Quantum field theory Exercises 6. Solutions 2005-12-05

• Exercise 6.1.

To vary this action we should vary over ϕ and ϕ^* as independent variables. Alternatively, one can reexpress the action in terms of real and imaginary parts of ϕ , and use them as independent real fields—this leads to the same result.

Variation over ϕ^* gives

$$\delta S = \int d^4 x \left(\partial^\mu \phi \partial_\mu \delta \phi^* - \lambda (\phi \phi^* - v^2) \phi \delta \phi^* \right)$$

= $\int d^4 x \left(-\partial_\mu \partial^\mu \phi \delta \phi^* - \lambda (\phi \phi^* - v^2) \phi \delta \phi^* \right) + \text{boundary term}.$

The boundary term disappears if we require zero variation on the boundary, and the rest gives (we want δS be zero for any function $\delta \phi^*(x)$) the equation of motion

$$\partial_{\mu}\partial^{\mu}\phi + \lambda(\phi\phi^* - v^2)\phi = 0$$
.

Variation over ϕ is completely analogous and results in the complex conjugate equation

$$\partial_{\mu}\partial^{\mu}\phi^* + \lambda(\phi\phi^* - v^2)\phi^* = 0$$
.

• Exercise 6.2.

The equation of motion for the action is

$$\partial^{\mu}\partial_{\mu}\phi + V'(\phi) = 0 ,$$

Where V' denotes the derivative of the potential $V(\phi)$ over field. Writing explicitly time and spatial derivatives we have

$$\ddot{\phi} - \partial_i \partial_i \phi + V'(\phi) = 0$$
.

Let us now write the time derivative of the energy

$$egin{aligned} &rac{dE}{dt} = \int d^3x \left(\dot{\phi} \ddot{\phi} + \partial_i \phi \partial_i \dot{\phi} + V'(\phi) \dot{\phi}
ight) \ &= \int d^3x \left(\dot{\phi} \ddot{\phi} - (\partial_i \partial_i \phi) \dot{\phi} + V'(\phi) \dot{\phi}
ight) + \int _{ ext{boundary}} d^2 \Sigma_i (\partial_i \phi) \dot{\phi} \ , \end{aligned}$$

where we integrated the second term by parts, and $d^2\Sigma_i$ — surface element (a vector ortogonal to a infinitesimal surface element with the length equal to the surface). The boundary term disappears if we are working in the infinite space, or if we require that field derivatives vanish at the boundary (note, that if these are not the case, then it corresponds to the energy flow through the boundary, and the energy is not conserved inside the boundary).

Changing the $\ddot{\phi}$ in the last expression using the equation of motion (and setting boundary term to zero) we get

$$\frac{dE}{dt} = \int d^3x \left(\dot{\phi}(\partial_i \partial_i \phi - V'(\phi)) - (\partial_i \partial_i \phi) \dot{\phi} + V'(\phi) \dot{\phi} \right) = 0 \,.$$

We checked that the energy is conserved on equations of motion.

• Exercise 6.3.

To find the dimension of the scalar field let us look at the kinetic term

$$S = \int d^4x \frac{1}{2} \partial^\mu \phi \,\partial_\mu \phi \;,$$

The whole integral should be dimensionless, $[S] \sim 1$. Integration measure is 4th power of length, or inverse 4th power of mass $[d^4x] \sim m^{-4}$, each derivative gives one dimension of mass $[\partial_{\mu}] \sim m$. We get

$$1 \sim m^{-4} m^2 [\phi]^2$$
,

so dimension of the field is $[\phi] \sim m$.

Analogous argument in d dimensions gives $[\phi] \sim m^{(d-2)/2}$.