**Quantum field theory**
**Exercises 1. Solutions**
2005-10-31

- **Exercise 1.1.**

In SI: $\hbar \simeq 1.055 \cdot 10^{-34}$ J s; $c = 299792458 \ \mathrm{m\,s^{-1}}$
Using electron charge we get: 1 GeV $= 10^9$ eV $\simeq 10^9 \cdot 1.602 \cdot 10^{-19}$ J $= 1.602 \cdot 10^{-10}$ J

$$\Rightarrow \qquad \hbar \simeq \frac{1.055 \cdot 10^{-34}}{1.602 \cdot 10^{-10}} \ \mathrm{GeV\,s} \simeq 6.582 \cdot 10^{-25} \ \mathrm{GeV\ s}$$
$$\hbar c \simeq 1.055 \cdot 10^{-34} \ \mathrm{J\ s} \cdot 2.998 \cdot 10^8 \ \mathrm{m\,s^{-1}} \simeq 1.973 \cdot 10^{-16} \ \mathrm{GeV\,m}$$

(and both $\hbar = \hbar c = 1$)

1. 1 m = $\frac{1}{1.973 \cdot 10^{-16}\ \mathrm{GeV}} = 5.067 \cdot 10^{15}\ \mathrm{GeV}^{-1}$ $\quad \Rightarrow$ 1 cm = $5.067 \cdot 10^{13}\ \mathrm{GeV}^{-1}$
2. 1 s = $\frac{1}{6.582 \cdot 10^{-25}\ \mathrm{GeV}} = 1.519 \cdot 10^{24}\ \mathrm{GeV}^{-1}$
3. 1 kg = $8.988 \cdot 10^{16}\ \mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}^2} = 8.988 \cdot 10^{16}\ \mathrm{J} = \frac{8.988 \cdot 10^{16}}{1.602 \cdot 10^{-10}}\ \mathrm{GeV} = 5.610 \cdot 10^{26}\ \mathrm{GeV}$
   $\Rightarrow$ 1 g = $5.610 \cdot 10^{23}$ GeV
4. Boltzmann constant $k = 1.381 \cdot 10^{-23}\ \mathrm{J\,K^{-1}} = 8.617 \cdot 10^{-14}\ \mathrm{GeV\,K^{-1}}$, in $\hbar = c = 1$ system $k \equiv 1$ also, $\Rightarrow$ 1 K = $8.617 \cdot 10^{-14}$ GeV
5. We are using Heaviside system of units, i.e. $\varepsilon_0 = \mu_0 = 1$ if we start from SI—the Coulomb law is $F = \frac{1}{4\pi}\frac{q_1 q_2}{r^2}$. Thus charge is dimensionless. Best thing to remember is the fine structure constant $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \simeq \frac{1}{137}$. $\Rightarrow e = \sqrt{4\pi\alpha} = 0.303$.
6. There is $1/4\pi$ in the Coulomb law and analogous law for magnetic field, but now extra $4\pi$ in the Ampere law $\mathbf{F} = [\mathbf{B} \times \mathbf{I}]L$

$$\begin{aligned} 1\ \mathrm{G} &= 10^{-4}\ \mathrm{T} = 10^{-4}\ \mathrm{kg\,C^{-1}\,s^{-1}} \\ &= 10^{-4} \frac{5.610 \cdot 10^{26}\ \mathrm{GeV}}{1.519 \cdot 10^{24}\ \mathrm{GeV}^{-1}(\sqrt{4\pi/137}/1.6 \cdot 10^{-19})} = 1.95 \cdot 10^{-20}\ \mathrm{GeV}^2 \end{aligned}$$

7. $G_N = 6.67 \cdot 10^{-11}\ \mathrm{kg^{-1}\,m^3\,s^{-2}} = 6.67 \cdot 10^{-11} \frac{(5.067 \cdot 10^{15}\ \mathrm{GeV}^{-1})^3}{5.610 \cdot 10^{26}\ \mathrm{GeV} \cdot (1.519 \cdot 10^{24}\ \mathrm{GeV}^{-1})^2}$
   $\Rightarrow \qquad G_N \simeq 6.703 \cdot 10^{-39}\ \mathrm{GeV}^{-2}$
8. $H = 100\mathrm{km\,s^{-1}\,Mpc^{-1}}$
   $1\mathrm{Mpc} = 3.26\mathrm{LY} = 10^6 \cdot 3.26 \cdot 2.998 \cdot 10^8 \mathrm{m\,s^{-1}} \cdot 3600 \cdot 24 \cdot 365.25\,\mathrm{s} \simeq 3.084 \cdot 10^{22}\mathrm{m}$
   $H = 10^5 \mathrm{m\,s^{-1}} \cdot \frac{1}{3.084 \cdot 10^{22}\mathrm{m}} = 3.242 \cdot 10^{-18}\mathrm{s}^{-1}$
   $\Rightarrow \qquad H \simeq 3.242 \cdot 10^{-18} \cdot 1.519 \cdot 6.583 \cdot 10^{-25}\ \mathrm{GeV} \sim 2 \cdot 10^{-42}\ \mathrm{GeV}$
9. $\rho_{\mathrm{crit}} = 3H^2/(8\pi G_N) \sim 8 \cdot 10^{-47}\ \mathrm{GeV}^4$

• **Exercise 1.2.**

The only dimensional quantity you have for a gas of massles particles is its temperature $T$ (or, in usual notations, $kT$ for energy). Thus energy density should by something like $\rho_\gamma \sim T^4$. Using result from previous exercise $\rho_\gamma \sim (2.725 \cdot 8.617 \cdot 10^{-14}\ \text{GeV})^4 = 3 \cdot 10^{-51}\ \text{GeV}^4$. This is about $4 \cdot 10^{-5}$ of the critical density $\rho_{\text{crit}}$. And experimentally we know that overall universe density is quite close to critical, so photons contribute a very small part of the density of the Universe.

• **Exercise 1.3.**

A temperature $T \simeq 4.5 \cdot 10^6$K corresponds to an energy $\simeq 388$ eV. For a relativistic particle at the equilibrium temperature $T$, the average energy is $E_\gamma \simeq 3kT \sim O(1)$ keV. Since $E_\gamma \ll m_e \ll m_p$, we can use the Thomson phormula for the crossection of scattering on electrons and the same formula for protons, only with $m_p$ instead of $m_e$. Therefore, for protons we have $\sigma(\gamma p \to \gamma p) \simeq 8\pi\alpha^2/(3m_p^2)$. This is smaller than $\gamma e \to \gamma e$ cross-section by a factor $m_e^2/m_p^2$, and therefore contribution of scattering off the protons is neglidgible. Because of electric charge neutrality number density for electrons and protons is the same and is $n = \rho/(m_e + m_p) \simeq \rho/m_p \simeq 0.8 \cdot 10^{24} cm^{-3}$. Inserting the numerical value for the Thompson cross-section, $\sigma(\gamma e \to \gamma e) \simeq 6.65 \cdot 10^{-25}\text{cm}^2$, we find $l \simeq 1.8$cm (more accurate modelling of the Sun gives $l \simeq 0.5$cm). The photons therephore perform a random walk of step $l$ inside the Sun. For a random walk in one dimension, after $N$ steps we have $\langle x^2 \rangle = Nl^2$. In three dimensions a radial distance $R_\odot$ is covered in $N$ steps with $R_\odot^2 = (1/3)Nl^2$ because, if we denote by $x$ the axis along which the photon finally escaped, not all steps have been performed along the $x$ direction. Rather in each step $\langle x^2 + y^2 + z^2 \rangle$ increased by $l^2$, so $\langle x^2 \rangle$ effectively performs a random walk of step $l^2/3$. Therefore we get an escape time $t = Nl/c = 4r_\odot^3/(lc) \simeq 3 \cdot 10^5$yr.

• **Exercise 1.4.**

Let us search for the expression in the form

$$dN = An_1 n_2 dV\, dt\ .$$

$dN$ obviously does not depend on frame of reference, $dV\,dt$ is invariant by definition. So we should find an invariant expression $An_1n_2$, which reduces to $\sigma v_1 n_1 n_2$ in the rest frame of the particles of type 2.

Let us notice first that the volume element changes with change to reference frame of speed $v$ like

$$V \to V' = V\sqrt{1-v^2}$$

(the "length" of the box reduces by this factor, and two transverse coordinates does not change with frame of reference). Thus, the number density changes like

$$n \to n' = \frac{n}{\sqrt{1-v^2}}\ ,$$

exactly like the energy of a particle. So, statement that $An_1n_2$ is invariant is equivalent to the statement that

$$A\frac{E_1E_2}{p_1^\mu p_{2\mu}} = A\frac{E_1E_2}{E_1E_2 - \mathbf{p}_1\mathbf{p}_2} = \text{inv}$$

(the denominator here is invariant—it is a scalar product of four-vectors). In rest frame for particle "2" its energy $E_2 = m_2$, and momenum is zero $\mathbf{p}_2 = 0$, so then this invariant is just equal to $A$ in this frame. At the same time it is $\sigma v_{\text{rel}}$ in this frame. So, we have

$$A = \sigma v_{\text{rel}} \frac{p_1^\mu p_{2\mu}}{E_1E_2}\ .$$

To get it into a final form we note, that invariant expression $p_1^\mu p_{2\mu}$ in the rest frame of particle "2" is

$$p_1^\mu p_{2\mu} = E_1 E_2 = \frac{m_1}{\sqrt{1 - v_{\rm rel}^2}} m_2 \,.$$

Thus we get

$$v_{\rm rel} = \sqrt{1 - \frac{m_1^2 m_2^2}{(p_1^\mu p_{2\mu})^2}} \,,$$

or, after expressing the energies and momenta of the particles from their speeds $\mathbf{v}_1$, $\mathbf{v}_2$ and some algebra

$$v_{\rm rel} = \frac{\sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - [\mathbf{v}_1 \times \mathbf{v}_2]^2}}{1 - (\mathbf{v}_1 \cdot \mathbf{v}_2)}$$

(notice that this expression is symmetric under exchange of $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e. it is not important in rest frame of which particle you define relative velocity).

Collectiong everything we get the general formula for number of scattering events

$$\begin{aligned} dN &= \sigma \frac{\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2}}{E_1 E_2} n_1 n_2 \, dV \, dt \\ &= \sigma \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - [\mathbf{v}_1 \times \mathbf{v}_2]^2} n_1 n_2 \, dV \, dt \end{aligned}$$

(last one is by W. Pauli, 1933).