Lecture # 9 11.11.2008

Main points of # 8

- found that $R^{\cdot}{}_{\cdot\cdot\cdot} \Longleftrightarrow$ flat space-time
- discussed left hand side of E. equation
- formulated action principle for fields
- considered ED as mechanical system and scalar field as — " —

Today:

- Lagrangians in curved space-time
- Einstein-Hilbert action
- Einstein equations
- Energy-momentum tensor for electrodynamics → to exercise
- energy-momentum tensor for system of point-like particles

Generalisation to curved space time: (2)

(i) in the action, replace d^4x by $\sqrt{-g}\, d^4x$ – invariant volume

(ii) in Lagrangian, replace ordinary derivatives by covariant derivatives

$\Downarrow$

in equations of motion, replace $\eta_{\mu\nu} \to g_{\mu\nu}$, and ordinary derivatives by covariant derivatives

Examples:

– scalar field,

$$S = \int \sqrt{-g}\, d^4x\, \mathcal{L}, \quad \mathcal{L} = \frac{1}{2} g^{\mu\nu} \varphi_{;\mu} \varphi_{;\nu} - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda \varphi^4}{4}$$

eq of motion:

$$\varphi^{;\mu}_{;\mu} - m^2 \varphi - \lambda \varphi^3 = 0$$

– Electrodynamics:

$$S = \int \sqrt{-g}\, d^4x\, \mathcal{L}, \quad \mathcal{L} = -\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$$

eq of motion:

$$F_{\mu\nu}{}^{;\nu} = 0$$

Action for gravitational field:

$\mathcal{L}$ must depend on the metric, and be a scalar function

The simplest possibility:

$$\mathcal{L} = -\frac{1}{16\pi G} R - \frac{\Lambda}{8\pi G}$$

Λ ← cosmological constant

R ← scalar curvature

G – we will see – Newtons constant

Other terms, as $R_{\mu\nu}R^{\mu\nu}$, etc can be added – we will see later that they are irrelevant for all practical purposes

Einstein equations follow from variation of Lagrangian with respect to the metric $g_{\mu\nu}$

Important variations:

$\delta g_{\mu\nu}$: tensor

$\delta g^{\mu\nu}$: not necessarily

$g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu$. $\quad g_{\mu\nu}\,\delta g^{\nu\rho} + g^{\nu\rho}\,\delta g_{\mu\nu} = 0$

$$g:\ \det g = \exp[\mathrm{Tr}\log g];$$

$$\delta g^{\nu\rho} g_{\mu\nu} g^{\mu\alpha} + g^{\nu\rho}\,\delta g_{\mu\nu}\, g^{\mu\alpha} = 0;$$

$$\delta g^{\alpha\rho} = -g^{\alpha\mu}\,\delta g_{\mu\nu}\, g^{\nu\rho}$$

$$\det[g+\delta g] = \exp[\mathrm{Tr}\,[\log(g+\delta g)]] =$$

$$= \exp[\mathrm{Tr}\{\log g + \log(1+g^{-1}\delta g)\}] \approx$$

$$\approx \exp[\mathrm{Tr}\,\log g + \mathrm{Tr}\,g^{-1}\delta g] =$$

$$= g\,(1+\mathrm{Tr}\,g^{-1}\delta g) = g + g\,g^{\mu\nu}\,\delta g_{\mu\nu} \Rightarrow$$

$$\delta g = g\,g^{\mu\nu}\,\delta g_{\mu\nu} = -g\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\,\delta g = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$\delta\int d^4x\,\sqrt{-g}\,R = \delta\int g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\,d^4x =$$

$$= \int\Big[R_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu} + R_{\alpha\beta}g^{\alpha\beta}\Big[-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\Big]$$

$$+ g^{\mu\nu}\sqrt{-g}\,\delta R_{\mu\nu}\Big]d^4x =$$

$$= \int\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big)\sqrt{-g}\,d^4x\,\delta g^{\mu\nu}$$

$$+ \int g^{\mu\nu}\,\delta R_{\mu\nu}\sqrt{-g}\,d^4x$$

To compute the last term, let's use the local lorentzian frame in this coordinate system

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} \text{, and}$$

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\frac{\partial \delta\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - g^{\mu\nu}\frac{\partial \delta\Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} =$$

$$= \frac{\partial}{\partial x^{\lambda}}\Big[\underbrace{g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\alpha}_{\mu\alpha}}_{\omega^{\lambda}}\Big] = \frac{\partial \omega^{\lambda}}{\partial x^{\lambda}}$$

This must be a scalar $\Rightarrow$

$$\omega^{\lambda}{}_{;\lambda}$$

But: $\omega^{\lambda}{}_{;\lambda} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}\,\omega^{\lambda}\right)$

Therefore,

$$\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\,d^4x = \int d^4x \underbrace{\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}\,\omega^{\lambda}\right)}$$

total derivative $\Rightarrow$

can be represented as a surface integral $\Rightarrow$

does not give a variation

So,

$$\delta S_E = \int d^4x \sqrt{-g} \left\{ \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\left(-\frac{1}{16\pi G}\right) - \frac{\Lambda}{8\pi G}\left(-\frac{1}{2} g_{\mu\nu}\right)\right\} \delta g^{\mu\nu} \Rightarrow$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = +16\pi G \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$$

where S_M is the matter action.

$$\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = \frac{1}{2} T_{\mu\nu} \Leftarrow \text{energy-momentum tensor}$$

Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Energy-momentum conservation: $T_{\mu}{}^{\nu}{}_{;\nu} = 0$

follows from $g_{\mu\nu;\rho} = 0$ and Bianchi identity

analogue: $\partial_\mu F^{\mu\nu} = j^\nu \Rightarrow \partial_\nu j^\nu = 0$ in electrodynamics

Examples:

real scalar field:

$$S = \int \sqrt{-g}\, d^4x \left[\frac{1}{2} g^{\mu\nu} \frac{\partial\varphi}{\partial x^\mu} \frac{\partial\varphi}{\partial x^\nu} - \frac{1}{2} m^2 \varphi^2\right]$$

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} \Rightarrow$$

$$\delta S = \int d^4x \left[\mathcal{L}\right]\left(-\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}\right) + \int \sqrt{-g}\, d^4x \cdot \frac{1}{2}\, \delta g^{\mu\nu} \frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} =$$

$$= \int \sqrt{-g}\, d^4x \left\{ \frac{1}{2} \frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{L} \right\} \delta g^{\mu\nu} \Rightarrow$$

$$T_{\mu\nu} = \left(\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} - g_{\mu\nu}\mathcal{L}\right),$$

the same as in flat space

Vector field: (move to exers)

$$S = \int d^4x \sqrt{-g}\left[-\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \qquad \right]$$

$$\delta S = \int d^4x \sqrt{-g}\left[-\frac{1}{2} g_{\alpha\beta}\mathcal{L}\, \delta g^{\alpha\beta} - \frac{1}{2}\delta g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} + \right.$$

$$\left. + \qquad \right] =$$

$$= \frac{1}{2}\int d^4x (\sqrt{-g})\left\{-g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \qquad - g_{\alpha\beta}\mathcal{L}\right\}\delta g^{\alpha\beta}$$

$$T_{\mu\nu} = -g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \qquad - g_{\mu\nu}\mathcal{L}$$

Interesting: $T_{00} = E^2 - \left(\frac{1}{2}E^2 - \frac{1}{2}B^2\right) = \frac{1}{2}E^2 + \frac{1}{2}B^2$ – energy density of EM field

Energy-momentum tensor for system of point-like particles

Action for point-like particle, moving along trajectory

$x^\mu = x^\mu(\tau)$:

$$S = -m \int d\tau \left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}$$

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \cdot 2$$

$$\delta S = m \int \left\{ (g_{\mu\nu} + \delta g_{\mu\nu}) dx^\mu dx^\nu \right\}^{1/2} - m \int \left\{ g_{\mu\nu} dx^\mu dx^\nu \right\}$$

$$= m \int \frac{1}{2} \frac{\delta g_{\mu\nu} dx^\mu dx^\nu}{\sqrt{g_{\mu\nu} dx^\mu dx^\nu}} =$$

$$(a + \delta a)^{1/2} = a^{1/2} \left[1 + \frac{\delta a}{a} \right]^{1/2} = a^{1/2} + \frac{1}{2} \frac{\delta a}{\sqrt{a}}$$

$$= m \int \frac{1}{2} \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \cdot d\tau \cdot d^3x \, dx^0 \, \delta(x^0 - x^0(\tau)) \, \delta^3(x^i - x^i(\tau))$$

introduce proper time: $d\tau = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$

$$\Rightarrow \quad \frac{\delta S}{\delta g_{\mu\nu}} = \int \frac{m}{2} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \, \delta^4(x^\mu - x^\mu(\tau)) \cdot d\tau$$

$$T_{\mu\nu} = + \frac{1}{\sqrt{-g}} \int g_{\mu\alpha} g_{\nu\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} d\tau \, \delta^4(x^\mu - x^\mu(\tau))$$

To prove that this is tensor:

consider transformation of δ-function:

$$dx^\mu \to \frac{\partial x^\mu}{\partial x^{\alpha'}} dx^{\alpha'} \Rightarrow$$

$$\delta^4(x^\mu - x^\mu(\tau)) \to \delta^4(x'^\alpha - x'^\alpha(\tau)) \cdot \frac{1}{\det \frac{\partial x^\mu}{\partial x^{\alpha'}}}$$

$$\sqrt{-g} = \det g_{\mu\nu} \to \sqrt{\left(\det \frac{\partial x^{\alpha'}}{\partial x^\alpha}\right)^2 \cdot \det g'_{\mu\nu}}$$

$$\Rightarrow \frac{1}{\sqrt{-g}} \delta^4(x^\mu - x^\mu(\tau)) \to$$

$$\to \frac{1}{\sqrt{-g'}} \underbrace{\frac{1}{\left(\det \frac{\partial x'}{\partial x}\right)} \delta^4(x' - x'^\alpha) \frac{1}{\det \frac{\partial x}{\partial x'}}}_{=1}$$

QED

for flat space-time:

$$\sqrt{g} \to 1; \quad g_{\mu\nu} \to \eta_{\mu\nu}; \quad \frac{dx^\mu}{d\tau} = p^\mu \Rightarrow$$

$$T_{\mu\nu} = m \int d\tau \, \frac{p_\mu}{m} \frac{p_\nu}{m} \, \delta^3(x^i - x^i(\tau)) \, \delta(x^0 - x^0(\tau))$$

$$d\tau = \sqrt{(1-v^2)} \, dx^0 = \frac{dx_0}{\gamma}$$

$$= \frac{p_\mu p_\nu}{E} \, \delta^3(x^i - x^i(t))$$

$$S_0: \quad T_{00} = E \, \delta^3(x^i - x^i(t))$$

$$T_{0i} = p_i \, \delta^3(x^i - x^i(t))$$

$$T_{ij} = \frac{p_i p_j}{E} \, \delta(x^i - x^i(t))$$

total energy:

$$P_0 = \int T_{00} \, d^3x = E$$

total momentum

$$P_i = \int T_{0i} \, d^3x = p_i$$