Quantum field theory
Exercises 1. Solutions
2005-10-31

• Exercise 1.1.
In SI: \( \hbar \simeq 1.055 \cdot 10^{-34} \) J s;
Using electron charge we get: \( 1 \) GeV = \( 10^9 \) eV \( \simeq 10^9 \cdot 1.602 \cdot 10^{-19} \) J = \( 1.602 \cdot 10^{-10} \) J

\[ \Rightarrow \quad \hbar \simeq 1.055 \cdot 10^{-34} \frac{1}{1.602 \cdot 10^{-10}} \text{ GeV s} \simeq 6.582 \cdot 10^{-25} \text{ GeV s} \]
\( \hbar c \simeq 1.055 \cdot 10^{-34} \text{ Js} \cdot 2.998 \cdot 10^8 \text{ m s}^{-1} \simeq 1.973 \cdot 10^{-16} \text{ GeV m} \)

(and both \( \hbar = \hbar c = 1 \))

1. \( 1 \) m = \( \frac{1}{1.973 \cdot 10^{-16}} \) GeV = \( 5.067 \cdot 10^{15} \) GeV\(^{-1} \) \( \Rightarrow 1 \) cm = \( 5.067 \cdot 10^{13} \) GeV\(^{-1} \)

2. \( 1 \) s = \( \frac{1}{6.582 \cdot 10^{-25}} \) GeV = \( 1.519 \cdot 10^{24} \) GeV\(^{-1} \)

3. \( 1 \) kg = \( 8.988 \cdot 10^{16} \) kg m\(^2\) \( \equiv \) 8.988 \( \cdot 10^{16} \) J = \( \frac{8.988 \cdot 10^{16}}{1.602 \cdot 10^{-19}} \) GeV = \( 5.610 \cdot 10^{26} \) GeV
\[ \Rightarrow 1 \) g = \( 5.610 \cdot 10^{23} \) GeV

4. Boltzmann constant \( k = 1.381 \cdot 10^{-23} \) J K\(^{-1} \) = 8.617 \( \cdot 10^{-14} \) GeVK\(^{-1} \), in \( h = c = 1 \) system \( k \equiv 1 \) also, \( \Rightarrow 1 \) K = 8.617 \( \cdot 10^{-14} \) GeV

5. We are using Heaviside system of units, i.e. \( \varepsilon_0 = \mu_0 = 1 \) if we start from SI—the Coulomb law is \( F = \frac{q_1 q_2}{4 \pi \varepsilon_0 r^2} \). Thus charge is dimensionless. Best thing to remember is the fine structure constant \( \alpha = \frac{e^2}{4 \pi \varepsilon_0 \hbar c} \simeq \frac{1}{137} \). \( \Rightarrow e = \sqrt{4 \pi \alpha} = 0.303 \).

6. There is \( 1/4\pi \) in the Coulomb law and analogous law for magnetic field, but now extra \( 4\pi \) in the Ampere law \( \mathbf{F} = \mathbf{[B} \times \mathbf{I}]L \)

\[ 1 \text{ G} = 10^{-4} \text{ T} = 10^{-4} \text{ kg C}^{-1} \text{s}^{-1} \]
\[ = 10^{-4} \frac{5.610 \cdot 10^{26} \text{ GeV}}{1.519 \cdot 10^{24} \text{ GeV}^{-1} (\sqrt{4\pi/137}/1.6 \cdot 10^{-19})} = 1.95 \cdot 10^{-20} \text{ GeV}^2 \]

7. \( G_N = 6.67 \cdot 10^{-11} \) kg\(^{-1}\)m\(^3\)s\(^{-2}\) = \( 6.67 \cdot 10^{-11} \frac{(5.067 \cdot 10^{15} \text{ GeV}^{-1})^3}{5.610 \cdot 10^{26} \text{ GeV} \cdot (1.519 \cdot 10^{24} \text{ GeV}^{-1})^2} \)
\[ \Rightarrow G_N \simeq 6.703 \cdot 10^{-39} \text{ GeV}^{-2} \]

8. \( H = 100 \text{ km s}^{-1} \text{ Mpc}^{-1} \)
\( 1 \text{ Mpc} = 3.26 \text{LY} = 10^6 \cdot 3.26 \cdot 2.998 \cdot 10^8 \text{m s}^{-1} \cdot 3.600 \cdot 24 \cdot 365.25 \text{s} \simeq 3.084 \cdot 10^{22} \text{m} \)
\( H = 10^5 \text{ms}^{-1} \cdot \frac{1}{3.084 \cdot 10^{22} \text{m}} \simeq 3.242 \cdot 10^{-18} \text{s}^{-1} \)
\[ \Rightarrow H \simeq 3.242 \cdot 10^{-18} \cdot 1.519 \cdot 6.583 \cdot 10^{-25} \text{ GeV} \sim 2 \cdot 10^{-42} \text{ GeV} \]

9. \( \rho_{\text{crit}} = 3H^2 / (8\pi G_N) \sim 8 \cdot 10^{-47} \text{ GeV}^4 \)
• Exercise 1.2.

The only dimensional quantity you have for a gas of massless particles is its temperature $T$ (or, in usual notations, $kT$ for energy). Thus energy density should by something like $\rho_\gamma \sim T^4$. Using result from previous exercise $\rho_\gamma \sim (2.725 \cdot 8.617 \cdot 10^{-14} \text{ GeV})^4 = 3 \cdot 10^{-51} \text{ GeV}^4$. This is about $4 \cdot 10^{-5}$ of the critical density $\rho_{\text{crit}}$. And experimentally we know that overall universe density is quite close to critical, so photons contribute a very small part of the density of the Universe.

• Exercise 1.3.

A temperature $T \simeq 4.5 \cdot 10^6 \text{K}$ corresponds to an energy $\simeq 388 \text{ eV}$. For a relativistic particle at the equilibrium temperature $T$, the average energy is $E_\gamma \simeq 3kT \sim O(1) \text{ keV}$. Since $E_\gamma \ll m_e \ll m_p$, we can use the Thomson formula for the cross section of scattering on electrons and the same formula for protons, only with $m_p$ instead of $m_e$. Therefore, for protons we have $\sigma(\gamma p \rightarrow \gamma p) \simeq 8\pi\alpha^2/(3m_p^2)$. This is smaller than $\gamma e \rightarrow \gamma e$ cross-section by a factor $m_e^2/m_p^2$, and therefore contribution of scattering off the protons is negligible. Because of electric charge neutrality number density for electrons and protons is the same and is $n = \rho/(m_e + m_p) \simeq \rho/m_p \simeq 0.8 \cdot 10^{24} \text{ cm}^{-3}$. Inserting the numerical value for the Thompson cross-section, $\sigma(\gamma e \rightarrow \gamma e) \simeq 6.65 \cdot 10^{-25} \text{ cm}^2$, we find $l \simeq 1.8\text{ cm}$ (more accurate modelling of the Sun gives $l \simeq 0.5\text{ cm}$). The photons therefore perform a random walk of step $l$ inside the Sun. For a random walk in one dimension, after $N$ steps we have $\langle x^2 \rangle = Nl^2$. In three dimensions a radial distance $R_\odot$ is covered in $N$ steps with $R_\odot^3 = (1/3)Nl^2$ because, if we denote by $x$ the axis along which the photon finally escaped, not all steps have been performed along the $x$ direction. Rather in each step $\langle x^2 + y^2 + z^2 \rangle$ increased by $l^2$, so $\langle x^2 \rangle$ effectively performs a random walk of step $l^2/3$. Therefore we get an escape time $t = Nl/c = 4r_\odot^3/(4c) \simeq 3 \cdot 10^3 \text{ yr}$.

• Exercise 1.4.

Let us search for the expression in the form

$$dN = An_1n_2dV dt.$$ 

$dN$ obviously does not depend on frame of reference, $dV dt$ is invariant by definition. So we should find an invariant expression $An_1n_2$, which reduces to $\sigma v_1n_1n_2$ in the rest frame of the particles of type 2.

Let us notice first that the volume element changes with change to reference frame of speed $v$ like

$$V \rightarrow V' = V\sqrt{1-v^2}$$

(the “length” of the box reduces by this factor, and two transverse coordinates does not change with frame of reference). Thus, the number density changes like

$$n \rightarrow n' = \frac{n}{\sqrt{1-v^2}},$$

exactly like the energy of a particle. So, statement that $An_1n_2$ is invariant is equivalent to the statement that

$$A \frac{E_1E_2}{p_1\mu p_2\mu} = A \frac{E_1E_2}{E_1E_2 - p_1p_2} = \text{inv}$$

(the denominator here is invariant—it is a scalar product of four-vectors). In rest frame for particle “2” its energy $E_2 = m_2$, and momentum is zero $p_2 = 0$, so then this invariant is just equal to $A$ in this frame. At the same time it is $\sigma v_{\text{rel}}$ in this frame. So, we have

$$A = \sigma v_{\text{rel}} \frac{p_1^\mu p_2^\mu}{E_1E_2}.$$
To get it into a final form we note, that invariant expression $p_1^{\mu} p_2^\mu$ in the rest frame of particle “2” is

$$p_1^{\mu} p_2^\mu = E_1 E_2 = \frac{m_1}{\sqrt{1 - v_{\text{rel}}^2}} m_2.$$ 

Thus we get

$$v_{\text{rel}} = \sqrt{1 - \frac{m_1^2 m_2^2}{(p_1^{\mu} p_2^\mu)^2}},$$

or, after expressing the energies and momenta of the particles from their speeds $v_1, v_2$ and some algebra

$$v_{\text{rel}} = \sqrt{\frac{(v_1 - v_2)^2 - [v_1 \times v_2]^2}{1 - (v_1 \cdot v_2)}}$$

(notice that this expression is symmetric under exchange of $v_1$ and $v_2$, i.e. it is not important in rest frame of which particle you define relative velocity).

Collecting everything we get the general formula for number of scattering events

$$dN = \sigma \frac{\sqrt{(p_1^{\mu} p_2^\mu)^2 - m_1^2 m_2^2}}{E_1 E_2} n_1 n_2 dV dt$$

$$= \sigma \frac{\sqrt{(v_1 - v_2)^2 - [v_1 \times v_2]^2}}{1 - (v_1 \cdot v_2)} n_1 n_2 dV dt$$

(last one is by W. Pauli, 1933).