Main points of #7

- Considered symmetry properties of the Riemann tensor
- Discussed how to construct local inertial coordinate system
- Proved Bianchi identity
- Defined parallel transport
- Found what happens with vector if it is transported parallel way around closed curve,

\[ \Delta V_x = \frac{1}{2} \int_{\gamma} \left( \sum_{\mu} \nabla_{\mu} \phi \right) \nabla_{\mu} \phi \, dt \]
Today:
- large curves
- Brief of the statement:
  \[ r - \ldots = 0 \implies x \text{, then there exist a coordinate system with } g_{\mu\nu}(x) = \eta_{\mu\nu} \]
- Left hand side of Einstein equations
- Action principle for fields
- Electrodynamics as a mechanical system

Long curve

\[ \Phi = \Phi_0 + \Phi_0 + \Phi \]

\[ \ell_0, \ell_2, \ell_3 \text{ (parts are cancelled away!) } \]
So, if $R$ is zero in some region, then if curve lies in this region,

$$\nabla \times A = 0 \text{ for any curve.}$$

If $R^\mu_{\nu\rho\sigma} = 0$ everywhere, then, if we fixed $V^\mu$ at some point $x_0$, we can define parallel transport of this $V^\mu$ to any point $x$, and this will not depend on path $\Rightarrow$

we can define in this way $V^\mu(x)$ in any point $x$. Since the curves are arbitrary, $(x)$ is valid

for any of them $\Rightarrow$ $V^\mu_{;\beta} = 0$

So, if $R^\nu_{\mu\rho\sigma} = 0$ everywhere, $\exists$ $V^\mu$, for which

$V^\mu_{;\beta} = 0$ everywhere

Contrary:

If we have vector field, for which

$V^\mu_{;\beta} = 0$, then

$R^\mu_{\nu\rho\sigma} V^\rho = 0$ everywhere

\[ (*) : \frac{d V^\mu}{d\tau} = \Gamma^\mu_{\rho\sigma} \frac{d x^\rho}{d\tau} V^\sigma \]
Now we can prove the sufficient condition (if $R^{\cdots} = 0$, then there exists a coordinate system in which $\xi^\mu = \eta^\mu$).

Steps:

1. Choose point $x_0$, and make a coordinate transformation such that $\xi^\mu (x_0) = \eta^\mu$.

Let the basis vectors in this point be

$$\xi^{(i)}\mu = \eta^{(i)}\mu$$

2. Make parallel transport of every vector all over the space-time. Since $R^{\cdots} = 0$, this is possible, and derived vectors depend on $x$, and does not depend on path.

3. Consider

$$\frac{\partial}{\partial x^\mu} \left[ \xi^{(i)\mu}(x) \xi^{(j)\nu}(x) \right] = \xi^{(i)\mu} \xi^{(j)\nu} \xi^{(0)\mu}$$

Since $\xi^{(i)\mu} = 0$, then $\xi^{(i)\mu}(x) \xi^{(j)\nu}(x) = 0$. Hence, 

$$(\xi^{(i)}(x) \xi^{(j)}(x)) = 0.$$
left hand side of E equations

We want to construct a second rank tensor [since we had $T^2 g_{00}$ in
Newtonian limit]

The (almost unique choice)

$$\Delta R_{\mu \nu} + \beta g_{\mu \nu} R + \gamma g_{\mu \nu},$$

with $\alpha, \beta$ and $\gamma$ - arbitrary constants.

To find these constants, and to get
the right - hand side of E.
equations, we will start another
important topic:

Action principle for fields
in analytical mechanics

$q_i$: generalized coordinate

$\dot{q}_i$: generalized velocity

Action principle: $S = \mathcal{L}(q_i, \dot{q}_i)$, and the system moves in such away that action $S = \int dt \mathcal{L}(q_i, \dot{q}_i)$ is minimal.

Field as generalized coordinate: associate with every point of space a generalized coordinate and its velocity: $q^x, \dot{q}^x = \text{classical field}$.

Better notations:

$\varphi(x, t) = \varphi(x^{\mu})$ - scalar field

$A^{\mu}(x^{\nu})$ - vector field

$\eta_{\mu\nu}(x^{\nu})$ - tensor field, metric
Equations of motion: from action principle, scalar function
\[ S = \int d^4x \sqrt{\gamma} \ L \ (\gamma, \partial \gamma) \]

Example:
\[ L (\nu, \partial \nu) \]

Minkowsky space-time

Principle of equivalence \implies\ invariance under all coordinate transformations.

The system moves in such a way that the action stays minimal,
\[ t = t_1 : \ \gamma (t_1, \nu) = \gamma (t) \]
\[ t = t_2 : \ \gamma (t_2, \nu) = \gamma (t) \]

for scalar field, or similar for any other field

Let us take for simplicity flat space-time and scalar field
Consider

\( \psi = \psi_0(\alpha, t) + 8\psi(x, t) \):

\( \psi_0 \) obeys boundary conditions, \( 8\psi(x, t) = 0 \)

at \( t = t_0 \) and \( t = t_2 \).

We also require that \( 8\psi(x, t) \rightarrow 0 \) at

\( x \rightarrow \infty \)

"local" variation of field.

\[ SS = \int d^3x \left[ \frac{\partial}{\partial \psi} \cdot 8\psi + \frac{\partial}{\partial [\partial_t \psi]} - \partial_{\alpha} 8\psi \right] = \]

\[ = \int d^3x \left[ \frac{\partial}{\partial \psi} \cdot 8\psi - \partial_\alpha \left[ \frac{\partial}{\partial \alpha} \right] 8\psi \right] + \]

\[ + \int d^3x \left[ \frac{\partial}{\partial \psi} \cdot 8\psi \right] \bigg|_{t_0}^{t_2} = 0 \text{ because of boundary condition}.

\[ = \int d^3\Sigma \cdot dt \cdot \left[ \frac{\partial}{\partial \psi} \cdot 8\psi \right] = 0 \text{ since}

\[ 8\psi \rightarrow 0 \]

\( \alpha \rightarrow \infty \)
\[ S = \int dx \left[ \frac{\partial L}{\partial \dot{\varphi}} - \dot{\varphi} \left( \frac{\partial L}{\partial \varphi} \right) \right] \Rightarrow \dot{\varphi} = 0 \]

Therefore,

\[ \frac{\partial L}{\partial \dot{\varphi}} - \dot{\varphi} \left( \frac{\partial L}{\partial \varphi} \right) = 0 \] - equations of motion

**Example**

\[ \ddot{\varphi} = \frac{1}{2} \omega_y \dot{\varphi} \dot{\varphi} - \frac{1}{2} \omega_x^2 \varphi^2 - \frac{2}{3} \varphi^3 \]

Variation of the action:

\[ \delta S = \int dx \left[ \frac{\partial L}{\partial \dot{\varphi}} \delta \dot{\varphi} - \frac{\partial L}{\partial \varphi} \delta \varphi \right] = \int dx \left[ \frac{\partial L}{\partial \varphi} \delta \varphi - \frac{\partial L}{\partial \dot{\varphi}} \delta \dot{\varphi} \right] \]

\[ = \int dx \left[ -\frac{1}{2} \omega_y \dot{\varphi} \delta \dot{\varphi} - \frac{1}{2} \omega_x^2 \varphi^2 \delta \varphi - 2 \varphi^2 \frac{\partial \varphi}{\partial \varphi} \delta \varphi \right] \Rightarrow \delta \varphi = 0 \] - equations of motion:

\[ \ddot{\varphi} + m^2 \varphi + 2 \varphi^3 = 0 \]
Electrodynamics as a system with infinite number of degrees of freedom.

Maxwell equations in the vacuum:

\[ \partial_{\mu} F^{\mu} = 0 \Rightarrow \text{div} \vec{E} = 0, \quad \frac{\partial \vec{E}}{\partial t} - \text{rot} \vec{B} = 0 \]

\[ \text{Eqs: } \partial_{\mu} F^{\mu} = 0 \Rightarrow \text{div} \vec{B} = 0, \quad \frac{\partial \vec{B}}{\partial t} + \text{rot} \vec{E} = 0 \]

Can we find Lagrangian from which these equations follow?

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \Rightarrow \text{Eqs: } \partial_{\mu} F^{\mu} = 0 \]

Let us try

\[ L = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \to \]

\[ \delta S = \int \partial \Pi \left[ -\frac{1}{4} [F_{\mu\nu} \delta F^{\mu\nu} + F^{\mu\nu} \delta F_{\mu\nu}] \right] = \]

\[ \int \partial \Pi \left[ -\frac{1}{2} F_{\mu\nu} \delta F^{\mu\nu} = \frac{1}{2} F_{\mu\nu} \left( \delta^\nu_\mu \delta A^\mu - 2 \delta^\nu_\mu \delta A^\mu \right) \right. \]

\[ \left. = \int \partial \Pi \left[ -\frac{1}{2} \partial^\nu F_{\mu\nu} \cdot \delta A^\mu \right] \right] \Rightarrow \]

\[ \partial^\nu F_{\nu\mu} = 0 - \text{QED} \]