Series 5

Coupled mean field approximation

In the mean field approximation the exact interactions between all the particles are replaced with mean values (i.e. mean field). Considering for example a system of N spins σ_i the mean field approximation consists into approximate the probability of one configuration as:

$$P(\{\sigma\}) \approx \prod_{i=1}^{N} P(\sigma_i)$$

We try to do a better approximation. We consider the Ising model in 1 dimensions and we assume that only the one step correlations are important:

$$P(\{\sigma\}) = P(\sigma_1|\sigma_2...\sigma_N)P(\sigma_2|\sigma_3...\sigma_N)...P(\sigma_N) \approx \prod_{i=1}^{N-1} P(\sigma_i|\sigma_{i+1}) = \prod_{i=1}^{N-1} \frac{P(\sigma_i,\sigma_{i+1})}{P(\sigma_{i+1})}$$

a) Write the average free energy, $F = <U> - T <S>$. The system is infinite (N is big) and boundary effects are not important.

- use the mean field approximation when calculating $<S>$;
- $<U>$ can be calculated exactly using $\sum_{s_j=\pm1} P(s_1,...,s_N) = P(s_1,...,s_{j-1},s_{j+1},...,s_N)$.

The solution should be written in term of the variables:

$$\rho_{++} = P(1,1), \quad \rho_{+-} = P(1,-1) = \rho_{-+} = P(-1,1), \quad \rho_{--} = P(-1,-1)$$

and

$$m = \rho_{++} - \rho_{--}$$
$$n = \rho_{++} + \rho_{--}$$

Can you guess the physical meaning of m?

b) The state of the system is the state which minimizes the free energy as a function of m and n. Find the equations which allow to calculate these variables. Is a phase transition predicted (as in the standard mean field approximation)?

c) Write the free energy of the equilibrium state (i.e. the parameters m and n found at the point b). Compare with the exact solution [$F = -k_B T \ln(2\cosh J\beta)$].

Remember the definitions:

$$<U> = -J \sum_{\{\sigma\}} P(\{\sigma\}) \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}$$
$$<S> = -k_B \sum_{\{\sigma\}} P(\{\sigma\}) \ln P(\{\sigma\})$$