Quantum field theory
Exercises 7. Solutions
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• **Exercise 7.1.**

Just applying the stadard expression for Euler–Lagrange equations will lead to wrong result, because in this case lagrangian depends also on second derivatives of the field. So one should repeat the variational procedure to find extremum of the action

$$\delta S = \int d^4x \left(-\frac{1}{2}\delta\phi\partial^\mu\partial_\mu\phi - m^2\phi\delta\phi - \frac{1}{2}\phi\delta(\partial^\mu\partial_\mu\phi) \right) .$$

Now, like in the standard derivation, we should integrate last term by parts to make it proportional to $\delta\phi$ (note that $\delta(\partial^\mu\partial_\mu\phi) = \partial^\mu\partial_\mu(\delta\phi)$). After repeating this two times we get

$$\delta S = \int d^4x \left(-\frac{1}{2}\delta\phi\partial^\mu\partial_\mu\phi - m^2\phi\delta\phi - \frac{1}{2}\partial^\mu\partial_\mu\phi\delta\phi \right) + \text{boundary terms} .$$

If fields decay fast enough the boundary terms are zero. So, collecting the terms and requiring $\delta S = 0$ for arbitrary $\delta\phi(x)$, we get

$$\partial^\mu\partial_\mu\phi + m^2\phi = 0 ,$$

i.e. the standard Klein–Gordon equation. One could expect this from the very beginning because the lagrangian density in the problem differs from those of the usual scalar field by full derivative $\partial^\mu(\frac{1}{2}\phi\partial_\mu\phi)$.

• **Exercise 7.2.**

1. d_ϕ must be equal to 1, i.e. mass dimension of ϕ. Then $\partial\phi/\partial x^\mu \to \partial\phi'/\partial x'^\mu = \mathrm{e}^{-2\alpha}\partial\phi/\partial x^\mu$ and $(\partial\phi)^2$ cancels the factor $\mathrm{e}^{4\alpha}$ coming from the measure d^4x. The corresponding current is found using the standard formula and is equal

$$j_D^\mu = (\phi + x^\nu\partial_\nu\phi)\partial^\mu\phi - \frac{1}{2}x^\mu\partial_\nu\partial^\nu\phi .$$

2. $\phi^2 \to \mathrm{e}^{-2\alpha}\phi^2$, so $d^4x\phi^2$ is not invariant.

3. $\phi^4 \to \mathrm{e}^{-4\alpha}\phi^4$, so $d^4x\phi^4$ is invariant. Note, that dialtations are a classical symmetry when there is no explicit mass scale in the theory (no dimesional parameters in the action). So, it is not spoiled bu $\lambda\phi^4$, as far as λ is dimensionless in 4 dimensions.