Lecture # 6 21.10.2008

Main points of #5

- equivalence principle:
  gravity can be removed in accelerating coordinate system is small vicinity of space-time point (or a line of free-fall)

- found equation for free fall:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

- metric $g_{\mu\nu}$ is in general arbitrary function of x^μ,
  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ but can be converted to $\eta_{\mu\nu}$ at any point x_0.
- action of particle in gravitational field: $S = \int ds = \int \left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{1/2} d\tau$
- geodesic: shortest (in terms of proper time τ) path from A to B

Today:

- Newton approximation
- gravitational red shift
- Curvature tensor
- flat versus curved space-time

Weak gravitational fields:

Newton approximation

Consider motion of the particle if its speed is very small in static gravitational field, take $\mu = i$ in eq (*)

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

$ds \approx dt \Rightarrow$

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00} = 0 \;, \quad \text{since } \frac{dx^i}{ds} \approx \frac{dx^i}{dt} = v \ll 1$$

gravitational field: metric,

consider $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$, with $\delta g_{\mu\nu}$ small

$$\Rightarrow \Gamma^i_{00} = \frac{1}{2}\eta^{i\alpha}\left(\delta g_{\alpha 0,0} + \delta g_{\alpha 0,0} - \delta g_{00,\alpha}\right)$$

$$g^{\mu\nu} = \eta^{\mu\nu} + O(\delta g_{\mu\nu})$$

$= 0$, since the field is static $\Rightarrow$

$$\Gamma^i_{00} = -\frac{1}{2}\eta^{i\alpha}\frac{\partial}{\partial x^\alpha}\delta g_{00} = \frac{1}{2}\frac{\partial}{\partial x^i}\delta g_{00}$$

So, we get

$$\frac{d^2 x^i}{dt^2} + \frac{1}{2}\frac{\partial}{\partial x^i}\delta g_{00} = 0$$

equation of motion with the use of gravitational potential φ

$$\frac{d^2\vec{x}}{dt^2} + \vec{\nabla}\varphi = 0$$

(for point-like source

$$\varphi = -G_N \frac{M}{r}$$

So, we get that for static fields

$g_{00} \simeq 1 + 2\varphi$ other components of the metric cannot be fixed by this eqn. alone

Numerically, surface of Earth: $\varphi \simeq 10^{-9}$

$\Rightarrow$ deviation from Lorentz case is small.

Larger for sun (check!)

for neutron star (check!)

Change of the scale of time

Clock which do not move in arbitrary coordinate system in gravitational field:

$$ds^2 = d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu ,$$

$$dx^i = 0 \Rightarrow d\tau^2 = g_{00} dx^{0\,2} \Rightarrow$$

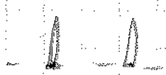
finite time interval: $\tau = \int \sqrt{g_{00}}\, dx^0$

true time — coordinate time

This cannot be observed if a process in <u>one</u> spacial point is considered, since time is changed not only for clock, but for all physical processes

Idea:

$\cdot 2$ ← atomic transition



$\cdot 2$ ← exactly the same atomic transition

Consider this in some coordinate system

let $\Delta\tau$: true time interval between pulses $\Rightarrow$

$$dx_1^0 = \frac{\Delta\tau}{\sqrt{g_{00}(x_1)}}\ ; \quad dx_2^0 = \frac{\Delta\tau}{\sqrt{g_{00}(x_2)}}$$

So: $$\frac{dx_1^0}{dx_2^0} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} = \frac{\nu_2}{\nu_1}$$

$$\nu_2 = \nu_1 \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} \approx \nu_1 \left(\frac{1+2\varphi(x_2)}{1+2\varphi(x_1)}\right)^{1/2} \approx$$

$$\approx \nu_1 \left(1 + \varphi(x_2) - \varphi(x_1)\right), \text{ or}$$

$$\frac{\nu_2 - \nu_1}{\nu} = \varphi(x_2) - \varphi(x_1)$$

Sun $\cdot \nu_s$ frequency from atom in the Earth $\nu_e \cdot$ Earth

$$\frac{\nu_s - \nu_e}{\nu} = \varphi_s - \varphi_E = -G_N \frac{M_\odot}{R_\odot} \approx -2 \cdot 10^{-6}$$

frequency we get from atom in the sun

negative: red shift

Not easy to see: it is difficult to disentangle the Doppler effects.

Experiments in Laboratory:

Messbauer effect

- Fe^{57}, 14.4 keV line

22.6 m $\qquad \Delta\varphi = -2.46 \cdot 10^{-15}$

resonance absorption, it may go out of resonance $\qquad$ frequences are the same.

width of the line: $\frac{\Gamma}{\nu} = 1{,}3 \cdot 10^{-12}$

<u>Principe of general covariance:</u> — another form of equivalence principle

- take any equation which is Lorentz-invariant
- write it in any coordinate system
- consider the metric as an arbitrary function, not necessarily the one which corresponds to Minkowski space

why? we can consider a frame, which eliminates gravity in a small vicinity of some point, where equation has Lorentzian character

Remark: in all formalism which we already developed is applicable for general metric $g_{\mu\nu}$, not necessarily related to the minkowskian one

Now our aim is to find equations for gravitational field itself.

Requirements :

- transformation properties of lhs must be the same as tr properties of rhs
- For weak static fields we must reproduce poisson equation,

$$\nabla^2 \varphi = G_N \cdot \rho$$

Hint : $g_{00} = 1 + 2\varphi \Rightarrow$

$$\nabla^2 \varphi = \frac{1}{2} \nabla^2 g_{00} \Rightarrow$$

most probably the equation must have the form [second order derivatives] $g_{\mu\nu} = T_{\mu\nu}$

second rank tensors

Problem for today and next week - determine left-hand side

We should construct a tensor from 1st and second derivatives of the metric

What do we know:

$g_{\mu\nu;\alpha} = 0$ - cannot be used.

Go to page 11
Leave pages 9,10 for exercises

Idea: use affine connection, Γ (Christ. symbols) and try to construct tensor from it

Let's use definition:

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$

where ξ^{α} are local-inertial coordinates (associated with free fall)

$$\xi^{\alpha} = \xi^{\alpha}(x')$$

Let us use now other coordinate system, $x' = x'(x) \Rightarrow$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} =$$

$$= \frac{\partial x^{\lambda}}{\partial x'^{\beta}} \frac{\partial x'^{\beta}}{\partial \xi^{\alpha}} \cdot \frac{\partial}{\partial x^{\mu}} \left[\frac{\partial \xi^{\alpha}}{\partial x'^{\rho}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} \right]$$

$$= \frac{\partial x^{\lambda}}{\partial x'^{\beta}} \frac{\partial x'^{\beta}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\gamma} \partial x'^{\rho}} \frac{\partial x'^{\gamma}}{\partial x^{\mu}} \frac{\partial x'^{\rho}}{\partial x^{\nu}}$$

$$+ \frac{\partial x^{\lambda}}{\partial x'^{\beta}} \underbrace{\frac{\partial x'^{\beta}}{\partial \xi^{\alpha}} \cdot \frac{\partial \xi^{\alpha}}{\partial x'^{\rho}}}_{\delta^{\beta}_{\rho}} \frac{\partial^2 x'^{\rho}}{\partial x^{\mu} \partial x^{\nu}} =$$

$$= \underbrace{\frac{\partial x^{\lambda}}{\partial x'^{\beta}} \frac{\partial x'^{\gamma}}{\partial x^{\mu}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} \Gamma'^{\beta}_{\gamma\rho}}_{\text{like tensor}} + \underbrace{\frac{\partial x^{\lambda}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\beta}}{\partial x^{\mu} \partial x^{\nu}}}_{\text{extra piece}}$$

Idea: get some combination of Γ which transforms like tensor

write:

$$\frac{\partial^2 x'^{\beta}}{\partial x^{\mu} \partial x^{\nu}} = \frac{\partial x'^{\beta}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu\nu} - \frac{\partial x'^{\gamma}}{\partial x^{\mu}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} \Gamma'^{\beta}_{\gamma\rho}$$

Consider

$$\frac{\partial^3 x'^\beta}{\partial x^\mu \partial x^\nu \partial x^\alpha} = \frac{\partial^2 x'^\beta}{\partial x^\lambda \partial x^\alpha} \cdot \Gamma^\lambda_{\mu\nu} + \frac{\partial x'^\beta}{\partial x^\lambda} \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\alpha}$$

$$- \frac{\partial^2 x'^\gamma}{\partial x^\mu \partial x^\alpha} \frac{\partial x'^\rho}{\partial x^\nu} \Gamma'^\beta_{\gamma\rho} - \frac{\partial x'^\gamma}{\partial x^\mu} \frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\nu} \Gamma'^\beta_{\gamma\rho}$$

$$- \frac{\partial x'^\gamma}{\partial x^\mu} \frac{\partial x'^\rho}{\partial x^\nu} \frac{\partial \Gamma'^\beta_{\gamma\rho}}{\partial x'^\delta} \frac{\partial x'^\delta}{\partial x^\alpha}$$

Further steps:
- use (*) to replace second derivatives
- write the same equation with $\mu \leftrightarrow \alpha$
- take the difference between them

Result:

$$\frac{\partial x'^\tau}{\partial x^\lambda} R^\lambda_{\mu\nu\varkappa} - \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \frac{\partial x'^\eta}{\partial x^\varkappa} R'^\tau_{\rho\sigma\eta} = 0$$

or
$$R'^\tau_{\rho\sigma\eta} = \frac{\partial x'^\tau}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} \frac{\partial x^\varkappa}{\partial x'^\eta} R^\lambda_{\mu\nu\varkappa}$$

where

$$R^\lambda_{\mu\nu\varkappa} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\varkappa} - \frac{\partial \Gamma^\lambda_{\mu\varkappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\varkappa\eta} - \Gamma^\eta_{\mu\varkappa} \Gamma^\lambda_{\eta\nu}$$

Riemann curvature tensor

(different in sign with Landau–Lifshits, but like in Weinberg)

Another, shorter way to find R.

Consider some vector field V_μ and its second covariant derivatives:

$$V_{\mu;\nu;\rho} - V_{\mu;\rho;\nu}$$

Computation: $V_{\mu;\nu} \equiv T_{\mu\nu} = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^{\varkappa}_{\mu\nu} V_\varkappa$

$$T_{\mu\nu;\rho} = \frac{\partial T_{\mu\nu}}{\partial x^\rho} - \Gamma^{\lambda}_{\mu\rho} T_{\lambda\nu} - \Gamma^{\lambda}_{\nu\rho} T_{\mu\lambda} =$$

$$= \frac{\partial}{\partial x^\rho}\left[\frac{\partial V_\mu}{\partial x^\nu} - \Gamma^{\varkappa}_{\mu\nu} V_\varkappa\right] - \Gamma^{\lambda}_{\mu\rho}\left[\frac{\partial V_\lambda}{\partial x^\nu} - \Gamma^{\varkappa}_{\lambda\nu} V_\varkappa\right]$$
$$- \Gamma^{\lambda}_{\nu\rho}\left[\frac{\partial V_\mu}{\partial x^\lambda} - \Gamma^{\varkappa}_{\mu\lambda} V_\varkappa\right]$$

$$= \frac{\partial^2 V_\mu}{\partial x^\rho \partial x^\nu} - \frac{\partial \Gamma^{\varkappa}_{\mu\nu}}{\partial x^\rho} V_\varkappa - \Gamma^{\varkappa}_{\mu\nu}\frac{\partial V_\varkappa}{\partial x^\rho} - \Gamma^{\lambda}_{\mu\rho}\frac{\partial V_\lambda}{\partial x^\nu} - \Gamma^{\lambda}_{\nu\rho}\frac{\partial V_\mu}{\partial x^\lambda}$$
$$+ \left(\Gamma^{\lambda}_{\mu\rho}\Gamma^{\varkappa}_{\lambda\nu} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\varkappa}_{\mu\lambda}\right) V_\varkappa$$

symmetric $\{\nu,\rho\}$

symmetric $\{\nu,\rho\}$

symmetric $\{\nu,\rho\}$

symmetric $\{\nu,\rho\}$

Difference $V_{\mu;\nu;\rho} - V_{\mu;\rho;\nu} =$

$$= \frac{\partial \Gamma^{\alpha}_{\mu\rho}}{\partial x^{\nu}} V_{\alpha} - \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\rho}} V_{\alpha} + \left(\Gamma^{\lambda}_{\mu\rho} \Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu} \Gamma^{\alpha}_{\lambda\rho}\right) \cdot V_{\alpha}$$

$$= R^{\alpha}_{\mu\rho\nu} V_{\alpha}$$

$R^{\alpha}_{\mu\rho\sigma}$ is a tensor, since left-h. side is a tensor.

Other non-trivial tensors we can get from $R^{\alpha}{}_{\mu\rho\sigma}$: Ricci: $R_{\mu\alpha} = R^{\lambda}{}_{\mu\lambda\alpha}$; $R = g^{\mu\nu} R_{\mu\nu}$

<u>Symmetry properties</u> of Riemann tensor:

Define $R_{\lambda\mu\nu\alpha} = g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu\alpha}$

use expression for $\Gamma^{\mu}_{\nu\rho}$ through derivatives of the metric tensor [for $\frac{\partial \Gamma}{\partial x}$]

get result:

$$R_{\lambda\mu\nu\alpha} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\alpha} \partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\alpha}}{\partial x^{\nu} \partial x^{\mu}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\lambda}}\right]$$

$$+ g_{\eta\sigma}\left[\Gamma^{\eta}_{\nu\lambda} \Gamma^{\sigma}_{\mu\alpha} - \Gamma^{\eta}_{\alpha\lambda} \Gamma^{\sigma}_{\mu\nu}\right]$$