Quantum field theory
Exercises 3. Solutions
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• Exercise 3.1. Two body decays

Let us go to the most convenient frame of reference, i.e. rest frame of the decaying particle.

- Momentum conservation gives $\mathbf{p}_1 + \mathbf{p}_2 = 0$, and energy conservation $E_1 + E_2 = M$. Together with usual relativistic expressions for energies of the particles $E_i^2 = \mathbf{p}_i^2 + m_i^2$ this leads to determination of th energies of both particles. So the only free parameters describe the *direction* of the $\mathbf{p}_1$, i.e. two polar angles θ, ϕ.

- The expression for the phase space in this frame is

$$d\Phi^{(2)} = (2\pi)^4 \delta(M - E_1 - E_2)\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2)\frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} .$$

We have six integrations, four of them will be removed by delta-functions (we again arrive to the conclusion that there are two parameters describing the final state). Performing integration over $\mathbf{p}_2$ is trivial and leads to

$$d\Phi^{(2)} = \frac{1}{(2\pi)^2}\delta(M - E_1 - E_2)\frac{1}{4E_1E_2}d^3\mathbf{p}_1 ,$$

where $E_1^2 = m_1^2 + \mathbf{p}_1^2$ and $E_2^2 = m_2^2 + \mathbf{p}_2^2 = m_2^2 + \mathbf{p}_1^2$, because spatial δ-function implies $\mathbf{p}_2 = -\mathbf{p}_1$. Let us change to the polar coordinates for the momentum of the first particle $d^3\mathbf{p}_1 = p_1^2 dp_1 d\Omega$, where $p_1 \equiv |\mathbf{p}_1|$ is absolute value of the momentum and $d\Omega = \sin\theta d\theta d\phi$ is infenitesimal solid angle. Writing explicitly the integral over the p_1

$$d\Phi^{(2)} = \frac{1}{(2\pi)^2}d\Omega\int_0^\infty \delta(M - E_1 - E_2)\frac{1}{4E_1E_2}d^3 p_1^2 dp_1 .$$

Now we use the identity

$$\delta(f(x)) = \sum_j \frac{1}{f'(x_0^j)}\delta(x - x_0^j) ,$$

where sum is over all zeros x_0^j of the function $f(x)$, $f(x_0^j) = 0$. In our case there is only one zero (as far as $p_1 \geq 0$), so

$$d\Phi^{(2)} = \frac{1}{32\pi^2 M^2}\sqrt{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)}\, d\Omega .$$

Identical particles. An important note—if the particles are identical (indistinguishable) then, apart from just setting $m_1 = m_2$, one should integrate only over *half* of the directions $d\Omega$! Or usually one integrates over the whole set of directions and multiplies the answer by $1/2$.

- The decay rate in the rest frame is

$$\Gamma = \int \frac{|\mathscr{M}_{fi}|^2}{2M}d\Phi^{(2)} = \int \frac{|\mathscr{M}_{fi}|^2}{2M}\frac{1}{32\pi^2 M^2}\sqrt{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)}\, d\Omega .$$

If the matrix element is independent on the direction of flight of the decay products (for example, this is the case when the initial particle is scalar) the integration is just 4π, so the decay rate is

$$\Gamma = \frac{|\mathscr{M}_{fi}|^2}{16\pi M^3}\sqrt{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)} \ .$$

• **Exercise 3.2. Three body decay**

We will work in the rest frame of the decaying particle, i.e. $p = (M, 0)$. The three body phase space is

$$d\Phi^{(3)} = \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2}\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3}(2\pi)^4\delta^{(4)}(p_1 + p_2 + p_3 - p) \ .$$

First we perform integration over the $d^3\mathbf{p}_3$ using the spatial part of the δ-function

$$d\Phi^{(3)} = \frac{1}{8(2\pi)^5}\frac{d^3\mathbf{p}_1\, d^3\mathbf{p}_2}{E_1 E_2 E_3}\delta(E_1 + E_2 + E_3 - M) \ ,$$

where again $\mathbf{p}_3 \equiv -(\mathbf{p}_1 + \mathbf{p}_2)$, and $E_i \equiv \sqrt{m_i^2 + \mathbf{p}_i^2}$. We see, that momenta of all the decay products are in one plane, and the configuration is defined by two angles describing the position of the plane, one more angle to define some direction on the plane (say, direction of $\mathbf{p}_1$) and two more parameters, for example E_1 and angle θ_{12} between $\mathbf{p}_1$ and $\mathbf{p}_2$. If we analyze the decay of a spin-0 particle, or average over its spin states, and there is no external fields, then there is no preferred direction for the decay, and the amplitude should not depend on the first three angles, so, changing to polar coordinates for $\mathbf{p}_1$ and $\mathbf{p}_2$ we have

$$d\Phi^{(3)} = \frac{1}{8(2\pi)^5}\frac{4\pi p_1^2 dp_1}{E_1 E_2 E_3}2\pi p_2^2 dp_2 d\cos\theta_{12}\delta(E_1 + E_2 + E_3 - M) \ ,$$

whew the first factor 4π appeared form integration over all directions of $\mathbf{p}_1$, and the second 2π corresponds to the integration over rotations of $\mathbf{p}_2$ around the axis of $\mathbf{p}_1$. Using identities

$$\begin{gathered}
E_1^2 = p_1^2 + m_1^2 \quad \Rightarrow \quad E_1 dE_1 = p_1 dp_1 \\
E_2^2 = p_2^2 + m_2^2 \quad \Rightarrow \quad E_2 dE_2 = p_2 dp_2 \\
E_3^2 = (\mathbf{p}_1 + \mathbf{p}_2)^2 + m_3^2 = p_1^2 + p_2^2 + 2p_1 p_2\cos\theta_{12} + m_3^2 \\
\Rightarrow \quad E_3 dE_3 = p_1 p_2 d\cos\theta_{13} \text{ for fixed } p_1, p_2 \ .
\end{gathered}$$

Now let is perform the integration over $d\cos\theta_{13}$ using the last identity

$$d\Phi^{(3)} = \frac{1}{32\pi^3}dE_1 dE_2 dE_3\delta(E_1 + E_2 + E_3 - M) = \frac{1}{32\pi^3}dE_1 dE_2 \ .$$

Using the results from the exercise 2.1 we can write $ds = -2MdE_1$, $dt = -2MdE_2$, so

$$d\Phi^{(3)} = \frac{ds\, dt}{16M^2(2\pi)^3} \ ,$$

for the case of decaying spin-0 particle without external fields (i.e. no preferred direction in space)

• **Exercise 3.3.**

Writing explicitly the expressions for $d\Phi^{(j)}$ and $d\Phi^{(n-j+1)}$ we get

$$d\Phi^{(n)}(P;p_1,\ldots,p_n) = \int_0^\infty \frac{d\mu^2}{2\pi} \left(\prod_{i=1}^{j} \frac{d^3 p_i}{(2\pi)^3 2E_i}\right) (2\pi)^4 \delta^{(4)}(p_1+\cdots+p_j-q)$$
$$\times \left(\prod_{i=j+1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}\right) \frac{d^3 q}{(2\pi)^3 2q^0} (2\pi)^4 \delta^{(4)}(p_{j+1}+\cdots+p_n+q-P)\ .$$

The first δ-function forces $q = p_1+\cdots+p_j$, so we can insert this into the second δ-function:

$$d\Phi^{(n)}(P;p_1,\ldots,p_n) = \int_0^\infty \frac{d\mu^2}{2\pi} \left[\left(\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}\right) (2\pi)^4 \delta^{(4)}(p_1+\cdots+p_n-P)\right]$$
$$\times \frac{d^3 q}{(2\pi)^3 2q^0} (2\pi)^4 \delta^{(4)}(p_1+\cdots+p_j-q)\ .$$

Now we use the identity, following from the definition of μ^2,

$$\frac{d^3 q}{2q^0} = d^4 q\, \delta(q^2-\mu^2)\theta(q^0)\ ,$$

where $\theta(x)$ is the step function zero for negative arguments and 1 for positive. Then

$$d\Phi^{(n)}(P;p_1,\ldots,p_n) = \left[\left(\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}\right) (2\pi)^4 \delta^{(4)}(p_1+\cdots+p_n-P)\right]$$
$$\times \int_0^\infty \frac{d\mu^2}{2\pi} \int d^4 q\, \delta(q^2-\mu^2)\theta(q^0)\delta^{(4)}(p_1+\cdots+p_j-q)\ .$$

The last two integrals (perform first integration over μ^2) gives 1. What is left is exactly the expression for $d\Phi^{(n)}$.