Quantum field theory
Exercises 4. Solutions
2005-11-21

• **Exercise 4.1.**

- In the rest fram $E = m$, $\mathbf{p} = 0$. Performing a boost with rapidity η (along x axis for example) we have $E = m\cosh\eta$, $p_x = m\sinh\eta$. Thus, $(E+p)/(E-p) = \mathrm{e}^{2\eta}$.
- Performing a second boost with rapidity η' along x we get $E \to E\cosh\eta' + p\sinh\eta'$ and $p \to E\sinh\eta' + p\cosh\eta'$, so

$$\begin{aligned} \mathrm{e}^{2\eta} &\to \frac{(E\cosh\eta' + p\sinh\eta') + (E\sinh\eta' + p\cosh\eta')}{(E\cosh\eta' + p\sinh\eta') - (E\sinh\eta' + p\cosh\eta')} \\ &= \mathrm{e}^{2\eta'}\frac{E+p}{E-p} = \mathrm{e}^{2(\eta+\eta')} \,. \end{aligned}$$

• **Exercise 4.2.**

Let $v^0 = \xi_R^\dagger \psi_R$ and $v^i = \xi_R^\dagger \sigma^i \psi_R$. Using the rules for spinor transformations under infinitesimal boosts and rotations

$$\psi_R \to \mathrm{e}^{(-i\theta_i+\eta_i)\frac{\sigma^i}{2}}\psi_R = (1 + (-i\theta_i + \eta_i)\frac{\sigma^i}{2} + \ldots)\psi_R \,,$$

we get for v^0

$$\begin{aligned} v^0 &\to \xi_R^\dagger(1 + (+i\theta_i + \eta_i)\frac{\sigma^i}{2})(1 + (-i\theta_i + \eta_i)\frac{\sigma^i}{2})\eta_R = \xi_R^\dagger(1 + \eta_i\sigma^i + \ldots)\eta_R \\ &= v^0 + \eta_i v^i + \ldots \end{aligned}$$

and for v^i

$$\begin{aligned} v^i &\to \xi_R^\dagger(1 + (i\theta_j + \eta_j)\frac{\sigma^j}{2})\sigma^i(1 + (-i\theta_k + \eta_k)\frac{\sigma^k}{2})\eta_R \\ &= \xi_R^\dagger(\sigma^i + \frac{i}{2}\theta_j[\sigma^j, \sigma^i] + \frac{1}{2}\eta_j\{\sigma^j, \sigma^i\} + \ldots)\eta_R = \xi_R^\dagger(\sigma^i - \theta_j\varepsilon^{jik}\sigma^k + \eta_j\delta^{ji} + \ldots)\eta_R \\ &= v^i - \theta_j\varepsilon^{jik}v^k + \eta_j\delta^{ji}v^0 + \ldots \,. \end{aligned}$$

This is exactly transformation properties of a (contravariant) four-vector under infinitesimal boost η_i (boost of rapidity $|\eta|$ in the direction of $\eta^i/|\eta|$). We also checked that it has correct properties under usual rotations θ^i.

Calculations for $\xi_L^\dagger \bar{\sigma}^\mu \eta_L$ are completely analogous.

You can also check everything explicitly for finite boosts, using the formula

$$\mathrm{e}^{\eta_i\sigma^i} = \cosh\eta + \frac{\eta_i}{\eta}\sigma^i\sinh\eta \,,$$

where $\eta = \sqrt{\eta^i\eta^i}$ is rapidity of the boost and η_i/η — its direction.

• **Exercise 4.3.**

Transformations of the double-contravariant tensor are

$$F^{\mu}\nu \to \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma} \ ,$$

where $\Lambda^{\mu}{}_{\rho} = \exp\{-(i/2)\omega_{\alpha\beta}(J^{\alpha\beta})^{\mu}{}_{\rho}\}$, and

$$(J^{\alpha\beta})^{\mu}{}_{\rho} = i(\eta^{\alpha\mu}\delta^{\beta}_{\rho} - \eta^{\beta\mu}\delta^{\alpha}_{\rho}) \ .$$

Then, to the first order

$$\delta F^{\mu\nu} = \omega^{\mu}{}_{\rho}F^{\rho\nu} - \omega^{\nu}{}_{\rho}F^{\rho\mu} \ .$$

Using the explicit form $\omega_{i0} = -\omega^{i0} = -\eta^{i}$ and $\omega_{ij} = \omega^{ij} = \varepsilon^{ijk}\theta^{k}$ one gets for **B** and **E**

$$\delta\mathbf{E} = -[\boldsymbol{\eta}\times\mathbf{B}] + [\boldsymbol{\theta}\times\mathbf{E}] \ ,$$
$$\delta\mathbf{B} = +[\boldsymbol{\eta}\times\mathbf{E}] + [\boldsymbol{\theta}\times\mathbf{B}] \ .$$