Lecture # 7 28.10.2008

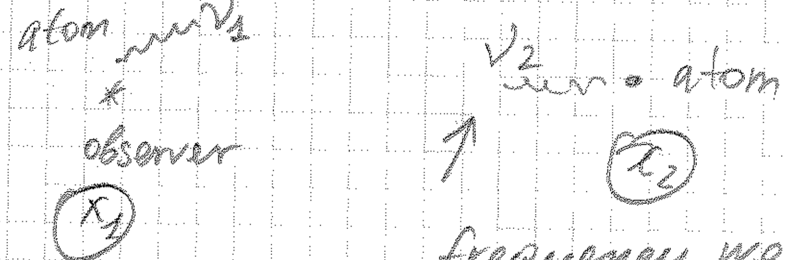
Main points of # 6

- for weak static gravitational fields

  $g_{00} = 1 + 2\varphi$, where $\varphi$ is Newtonian gravitational potential

- considered gravitational red shift:

$$\frac{\nu_2 - \nu_1}{\nu} = \varphi(x_2) - \varphi(x_1)$$



frequency we measure in point 1, but emitted at point $x_2$

- Found that $V_{\mu;\nu;\rho} - V_{\mu;\rho;\nu} = R^{\lambda}{}_{\mu\nu\rho} V_{\lambda}$

  where $R$ is the Riemann curvature tensor

- Defined $R_{\mu\rho} = R^{\lambda}{}_{\mu\lambda\rho}$ — Ricci tensor
- Defined $R = g^{\mu\rho} R_{\mu\rho}$ — scalar curvature

Today: - symmetry properties of

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\rho} R^{\rho}{}_{\mu\nu\kappa}$$

- Bianchi identities
- parallel transport, curved versus flat space-time

Properties:

(1) $\lambda \leftrightarrow \nu$
$\mu \leftrightarrow \varkappa$ $\quad R_{\lambda\mu\nu\varkappa} = R_{\nu\varkappa\lambda\mu}$ (symmetry

(2) $R_{\lambda\mu\nu\varkappa} = -R_{\mu\lambda\nu\varkappa} = R_{\mu\lambda\varkappa\nu} = -R_{\lambda\mu\varkappa\nu}$

(antisymmetry)

(3) Cyclic property

$$R_{\lambda\mu\nu\varkappa} + R_{\lambda\varkappa\mu\nu} + R_{\lambda\nu\varkappa\mu} = 0$$

(4) Ricci : $R_{\mu\varkappa} = g^{\lambda\nu} R_{\lambda\mu\nu\varkappa}$ , $R_{\mu\varkappa} = R_{\varkappa\mu}$

↗ unique tensor of the second kind we can get from Rieman

(5) $R = g^{\mu\nu} R_{\mu\nu}$ : unique scalar we can get from Riemann tensor

Number of independent components of $R_{\alpha\beta\mu\nu}$

$\alpha\beta$ - antisymmetric ⇒ 6 possibilities,

(01) (02) (03) (12) (13) (23)

$\mu\nu$ - antisym. the same - 6 possibilities

symmetry $\alpha\beta \Rightarrow \mu\nu$ :

$$\frac{6\cdot 7}{2} = \frac{42}{2} = 21 \text{ different possibilities}$$

(consider 01 → as one index) [ 6 with the same structure and 15 with different structures

Ciclic relation:

$$\underbrace{R_{0123} + R_{0312} + R_{0231} = 0}_{\text{condition for}} \longrightarrow \text{15 with different structures}$$

So: 20 components = 21 - 1

# of invariant: take a local Lorentz coordinate system and perform there Lor transf, characterized by 6 parameters

as a result, we get

14 = 20 - 6 independent numbers

If $R_{ik} = 0$, then the # of algebraically independent components of $R_{....}$ is $10 = 20 - 10$

10: is a # of components of $R_{ik}$ $\Rightarrow$ the number of invariants in this case is $10 - 6 = \underline{\underline{4}}$

<u>For antisymmetric tensor</u>:

$F_{mn}$

Consider eigenvalues of $F_{mn}$ – they do not depend on coordinate system

# of (different) eigenvalues $= \underline{\underline{2}}$

For symmetric tensor:

# of different eigenvalues $= \underline{\underline{4}}$

Bianchi identities.

Besides algebraic symmetries, the curvature tensor satisfies the Bianchi identity.

It reads:

$$\left[R_{\lambda\mu\nu\varkappa;\eta} + R_{\lambda\mu\eta\nu;\varkappa} + R_{\lambda\mu\varkappa\eta;\nu}\right]=0$$

( cyclic interchange of three last indices)

Consequences: (remove two indices, and use the fact that $g_{\mu\nu;\rho}=0$)

$$g^{\lambda\nu}\left[\qquad\right]=0 \Rightarrow$$

$$R^{\lambda}{}_{\mu\lambda\varkappa;\eta} - R^{\lambda}{}_{\mu\lambda\eta;\varkappa} + R^{\lambda}{}_{\mu\varkappa\eta;\lambda}=0$$

$\nearrow$ due to antisymmetry

Or, we can write this via Ricci tensor as:

$$\left[R_{\mu\varkappa;\eta} - R_{\mu\eta;\varkappa} + R^{\lambda}{}_{\mu\varkappa\eta;\lambda}\right]=0$$

Once more

$$g^{\mu\varkappa}\left[\qquad\right] = R_{;\eta} - \underbrace{R^{\mu}{}_{\eta;\mu} - R^{\nu}{}_{\eta;\nu}}_{\text{the same}} = 0 \quad \Rightarrow$$

$$R^{\mu}{}_{\eta;\mu} - \frac{1}{2}R_{;\eta}=0$$

This can be rewritten as:

$$\left(R^{\mu}{}_{\eta} - \frac{1}{2}\delta^{\mu}_{\eta} R\right)_{;\mu} = 0$$

or as

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0$$

The proof:

Let us take some point $x_0$ and introduce there local Lorentzian coordinate system.

in this system $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$;

$$\Gamma^{\mu}_{\nu\rho}(x_0) = 0$$

This is always possible from physics point of view (equivalence principle)

Mathematically:

(i) take $g_{\mu\nu}(x_0)$ and compute $\Gamma^{\mu}_{\nu\rho}(x_0)$

(ii) introduce new coordinates as

$$x'^{\mu} = x^{\mu} + \frac{1}{2}\Gamma^{\mu}_{\nu\rho}(x_0)(x-x_0)^{\nu}(x-x_0)^{\rho}$$

and compute $\Gamma'^{\mu}_{\nu\rho}(x^0)$ in a new coordinate system with the use of transformation law for $\Gamma$

(iii) introduce new coordinates $x''^{\mu} = O^{\mu}{}_{\nu}\, x'^{\nu}$, where $O$ is an orthogonal matrix which diagonalize $g_{\mu\nu}$:

$$O\, G\, O^T = \text{diag} = \begin{pmatrix} -\lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix}$$

(iv) rescale coordinates as:

$$x'''^{\mu} = \sqrt{\lambda^{\mu}}\, \tilde{x}^{\mu}$$

Then, in terms of $\tilde{x}^{\mu}$ $\tilde{g}_{\mu\nu}(x_0) = \eta_{\mu\nu}$

and $\tilde{\Gamma}^{\mu}_{\nu\rho}(x_0) = 0$

Let us compute now $R_{\lambda\mu\nu\kappa}$ at point $x_0$ and its derivatives:

$$R_{\lambda\mu\nu\kappa}(x_0) = \frac{1}{2}\left[ \frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}} \right] + g_{\eta\sigma}\left(\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu}\right.$$

The second term, <u>and</u> its derivative with respect to $x^\eta$ can be dropped at point $x^0$; in our coordinate system. The covariant derivative of first term can be replaced by ordinary derivative (since $\Gamma^{\cdot}_{\cdot\cdot}(x_0)=0$ $\Rightarrow$ in this coord system

$$R_{\lambda\mu\underbrace{\nu\kappa;\eta}} = \frac{1}{2}\Big[ \frac{\partial^3 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu \partial x^\eta} - \frac{\partial^3 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda \partial x^\eta} - \frac{\partial^3 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu \partial x^\eta} + \frac{\partial^3 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda \partial x^\eta}\Big]$$

Now, write the same for cyclic permutation of $[\nu\kappa\eta]$, and add the results - due to the fact that ordinary derivatives commute, the result is zero.

Since we started from a tensor, and arbitrary point, the result is zero everywhere.

# Curved versus flat space-time

Question:

The space-time with metric $g_{\mu\nu}$ is Minkowskian (flat) if one can find transformation of coordinates $\xi^{\alpha}(x)$, such that

$$\eta^{\alpha\beta} = \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} g^{\mu\nu}$$

for all $x$.

Theorem: the necessary and sufficient conditions to have flat space-time are

(i) $\underline{R}^{\lambda}_{\mu\nu\varkappa} = 0 \quad \forall x$

(ii) for some $x$ $g^{\mu\nu}(x)$ has one negative eigenvalue and 3 positive eigenvalies

necessary condition: obvious
suppose we found such a coordinate system. in it metric is $\eta \Rightarrow R_{\dots} = 0 \Rightarrow$ it is equal to zero in any other coord system.

Theorem from linear algebra:

Suppose

$$G = D^{T}\eta D \quad \overset{\swarrow \text{ transpose}}{} , \text{ where } \det D \neq 0$$

Then $G$ and $\eta$ have the same number of positive, negative and zero eigenvalues

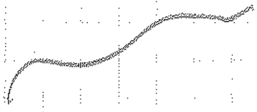
Sufficient condition:

Proof is given below

# Parallel transport and Riemann tensor

What is parallel transport of a vector?

in Euclidean space, cartesian coordinates

 curve, $x^\mu = x^\mu(\tau)$

Some vector — ∥ transport if

$\frac{dV_\alpha}{d\tau}=0$, or $\frac{\partial V_\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tau}=0$ $\quad V_{\alpha,\mu}\frac{dx^\mu}{d\tau}=0$

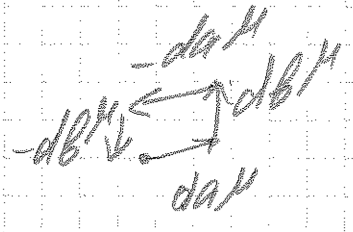
in arbitrary coordinate system:

$V_{\alpha;\mu}\frac{dx^\mu}{d\tau}=0$, or $\frac{\partial V_\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tau} - \Gamma^\lambda_{\alpha\mu} V_\lambda \frac{dx^\mu}{d\tau}=0$,

or $\boxed{\frac{dV_\alpha}{d\tau} = \Gamma^\lambda_{\alpha\mu}\frac{dx^\mu}{d\tau}V_\lambda}$ (*)

By definition: parallel transport is given by (*) in any space-time.

Consider ∥ transport along very small closed countour:



$0 \le \tau \le \varepsilon$ , go around

Shall we get the same vector?

$$x_0^\mu;\quad x_0^\mu + da^\mu;\quad x_0^\mu + da^\mu + db^\mu,\quad x_0^\mu + db^\mu$$

$$\Delta V_\alpha = \oint \Gamma^\lambda_{\alpha\mu} V_\lambda \frac{dx^\mu}{d\tau}\, d\tau$$

Let us compute this integral for very small contour, in the lowest order

$$x_0^\mu = x^\mu(0)$$

$$V_\lambda(x) = V_\lambda(0) + \frac{dV_\lambda}{\partial x^\nu}(x^\nu - x_0^\nu) = V_\lambda(0) + \Gamma^\varkappa_{\lambda\nu}(x^\nu - x_0^\nu) V_\varkappa(0)$$

$$+ \dots$$

$$\Gamma^\lambda_{\alpha\mu} = \Gamma^\lambda_{\alpha\mu}(0) + \frac{\partial \Gamma^\lambda_{\alpha\mu}}{\partial x^\nu}(x^\nu - x_0^\nu) + \dots$$

$$\Delta V_\alpha = \oint \Gamma^\lambda_{\alpha\mu}(0) \cdot V_\lambda(0) \frac{dx^\mu}{d\tau}\, d\tau = 0$$ – equal to zero since $\oint dx^\mu = 0$

$$+ \oint \left[ \frac{\partial \Gamma^\varkappa_{\alpha\mu}}{\partial x^\nu} + \Gamma^\varkappa_{\lambda\nu} \Gamma^\lambda_{\alpha\mu} \right] V_\varkappa (x^\nu - x_0^\nu) \frac{dx^\mu}{d\tau} \cdot d\tau$$

$$+ \dots$$

$$\Delta V_\alpha = \left[\frac{\partial \Gamma^{\varkappa}_{\alpha\mu}}{\partial x^\nu} + \Gamma^{\varkappa}_{\lambda\nu}\Gamma^{\lambda}_{\alpha\mu}\right] V_\varkappa \underbrace{\oint x^\nu \frac{dx^\mu}{d\tau}}_{\text{not zero!}}$$

and antisymmetric with respect to $\nu$ & $\mu$ :

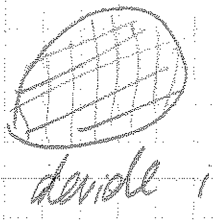
$$\oint x^\nu \frac{dx^\mu}{d\tau} = -\int x^\mu \frac{dx^\nu}{d\tau} \Rightarrow$$

$$\Delta V_\alpha = \frac{1}{2} R^{\varkappa}_{\alpha\mu\nu} V_\varkappa \underbrace{\oint x^\nu \frac{dx^\mu}{d\tau}}_{da^\nu db^\mu - da^\mu db^\nu}$$

for 

Conclusion: if we make ∥ transport of a vector along arbitrary closed curve around the point $x_0$, it will not change if and only if $R^{\varkappa}_{\alpha\mu\nu} = 0$ at this point.

For finite closed curve



devide in N small pieces ⇒

$$\Delta V_\mu = \sum_{n=1}^{N} (\Delta V_\mu)_n$$