^{(3)}(\mathbf{x}))|0\rangle = -i\delta^{(4)}(x)\ .
\end{aligned}$$