Quantum field theory
Exercises 5. Solutions
2005-11-28

• **Exercise 5.1.**

Average energy of the CMB photon is $E_\gamma = 2.3 \cdot 10^{-13}$ GeV. To find the center of mass energy E_{cm} let us construct the Lorentz invariant $s = (P+q)^2$ (here P and q are four-momenta of the proton and photon, respectively). In the c.m. frame ($\mathbf{p}+\mathbf{q} = 0$) it is just $s = (E_{cm}+E_{\gamma,cm})^2$, i.e. square of the total energy, so we should simply calculate it in the original frame. We have

$$\begin{aligned} s &= (P+q)^2 = (E+E_\gamma)^2 - (\mathbf{p}+\mathbf{q})^2 = E^2 + E_\gamma^2 + 2EE_\gamma - \mathbf{p}^2 - \mathbf{q}^2 - 2\mathbf{pq} \\ &= M^2 + 2EE_\gamma - 2\mathbf{pq} \ . \end{aligned}$$

We are interested in the maximum value, which is obtained when $\mathbf{p}$ and $\mathbf{q}$ are in opposite directions. Also, we may notice that $|\mathbf{q}| = E_\gamma$ ($m_\gamma = 0$), and for large $E \gg M$ also $|\mathbf{p}| \simeq E$. Then

$$s = M^2 + 2EE_\gamma + 2|\mathbf{p}||\mathbf{q}| = M^2 + 4EE_\gamma \ .$$

Or, finally (substracting proton rest mass to get only kinetic energy)

$$\begin{aligned} E_{cm}^{kin} &= \sqrt{s} - M = \sqrt{M^2 + 4EE_\gamma} - M \\ &= M\left(\sqrt{1 + 4E_\gamma \frac{E}{M}} - 1\right) \ . \end{aligned}$$

Comparing E_{cm}^{kin} with the pion mass $m_{\pi^0} = 135$ MeV we get that for $E \sim 3 \cdot 10^{20}$ eV they coincide. Physically, this means that at energies of this order pions (which are the lightest hadrons) can be produced in collisions of protons in cosmic rays with CMB photons, which prevents the protons to travel freely in the Universe. This effect is called *GZK cutoff* (Greisen, Zatsepin, Kuz'min), which says that we should not see cosmic rays with energies larger than $6 \cdot 10^{19}$ eV (we lost a factor of 5 in the estimates, we should have taken into account energy distribution of the photons).

• **Exercise 5.2.**

- Using definitions of σ^μ and $\bar{\sigma}^\mu$ we get for 00 component

  $$\sigma^0\bar{\sigma}^0 + \sigma^0\bar{\sigma}^0 = \mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1} = 2\mathbb{1} \ .$$

  For $0i$ component (the same for $i0$)

  $$\mathbb{1}\sigma^i + \mathbb{1}(-\sigma^i) = 0$$

  For ij component

  $$\sigma^i(-\sigma^j) + \sigma^j(-\sigma^i) = -\{\sigma^i, \sigma^j\} = -2\delta^{ij}\mathbb{1} \ .$$

- For exchanged $\sigma \leftrightarrow \bar{\sigma}$ the calculation is analogous.

- 00 component

  $$\mathrm{tr}(\mathbb{1}\mathbb{1}) = 2 \ .$$

  $0i$ component

  $$\mathrm{tr}(\mathbb{1}(-\sigma^i)) = 0 \ .$$

  ij component

  $$\mathrm{tr}(\sigma^i(-\sigma^j)) = -\,\mathrm{tr}(\delta^{ij}\mathbb{1} + i\varepsilon^{ijk}\sigma^k) = -\,\mathrm{tr}(\delta^{ij}\mathbb{1}) = -2\delta^{ij} \ .$$