Lecture # 8 4.11.2008 (1)

Main points of #7

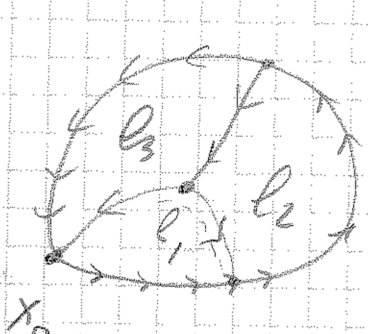
- Considered symmetry properties of the Riemann tensor
- Discussed how to construct local inertial coordinate system
- Proved Bianchi identity
- Defined parallel transport
- found what happens with the vector if it is transported parallel way around closed curve,

$$\Delta V_\alpha = \frac{1}{2} R^{\varkappa}_{\ \alpha\mu\nu} V_\varkappa \oint_{x_0} x^\nu \frac{dx^\mu}{d\tau}$$

Today:

- large curves
- Proof of the statement:
  if $R^{\cdot}_{\cdot\cdot\cdot}=0$ $\forall x$, then there exist
  a coordinate system with $g_{\mu\nu}(x)=\eta_{\mu\nu}$
- Left hand side of Einstein equations
- Action principle for fields
- Electrodynamics as a mechanical system

Large curve:



$$\oint_{l_0} = \oint_{l_1} + \oint_{l_2} + \oint_{l_3}$$

(internal parts are cancelled away!)

So, if R is zero in some region, then if curve lies in this region,

$\Delta V_\mu = 0$ for any curve.

If $R^\alpha_{\mu\rho} = 0$ everywhere, then, if we fixed V_μ at some point x_0, we can define parallel transport of this V to any point x, and this will not depend on path $\Rightarrow$ we can define in this way $V_\mu(x)$ in any point. Since ~~th~~ curves are arbitrary, $(*)$ is valid for any of them $\Rightarrow$ $V_{\alpha;\beta} = 0$

So. if $R^{\cdot}_{\cdot\cdot\cdot} = 0$ everywhere, $\exists$ V_α, for which $V_{\alpha;\beta} = 0$ everywhere

Contrary:

If we have vector field, for which

$V_{\alpha;\beta} = 0$, then

$$R^\alpha_{\mu\rho} V_\alpha = 0 \text{ everywhere}$$

$$(*):\ \frac{dV_\alpha}{d\tau} = \Gamma^\lambda_{\alpha\mu} \frac{dx^\mu}{d\tau} V_\lambda$$

Now we can prove the sufficient (4)

condition (if $R^{\cdot}_{\cdot\cdot\cdot} = 0$, then there exists coordinate system in which $g_{\mu\nu} = \eta_{\mu\nu}$

Steps:

\# 1 choose point x_0, and make a coordinate transformation such that

$g_{\mu\nu}(x_0) = \eta_{\mu\nu}$

let the basis vectors in this point are

$\xi^{(i)}_{\mu}$, $i = 0,1,2,3$; $\mu = 0,1,2,3$

$\xi^{(i)}_{\mu}\, \xi^{(j)\mu} = \eta^{ij}$

\# 2 make parallel transport of every vector all over the space-time. Since $R^{\cdot}_{\cdot\cdot\cdot}$ this is possible, and derived vectors depend on x, and does not depend on path

\# 3 Consider

$$\frac{\partial}{\partial x^{\alpha}}\left[\xi^{(i)}_{\mu}(x)\, \xi^{(j)\mu}(x)\right] = \xi^{(i)}_{\mu;\alpha}\, \xi^{(j)\mu} +$$

$$+\ \xi^{(i)}_{\mu}\, \xi^{(j)\mu}_{;\alpha} = 0, \text{ since } \xi^{i}_{\mu;\nu} = 0 \Rightarrow \xi^{(i)}_{\mu}(x)\, \xi^{(j)\mu}(x)$$

Left hand side of E equations.

We want to construct a second rank tensor [since we had $\vec{\nabla}^2 g_{00}$ in Newtonian limit]

The (almost unique choice)

$$\alpha R_{\mu\nu} + \beta g_{\mu\nu} R + \gamma g_{\mu\nu},$$

with α, β and γ - arbitrary constants.

To find these constants, and to get the right-hand side of E. equations, we will start another important topic:

<u>Action principle for fields</u>

in analytical mechanics:

q_i : generalized coordinate

$\dot{q}_i$: generalized velocity

Action principle: $\mathcal{L} = \mathcal{L}(q_i; \dot{q}_i)$,

and the system moves in such a way

that action $S = \int dt\, \mathcal{L}(q_i; \dot{q}_i)$

is minimal.

Field as generalized coordinate:

associate with every point of space a

generalized coordinate and its velocity:

$q_{\vec{x}}$, $\dot{q}_{\vec{x}}$ = classical field.

Better notations:

$\varphi(\vec{x}, t) = \varphi(x^\mu)$ – scalar field

$A^\mu(x^\nu)$ – vector field

$g_{\mu\nu}(x^\alpha)$ – tensor field, metric

Equations of motion: from action principle,

$$S = \int d^4x \; \overbrace{\mathcal{L}(\varphi, \partial_\mu \varphi)}^{\text{scalar function}}$$

This Example: $\mathcal{L}(A_\mu, A_{\mu;\nu})$

Minkowsky space-time

Principle of equivalence $\Rightarrow$ invariance under all coordinate transformations.

The system moves in such a way that the action stays minimal,

$$t = t_1 : \quad \varphi(\vec{x}, t_1) = \varphi_1(\vec{x})$$

$$t = t_2 : \quad \varphi(\vec{x}, t_2) = \varphi_2(\vec{x})$$

for scalar field, or similar for any other field

Let us take for simplicity flat space-time and scalar field

Consider

$$\varphi = \varphi_0(x,t) + \delta\varphi(x,t) \quad :$$

φ_0 obeys boundary conditions, $\delta\varphi(x,t) = 0$ at $t = t_1$ and $t = t_2$

We also require that $\delta\varphi(x,t) \to 0$ at $\vec{x} \to \infty$

„local" variation of field.

$$\delta S = \int d^4x \left[\mathcal{L}(\varphi + \delta\varphi, \partial_\mu\varphi + \partial_\mu\delta\varphi) - \mathcal{L}(\varphi, \partial_\mu\varphi) \right] =$$

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi} \cdot \delta\varphi + \frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]} \cdot \partial_\mu\delta\varphi \right] =$$

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi} \cdot \delta\varphi - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\right] \cdot \delta\varphi \right] +$$

$$+ \int d^3x \left\{ \frac{\partial\mathcal{L}}{\partial[\partial_0\varphi]} \cdot \delta\varphi \right\} \Big|_{t_1}^{t_2} = 0$$ because of boundary condition

$$+ \oint d\vec{S} \cdot dt \cdot \left\{ \frac{\partial\mathcal{L}}{\partial[\vec{\partial}\varphi]} \cdot \delta\varphi \right\} = 0$$ since $\delta\varphi \to 0$ at $x \to \infty$

surface integral, larger radius $\Rightarrow$

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right] \right] \cdot \delta\varphi = 0$$

Therefore,

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right] = 0$$ – equations of motion

Example

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4} \varphi^4$$

Variation of the action:

$$\delta S = \int d^4x \{ \partial_\mu \varphi \, \partial^\mu \delta\varphi - m^2 \varphi \, \delta\varphi - \lambda \varphi^3 \delta\varphi \} =$$

$$= \int d^4x \{ -\partial_\mu \partial^\mu \varphi - m^2 \varphi - \lambda \varphi^3 \} \delta\varphi \Rightarrow$$

equations of motion:

$$\partial_\mu \partial^\mu \varphi - m^2 \varphi - \lambda \varphi^3 = 0$$

— Electrodynamics as a Lagrangian system with infinite number of degrees of freedom

Maxwell equations in the vacuum

$$\partial_\mu F^{\mu\nu} = 0 \Rightarrow \operatorname{div} \vec{E} = 0 \quad \frac{\partial \vec{E}}{\partial t} - \operatorname{rot} \vec{B} = 0$$

$$\varepsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0 \Rightarrow \operatorname{div} \vec{B} = 0, \quad \frac{\partial \vec{B}}{\partial t} + \operatorname{rot} \vec{E} = 0$$

Can we find Lagrangian from which these equations follow?

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow \varepsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0$$

automatically

Let us try

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Rightarrow$$

$$\delta S = \int d^4x \left\{ -\frac{1}{4} \left[F_{\mu\nu} \delta F^{\mu\nu} + F^{\mu\nu} \delta F_{\mu\nu} \right] \right\} =$$

$$\int d^4x \left\{ -\frac{1}{2} F_{\mu\nu} \delta F^{\mu\nu} \right\} = \int \left\{ -\frac{1}{2} F_{\mu\nu} \left[\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu \right] \right\} =$$

$$= \int d^4x \left[-\frac{1}{2} \partial^\mu F_{\mu\nu} \cdot \delta A^\nu \cdot 2 \right] \Rightarrow$$

$$\partial^\mu F_{\mu\nu} = 0 \text{ — QED}$$