Quantum field theory
Exercises 8. Solutions
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• Exercise 8.1.

Let us first obtain the zero momentum solution given in the problem. First, let us notice, that *any* solution of Dirac equation also satisfies the Klein–Gordon equation, and thus should have the form of plane waves with dispersion relation $p^2 = m^2$. Really, if we act on the Dirac equation by the conjugate operator, we get

$$0 = (-i\partial\!\!\!/ - m)(i\partial\!\!\!/ - m)\psi = (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = (\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2)\psi = 0\,,$$

where we used the equation $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + i\sigma^{\mu\nu}$, and the fact that $\partial_\mu\partial_\nu$ is symmetric.

So, generic solution of the Dirac equation should be the sum of the plane waves with all possible $p^2 = m^2$. For each momentum it is

$$\psi(x) = u(p)\mathrm{e}^{-ip_\mu x^\mu} \tag{1}$$

for so called *positive energy* part or

$$\psi(x) = v(p)\mathrm{e}^{ip_\mu x^\mu}$$

for *negative energy* part. The Dirac equation givus us further constraints on the spinors $u(p)$ and $v(p)$.

Let us focus on the positive frequency part. Substituting (1) into the Dirac equation one gets

$$(p\!\!\!/ - m)u(p) = 0\,.$$

In Weyl parametrisation of the gamma matrices, for $p^\mu = (m,\vec{0})$ this just leads to the requirement that upper two components of $u(p)$ are equal to the lower ones, $u_L = u_R$. The choice of normalisation $\sqrt{m}\xi$, given in the problem, is arbitrary (for linear equation with zero right part we can get a solution only up to the overall normalisation) and is a usual convention.

Let us now perform the boost with repidity η (along the z axis, for definiteness)

$$\begin{aligned}
u(p) &= \exp\left\{-\frac{1}{2}\eta\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right\}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \left\{\cosh\frac{\eta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \sinh\frac{\eta}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right\}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \begin{pmatrix}\mathrm{e}^{\eta/2}\left(\frac{1-\sigma^3}{2}\right) + \mathrm{e}^{-\eta/2}\left(\frac{1+\sigma^3}{2}\right) & 0\\ 0 & \mathrm{e}^{\eta/2}\left(\frac{1+\sigma^3}{2}\right) + \mathrm{e}^{-\eta/2}\left(\frac{1-\sigma^3}{2}\right)\end{pmatrix}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\,.
\end{aligned}$$

For rapidity one has the relations, following from Lorentz transformations,

$$\mathrm{e}^{\eta/2} = \sqrt{E + p^3}/\sqrt{m}\,,\quad \mathrm{e}^{-\eta/2} = \sqrt{E - p^3}/\sqrt{m}\,.$$

This leads to the following result for a particle with positive energy moving along the z axis

$$u(p) = \begin{pmatrix}\begin{pmatrix}\sqrt{E-p^3} & 0\\ 0 & \sqrt{E+p^3}\end{pmatrix}\xi\\ \begin{pmatrix}\sqrt{E+p^3} & 0\\ 0 & \sqrt{E-p^3}\end{pmatrix}\xi\end{pmatrix}\,.$$

One can verify, that for general case it can be written in the form

$$u(p) = \begin{pmatrix} \sqrt{p^\mu \sigma_\mu}\xi \\ \sqrt{p^\mu \bar{\sigma}_\mu}\xi \end{pmatrix} ,$$

where square root of the matrix is understood as rotating the matrix to the diagonal form, taking square root of each of the eigenvalues, and then rotating it back.

A similar calculation for the negative energy solution leads to

$$v(p) = \begin{pmatrix} \sqrt{p^\mu \sigma_\mu}\xi \\ -\sqrt{p^\mu \bar{\sigma}_\mu}\xi \end{pmatrix} ,$$

• **Exercise 8.2.**

Let us proove that trace of an odd number of gamma matrices γ^μ is zero.

$$\operatorname{tr}\gamma^\mu = \operatorname{tr}(\gamma^\mu\gamma^5\gamma^5) = -\operatorname{tr}(\gamma^5\gamma^\mu\gamma^5) = -\operatorname{tr}(\gamma^\mu\gamma^5\gamma^5) = -\operatorname{tr}\gamma^\mu ,$$

where we used the fact $\gamma^5\gamma^5 = 1$, then anticommuted γ^μ with γ^5, then used cyclic permutation under the trace. So, the only possibility is $\operatorname{tr}\gamma^\mu = 0$.

Absolutely analogous arguments works for any odd number of gamm matrices, because after odd number of commutations with γ^5 wi get minus sign. That is $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\lambda = 0$.

To calculate trace of two gamma matrices we do

$$\operatorname{tr}\gamma^\mu\gamma^\nu = \operatorname{tr}(2\eta^{\mu\nu}\mathbf{1} - \gamma^\nu\gamma^\mu) = 2\eta^{\mu\nu}\operatorname{tr}(\mathbf{1}) - \operatorname{tr}(\gamma^\mu\gamma^\nu)$$

so, as far as in 4 dimensions $\operatorname{tr}\mathbf{1} = 1$,

$$\operatorname{tr}\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu} .$$

For four gamma matrices

$$\begin{aligned}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho &= 2\eta^{\mu\nu}\operatorname{tr}\gamma^\lambda\gamma^\rho - \operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\rho = 2\eta^{\mu\nu}\operatorname{tr}\gamma^\lambda\gamma^\rho - 2\eta^{\mu\lambda}\operatorname{tr}\gamma^\nu\gamma^\rho + \operatorname{tr}\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\rho \\ &= 2\eta^{\mu\nu}\operatorname{tr}\gamma^\lambda\gamma^\rho - 2\eta^{\mu\lambda}\operatorname{tr}\gamma^\nu\gamma^\rho + 2\eta^{\mu\rho}\operatorname{tr}\gamma^\nu\gamma^\lambda + \operatorname{tr}\gamma^\nu\gamma^\lambda\gamma^\rho\gamma^\mu \\ &= 8\eta^{\mu\nu}\eta^{\lambda\rho} - 8\eta^{\mu\lambda}\eta^{\nu\rho} + 8\eta^{\mu\rho}\eta^{\nu\lambda} + \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\end{aligned}$$

so, finally

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda})$$

For traces with γ^5 we can use the definition $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Then

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^5 = i\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^0\gamma^1\gamma^2\gamma^3 = 0 ,$$

because for $\mu = \nu$ we just get $\operatorname{tr}\gamma^5 = 0$, and for $\mu \neq \mu$ we always get a trace of two different gamma matrices, which is zero.

The last nontrivial trace is

$$\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma^5) = -4i\varepsilon^{\mu\nu\lambda\rho} ,$$

because it is nonzero only for for different indices, antysimmetric under exchange of any two gamma matrices, and thus should be proportional to the epsilon-symbol. And the overall coefficient can be obtained just by calculating

$$i\operatorname{tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3) = -i\operatorname{tr}(\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3) = i\operatorname{tr}(\gamma^2\gamma^3\gamma^2\gamma^3) = i\operatorname{tr}(\gamma^3\gamma^3) = -i\operatorname{tr}(\mathbf{1})$$

• **Exercise 8.3.**

The two Lagrangians differ by a total derivative,

$$\mathscr{L}' = \mathscr{L} - (i/2)\partial_\mu(\bar{\psi}\gamma^\mu\psi) ,$$

so the lead to the same equations of motion for ψ and $\bar{\psi}$. With $\mathscr{L}$ we find $T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathscr{L}$. Note, that if the fields satisfy the equations of motion, then $\mathscr{L} = 0$, so we can write $T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi$. With $\mathscr{L}'$ we get $T\prime\mu\nu = T^{\mu\nu} - (i/2)\partial^\nu j^\mu$, where $j^\mu = \bar{\psi}\gamma^\mu\psi$. Note, that $\mathscr{L}' = 0$ on the equations of motion also. The conservation law $\partial_\mu T^{\mu\nu} = 0$ is not spoiled by this term because $\partial_\mu j^\mu = 0$ (use Dirac equations again here).

The conserved energy-momentum $P^\nu = \int d^3x T^{\nu 0}$ changes by $(-i/2)\int d^3x \partial^\nu j^0$. This is zero because, if ν is a spatial index it is a spatial derivative and the spatial integral vanishes, assuming the fields decrease sufficiently fast at infinity. For $\nu = 0$ we use $\partial_0 j^0 = -\partial_i j^i$ and again get a spatial derivative. Therefore $P^\nu = P'^\nu$.