• **Exercise 2.1.**

First let us notice that all $s$, $t$ and $u$ are Lorentz invariant by construction. So, we can choose any convenient frame of reference to work in, and as far as the result is expressed via $s$, $t$ and $u$ only it can be used in any other reference frame.

The simplest choice is the rest frame of the initial particle, where its energy $E = M$, and spatial momentum is zero $\mathbf{p} = 0$. Then immediately

\[
s = p^\mu p_\mu + p_1^\mu p_{1\mu} - 2p^\mu p_{1\mu} = M^2 + m_1^2 - 2ME_1 ,
\]
\[
t = M^2 + m_2^2 - 2ME_2 ,
\]
\[
u = M^2 + m_3^2 - 2ME_3 .
\]

It is also useful to write another expressions for $s$, $t$ and $u$ using four-momentum conservation $P = p_1 + p_2 + p_3$

\[
s = (p_2 + p_3)^2 , \quad t = (p_1 + p_3)^2 , \quad u = (p_1 + p_2)^2 .
\]

1. Using energy conservation $E_1 + E_2 + E_3 = M$ we get for the sum of the Mandelstam variables

\[
s + t + u = M^2 + m_1^2 + m_2^2 + m_3^2 .
\]

2. Let us first find the limits on $s$. The maximum value is obtained when $E_1$ is at minimum, i.e. $E_1 = m_1$, so

\[
s_{\text{max}} = (M - m_1)^2
\]

(this corresponds to decay when particle 1 is at rest, and particles 2 and 3 are flying in opposite directions). To find minimum $s$ value it is convenient to write it as $s = (p_2 + p_3)^2 = m_2^2 + m_3^2 + 2(E_2E_3 - (\mathbf{p}_2 \cdot \mathbf{p}_3))$. As far as it is Lorentz invariant, we can write it in any frame. Let us use centre of mass frame for the particle 2 and 3. There $\mathbf{p}_3 = -\mathbf{p}_3$ and $s = m_2^2 + m_3^2 + 2(E_2E_3 + |\mathbf{p}_2||\mathbf{p}_3|)$, so the minimum value is obtained for $\mathbf{p}_2 = \mathbf{p}_3 = 0$ (and $E_2 = m_2, E_3 = m_3$), leading to $s_{\text{min}} = (m_2 + m_3)^2$. In conclusion, the limits on $s$ are

\[
(m_2 + m_3)^2 \leq s \leq (M - m_1)^2 .
\]

Now, for each $s$ in this range we find the allowed region for $t$. Let us write $E_3^2 = \mathbf{p}_3^2 + m_3^2$ and use the energy and momentum conservation $E_3 = M - E_1 - E_2$, $\mathbf{p}_3 = -\mathbf{p}_1 - \mathbf{p}_2$ (as usual, in the initial particle rest frame). Then

\[
(M - E_1 - E_2)^2 = m_3^2 + \mathbf{p}_1^2 + \mathbf{p}_2^2 + 2(\mathbf{p}_1 \cdot \mathbf{p}_2) .
\]

The limiting cases correspond to

\[
(\mathbf{p}_1 \cdot \mathbf{p}_2) = \pm |\mathbf{p}_1||\mathbf{p}_2| = \pm \sqrt{(E_1^2 - m_1^2)(E_2^2 - m_2^2)} .
\]

Combining these two expressions and expressing squares of all momenta via energies, we get

\[
M^2 + 2E_1E_2 + m_1^2 + m_2^2 - m_3^2 - 2M(E_1 + E_2) = \pm 2\sqrt{(E_1^2 - m_1^2)(E_2^2 - m_2^2)} .
\]
The only thing left to do is to express $E_1$ and $E_2$ via $s$ and $t$

$$E_1 = \frac{M^2 + m_1^2 - s}{2M}, \quad E_2 = \frac{M^2 + m_2^2 - t}{2M}.$$  

If $m_1 = m_2 = 0$ all the curve with plus sign in the formula becomes $s + t = M^2 + m_3^2$, and with the minus sign $st = m_3^2 M^2$. If $m_3$ is also 0, this reduces to a triangle between the axes and $s + t = M^2$. 
