Lecture #5 14.10.2008

Main points of #4

- defined $E^{\alpha\beta\sigma\delta} = \frac{1}{\sqrt{-g}} \varepsilon^{\alpha\beta\gamma\delta}$

$$\varepsilon^{0123} = 1$$

- Considered physics case for Newtons law and for equality of inertial and gravitational mass
- Attempted to make a relativistic theory of gravity with the use of vector field : $A_\mu \Rightarrow G_\mu$ and scalar field :

$$\Box \varphi = \rho ; \quad m \frac{d^2 x^\gamma}{d s^2} = - \nabla^\gamma \varphi$$

(did not work)

Today:

- formulation of equivalence principle
- equation for free fall
- gravitational field and the metric
- free fall and the action principle, geodesic

Once more: equivalence principle

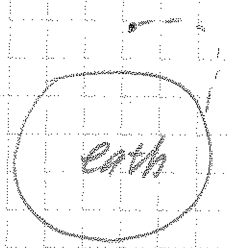
gravitational mass = inertial mass

$\Downarrow$

(i) take inertial frame + gravitational field

Start at $t=0$ and $\vec{x}=0$ with any mass — and any velocity

They all will move along the same trajectory



(ii) take vacuum (no gravitational field) and consider the motion of any body in non-inertial accelerating frame: all bodies will move exactly in the same way.

$\Downarrow$

non-inertial system of coordinate is equivalent to some gravitational field

Inverse statement:

gravitational field in some point of space and time can be excluded in a particular accelerating frame

Example:

falling lift: weightlessness!

Equations for free fall in a gravitational field

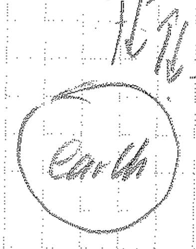
Take $\overbrace{\text{coordinate}}^{\text{accelerated}}$ system which falls together with the body:

$$\xi^\alpha = \xi^\alpha(x'^i) \quad ; \quad x'^i \text{ – Cartesian coordinates.}$$

In this coordinate system gravity is removed by acceleration – no forces act on particle $\Rightarrow$ particle moves with constant (zero) velocity, and the physics looks like physics in inertial frame

← coordinate system

← particle trajectory



$$\frac{d^2\xi^i}{dt^2}=0 \quad \underbrace{i=1,2,3}$$

Relativistic generalisation: $\frac{d^2\xi^\alpha}{ds^2}=0$ → $\alpha = 0,1,2,3$

where ds is the interval,

$$ds^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

Minkowsky metric!

(gravity was killed by acceleration!)

Equivalence principle!

Let us rewrite this equation in some other coordinate system (arbitrary) x^μ:

x^μ may be laboratory frame but may be some other as well.

Let us derive equations for coordinates of the particle in other coordinate system

$$\frac{d}{ds}\left[\frac{d\xi^\alpha}{ds}\right] = \frac{d}{ds}\left[\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{ds}\right] =$$

$$= \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{ds^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

multiply by $\frac{\partial x^\lambda}{\partial \xi^\alpha}$:

$$\underbrace{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu}}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\frac{\partial x^\lambda}{\partial x^\mu} = \delta^\lambda_\mu \Rightarrow$$

$$\frac{d^2 x^\lambda}{ds^2} + \underbrace{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}} \cdot \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

what is this object?

Introduce notations:

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \bar{\Gamma}^\lambda_{\mu\nu}$$

We will see that this is in fact the Christoffel symbol! affine connection

also, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \qquad (*)$$

minkowski

Proof that $\bar{\Gamma}^\lambda_{\mu\nu}$ is indeed the Chr. symbols: $\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}$

Take $(*)$ and find its derivative with respect to x^λ:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\lambda}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

$$+ \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}\eta_{\alpha\beta}$$

$$= \frac{\partial^2 \xi^\gamma}{\partial x^\mu \partial x^\lambda}\frac{\partial x^\rho}{\partial \xi^\gamma}\underbrace{\frac{\partial \xi^\alpha}{\partial x^\rho}\cdot\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}}_{g_{\rho\nu}}$$

$$+ \frac{\partial^2 \xi^\gamma}{\partial x^\lambda \partial x^\nu}\frac{\partial x^\rho}{\partial \xi^\gamma}\underbrace{\frac{\partial \xi^\beta}{\partial x^\rho}\frac{\partial \xi^\alpha}{\partial x^\mu}\eta_{\alpha\beta}}_{g_{\rho\mu}} =$$

$$= \overline{\Gamma}^{\rho}_{\mu\lambda} g_{\rho\nu} + \overline{\Gamma}^{\rho}_{\lambda\nu} g_{\rho\mu}$$

But this is exactly relation from the 3rd lecture $\Rightarrow$

$$\overline{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}$$

Once more, eq of motion:

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma^{\lambda}_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

This in analogue of relativistic Newton equation, which we had for electrodynamics

$$m \frac{d^2 x^\mu}{ds^2} = q F^{\mu}{}_{\nu} \frac{dx^\nu}{ds}$$

electric and magnetic field

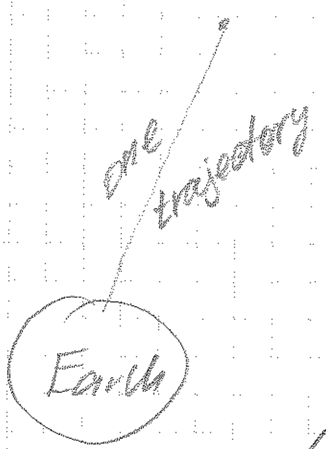
So, $\Gamma^{\lambda}_{\mu\nu}$ play a role of gravitational field.

As they are expressed through the metric $g_{\mu\nu}$, we can say that gravitational field is metric $g_{\mu\nu}$

Very important

The metric in gravitational and Laboratory frame, associated, e.g. with Earth CAN NOT be considered as a coordinate transformation of Minkowski space !!

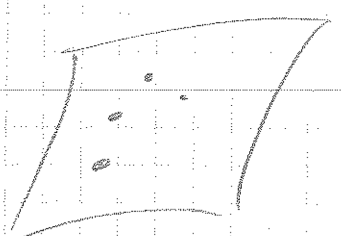
Why? We considered just one trajectory. On this line $x^\mu = x^\mu(t)$, and in the free-falling coord system we have Lorentzian metric



Coord of Lab frame

We have to cover all space by these lines and consider many coord. systems. When we do that, we get

$$g_{\mu\nu}(x)$$



↑ Plane through which our trajectories are going throug

8

We can choose a coorginate system in which the transformed $g_{\mu\nu}$ looks like $\eta_{\mu\nu}$ on a line, but not on the whole space.

Conclusion: the space-time in the precence of gravity is not minkowskinan

For Mink space-time there exist a coordinate transformation that changes $g_{\mu\nu}$ to $\eta_{\mu\nu}$ at any point $\Rightarrow$ In the precence of gravity this is not the case.

Guess: gravitational field is an arbitrary apriori function $g_{\mu\nu}$ - metric!
symmetric
(tensor of the second kind)

So: - scalar - bad
- vector - bad
- antisymmetric tensor - bad
- symmetric tensor - may be ok

Remark: what is equation of motion for a massless particle, like photon

Problem: $ds^2 = 0$ - we cannot use it for equation!

So - use $\xi \equiv \sigma$: time in locally Lorentzian coordinate system

$$\frac{d^2\xi^\alpha}{d\sigma^2} = 0; \quad \eta_{\alpha\beta} \frac{d\xi^\alpha}{d\sigma} \frac{d\xi^\beta}{d\sigma} = 0$$

in arbitrary coordinate system:

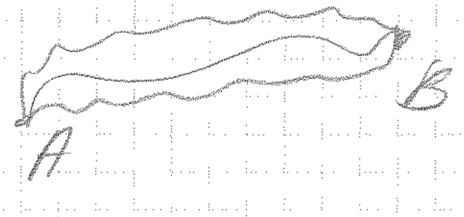
$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\sigma} \frac{dx^\lambda}{d\sigma} = 0;$$

$$g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = 0$$

Free fall and the action principle.

Analytical mechanics:

particle moves from point A to point B



in such a way that the action is minimal,

$$S = \int_0^T \mathcal{L}(x, \dot{x})\, dt$$

$\longrightarrow x(0) = A \;;\; x(T) = B$

From here one gets Lagrangian equations of motion:

$$S\big(x(t) + \delta x(t)\big) = \int_0^T \mathcal{L}(x + \delta x, \dot{x} + \delta\dot{x})\, dt =$$

$$= \int_0^T \mathcal{L}(x, \dot{x})\, dt + \underbrace{\int_0^T \left(\frac{\partial \mathcal{L}}{\partial x} \cdot \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\, \delta \dot{x}\right) dt}$$

$$= \int_0^T \delta x \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] dt$$

+ boundary term [equal to zero since]

What is the action for particle in gravitational field?

guess:

$\int ds \leftarrow$ interval — (or proper time)

the only lorentz-invariant expression we can have, which describes the particle.

Assume this is true and find equations of motion:

$$S = \int ds = \int \underbrace{\left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}}_{\frac{ds}{d\tau}} d\tau$$

τ: any variable fixing the trajectory

$$x^\mu = x^\mu(\tau)$$

$$x^\mu \to x^\mu + \delta x^\mu \Rightarrow$$

$$\delta S = \int \frac{1}{2} \frac{1}{\left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}} \cdot d\tau$$

$$\cdot \left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda} \delta x^\lambda \cdot \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 2 g_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] =$$

$$= \int \Big[\frac{1}{2} d\tau \cdot \frac{d\tau}{ds} \cdot \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad * ds \frac{1}{ds}$$

$$+ g_{\mu\nu} d\tau \cdot \frac{d\tau}{ds} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad * ds \frac{1}{ds} \Big]$$

$$= \int ds \Big\{ \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \delta x^\lambda \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$

$$+ g_{\lambda\nu} \frac{d\delta x^\lambda}{ds} \frac{dx^\nu}{ds} \Big\} =$$

$$= \int ds \cdot \delta x^\lambda \Big\{ \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$

$$- g_{\lambda\nu} \frac{d^2 x^\nu}{ds^2} -$$

$$- \frac{dx^\nu}{ds} \frac{\partial g_{\lambda\nu}}{\partial x^\rho} \frac{\partial x^\rho}{\partial s} \Big\}$$

= now: $\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} -$

$$- \frac{\partial g_{\lambda\nu}}{\partial x^\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds}$$

can be rewritten as (show!)

$$- g_{\lambda\nu} \Gamma^\nu_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}$$

So that eq of motion is indeed

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0 \qquad (*)$$

Physical sence: particle moves in gravitational field (free fall) in such a way that the proper time is minimal (or maximal)

We can say that it moves along shortest trajectory (or longest)

shortest = minimal free time

This type of trajectory is called geodesic