Series 3

Exercise 1.

The energy of the Ising model is given by:

$$H = -\sum_{\langle i,j\rangle} J s_i s_j - h\sum_{i=1}^{N} s_i. \tag{1}$$

We have also seen that the free energy per site, in the mean field approach, can be written:

$$f = \frac{F}{N} = -\frac{1}{\beta}\ln 2 - \frac{1}{\beta}\ln[\cosh(\beta Jmz + \beta h)] + Jm^2 z/2 \tag{2}$$

where z is the number of nearest neighbours of every site and $\beta = \frac{1}{k_B T}$.

a) Show that in the mean field approximation of the Ising model there's no phase transition if the magnetic field $h \neq 0$.

b) In the case where $h = 0$, express the *magnetic susceptibility* and the *specific heat* starting from the free energy, considering

$$\chi = \frac{\mathrm{d}m}{\mathrm{d}h} \tag{3}$$

$$c = \frac{\mathrm{d}\varepsilon}{\mathrm{d}T}, \tag{4}$$

where m is the magnetization defined as $m = \frac{1}{N}\sum_{i=1}^{N}\langle\ s_i\rangle$ and ε is the energy per site. Study their behaviour close to the critical point T_C.

Exercise 2.

Considering a chian of N spins $s_i = \pm 1$ in a uniform magnetic field H with periodic boundary conditions, $s_1 = s_{N+1}$. The energy of a configuration is given by

$$H = -J\sum_i \sigma_i\sigma_{i+1} - h\sum_i \sigma_i$$

1. Find an expression for the partition function

   $$Z = \sum_{\{\sigma\}} \exp(-\beta H(\{\sigma\}))$$

   as a function of the matrix

   $$V(\sigma,\sigma') = \exp\left[\beta J\sigma\sigma' + \frac{1}{2}\beta h(\sigma+\sigma')\right], \qquad \sigma,\sigma' = \pm 1$$

2. Write Z as a function of the eigenvalues of V and calculate these eigenvalues.

3. Compute the free energy per site $f(h,t)$, the magnetization $m = -\frac{\partial f}{\partial h}$ and the magnetic susceptibility $\chi = \frac{\mathrm{d}m}{\mathrm{d}h}\Big|_{h=0}$ in the limit $N \to \infty$.

4. Compare these results with the ones obtained with the mean field approximation (cf. cours and série 2).

5. Calculate the expression of the expectation value of a single spin $\langle s_j \rangle$ and of the two-point correlation function $\langle s_j s_m \rangle$ as a function of the matrix V.