Quantum field theory
Exercises 2. Solutions
2005-11-07

• **Exercise 2.1.**

First lest us notice that all s, t and u are Lorentz invariant by construction. So, we can choose any convenient frame of reference to work in, and as far as the result is expressed via s, t and u only it can be used in any other reference frame.

The simplest choice is the rest frame of the initial particle, where its energy $E = M$, and spatial momentum is zero $\mathbf{p} = 0$. Then immediately

$$\begin{aligned} s &= p^\mu p_\mu + p_1^\mu p_{1\mu} - 2p^\mu p_{1\mu} = M^2 + m_1^2 - 2ME_1 \ , \\ t &= M^2 + m_2^2 - 2ME_2 \ , \\ u &= M^2 + m_3^2 - 2ME_3 \ . \end{aligned}$$

It is also useful to write another expressions for s, t and u using four-momentum conservation $P = p_1 + p_2 + p_3$

$$s = (p_2 + p_3)^2 \ , \quad t = (p_1 + p_3)^2 \ , \quad u = (p_1 + p_2)^2 \ .$$

1. Using energy conservation $E_1 + E_2 + E_3 = M$ we get for the sum of the Mandelstam variables
$$s + t + u = M^2 + m_1^2 + m_2^2 + m_3^2 \ .$$

2. Let us first find the limits on s. The maximum value is obtained when E_1 is at minimum, i.e. $E_1 = m_1$, so
$$s_{\text{max}} = (M - m_1)^2$$
(this corresponds to decay when particle 1 is at rest, and particles 2 and 3 are flying in opposite directions). To find minimum s value it is convenient to write it as $s = (p_2 + p_3)^2 = m_2^2 + m_3^2 + 2(E_2E_3 - (\mathbf{p}_2 \cdot \mathbf{p}_3))$. As far as it is Lorentz invariant, we can write it in any frame. Let us use centre of mass frame for the particle 2 and 3. There $\mathbf{p}_3 = -\mathbf{p}_3$ and $s = m_2^2 + m_3^2 + 2(E_2E_3 + |\mathbf{p}_2||\mathbf{p}_3|)$, so the minimum value is obtained for $\mathbf{p}_2 = \mathbf{p}_3 = 0$ (and $E_2 = m_2$, $E_3 = m_3$), leading to $s_{\text{min}} = (m_2 + m_3)^2$. In conclusion, the limits on s are
$$(m_2 + m_3)^2 \leq s \leq (M - m_1)^2 \ .$$
Now, for each s in this range we find the allowed region for t. Let us write $E_3^2 = \mathbf{p}_3^2 + m_3^2$ and use the energy and momentum conservation $E_3 = M - E_1 - E_2$, $\mathbf{p}_3 = -\mathbf{p}_1 - \mathbf{p}_2$ (as usual, in the initial particle rest frame). Then
$$(M - E_1 - E_2)^2 = m_3^2 + \mathbf{p}_1^2 + \mathbf{p}_2^2 + 2(\mathbf{p}_1 \cdot \mathbf{p}_2) \ .$$
The limiting cases correspond to
$$(\mathbf{p}_1 \cdot \mathbf{p}_2) = \pm|\mathbf{p}_1||\mathbf{p}_2| = \pm\sqrt{(E_1^2 - m_1^2)(E_2^2 - m_2^2)} \ .$$
Combining these two expressions and expressing squares of all momenta via energies, we get
$$M^2 + 2E_1E_2 + m_1^2 + m_2^2 - m_3^2 - 2M(E_1 + E_2) = \pm 2\sqrt{(E_1^2 - m_1^2)(E_2^2 - m_2^2)} \ .$$

The only thing left to do is to express E_1 and E_2 via s and t

$$E_1 = \frac{M^2 + m_1^2 - s}{2M} , \qquad E_2 = \frac{M^2 + m_2^2 - t}{2M} .$$

If $m_1 = m_2 = 0$ all the curve with plus sign in the formula becomes $s + t = M^2 + m_3^2$, and with the minus sign $st = m_3^2 M^2$. If m_3 is also 0, this reduces to a triangle between the axes and $s + t = M^2$.