Main points of #4

- Defined \( E = \mu + \delta = \frac{1}{\sqrt{g}} \varepsilon^{\alpha \beta \gamma} \partial_{\alpha} p_{\beta} \varepsilon_{\gamma} = 1 \)

- Considered physics case for Newton's law and for equality of inertial and gravitational mass

- Attempted to make a relativistic theory of gravity with the use of vector field: \( \Theta_{\mu} \rightarrow G_{\mu} \)
  and scalar field: \( \Phi \rightarrow \phi, \quad m \frac{d^2 x^\mu}{dt^2} = -\sqrt{-g} \nabla^\mu \phi \)

(did not work)

Today:

- formulation of equivalence principle
- equation for free fall
- gravitational field and the metric
- free fall and the action principle, geodesic
Once more, equivalence principle

gravitational mass = inertial mass

(1) Take inertial frame + gravitational field and any velocity
Start at $t=0$ and $\vec{v}=0$ with any mass —
They all will move along the same trajectory

(ii) Take vacuum (no gravitational field) and consider the motion of any body in non-inertial accelerating frame:
all bodies will move exactly in the same way

non-inertial system of coordinate is equivalent to some gravitational field

Inverse statement:

gravitational field in some point of space and time can be excluded in a particular accelerating frame
Example:

falling lift: weightlessness!

Equations for free fall in a gravitational field. Take a coordinate system which falls together with the body:

$$\phi^\alpha = \phi^\alpha(x'^i) \quad x'^i - Cartesian \quad coordinate$$

In this coordinate system gravity is removed by acceleration - no forces act on particle \(\rightarrow\) particle moves with constant coordinate system.

The particle trajectory and the physics looks like physics in inertial frame.

Relativistic generalisation: \(\frac{d^2x}{dt^2} = 0\) where \(ds\) is the interval,

$$ds^2 = c^2 dt^2 - dx^2$$

Minkowsky metric!

(gravity was killed by acceleration!)

Equivalence principle!}
Let us rewrite this equation in some other coordinate system (arbitrary) $\alpha \mu$:

\[ dx^\alpha = ds \left[ \frac{\partial \xi^\alpha}{\partial x^\mu} \right] = \]

\[ = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial t^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \]

Multiply by $\frac{\partial x^\mu}{\partial \xi^\alpha}$:

\[ \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial^2 x^\mu}{\partial t^2} + \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \]

\[ \frac{\partial x^\mu}{\partial x^\alpha} = \delta^\mu_\alpha \Rightarrow \]

\[ \frac{d^2 x^\alpha}{ds^2} + \frac{\partial x^\alpha}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \]

**What is this object?**
Introduce notations:

\[
\frac{\partial x^a}{\partial \xi^\nu} \frac{\partial^2 \xi^\nu}{\partial x^b \partial x^a} = \Gamma^\nu_{\mu
u}
\]

We will see that this is in fact the Christoffel symbol. 
\text{affine connection}

also, 
\[ds^2 = g_{\mu\nu} dx^\mu dx^\nu\], where

\[g_{\mu\nu} = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{\partial \xi^\beta}{\partial x^\mu} \Gamma^\alpha_{\mu\beta} \quad (*)\]

\text{Minkowski}

Proof that \(\Gamma^\nu_{\mu\nu}\) is indeed the Christoffel symbols:

\[\Gamma^\nu_{\mu\nu} = \Gamma^\nu_{\mu\nu}\]

Take \((*)\) and find its derivative with respect to \(x^\alpha\):

\[
\frac{\partial g_{\mu\nu}}{\partial x^\alpha} = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\alpha} \frac{\partial \xi^\beta}{\partial x^\nu} \Gamma^\alpha_{\mu\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\nu \partial x^\alpha} \Gamma^\alpha_{\mu\beta}
\]

\[
= \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\alpha} \frac{\partial \xi^\beta}{\partial x^\nu} \Gamma^\alpha_{\mu\beta} \frac{\partial \xi^\beta}{\partial x^\nu} \Gamma^\alpha_{\mu\beta}
\]

\[
+ \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\nu \partial x^\alpha} \Gamma^\alpha_{\mu\beta} = \Gamma^\nu_{\mu\nu}
\]
\[ = \overline{\Gamma}^{\mu \nu \rho} \delta_{\rho \mu} + \overline{\Gamma}^{\rho \mu \nu} \delta_{\rho \mu} \]

But this is exactly relation from the 3rd lecture \( \Rightarrow \)

\[ \overline{\Gamma}^{\mu \nu}_\rho = \Gamma^{\mu \nu}_\rho \]

Once more, eq of motion:
\[ \frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \]

This is analogue of Newton equation, which we had for electrodynamics.
\[ m \frac{d^2 x^\mu}{ds^2} = q \int \frac{F^\mu}{\Phi} \frac{dx^\nu}{ds} \]

electric and magnetic field.

\[ \therefore \Gamma^\lambda_{\mu \nu} \text{ play a role of gravitational field.} \]

As they are expressed through the metric \( g_{\mu \nu} \), we can say that gravitational field is metric \( g_{\mu \nu} \).
Very important

The metric in gravitational and laboratory frame, associated, e.g. with Earth CAN NOT be considered as a coordinate transformation of Minkowski space!!

Why?

We considered just one trajectory. On this line $x^k = a^k(t)$, and in the lab frame of free-falling coord system we have Lorentzian metric.

We have to cover all space by these lines and consider many coord systems. When we do that, we get $G_{\mu \nu}(\alpha)$

Plane through which our trajectories are going through.
We can choose a coordinate system in which the transformed fluid looks like \( \mathbf{u} \) on a line, but not on the whole space.

Conclusion: the space-time in the presence of gravity is \textit{not} Minkowskian.

For Minkowski space-time there exist a coordinate transformation that changes \( \mathbf{u} \) to \( \mathbf{u} \) at any point. In the presence of gravity this is not the case.

Guess: gravitational field is an arbitrary \textit{a priori} function \( \mathbf{u} \)-\textit{metric}!

\[ \text{(tensor of the second kind)} \]

So: - scalar - bad
- vector - bad
- antisymmetric tensor - bad
- symmetric tensor may be ok
Remark: what is equation of motion for a massless particle, like photon?

Problem: $ds^2 = 0$ - we cannot use it for equation!

So - use $\xi \equiv \text{time in locally Lorentzian coordinate system}$

$$\frac{d^2 \xi}{d \sigma^2} + \eta^{\mu \nu} \frac{d^2 \phi^\mu}{d \phi^\nu} = 0$$

in arbitrary coordinate system:

$$\frac{d^2 \phi^\mu}{d \sigma^2} + \rho^{\mu \nu} \frac{d^2 \phi^\nu}{d \phi^\alpha} = 0$$

$$\Sigma \frac{d \phi^\mu}{d \sigma} \frac{d \phi^\nu}{d \sigma} = 0$$
Free fall and the action principle.

Analytical mechanics:
particle moves from point $A$ to point $B$

in such a way that the action is minimal,

$$S = \int_0^T \mathcal{L}(x, \dot{x}) \, dt$$

$\rightarrow x(0) = A; \ x(T) = B$

From here one gets Lagrangian equations of motion:

$$S(x(t), \delta x(t)) = \int_0^T \mathcal{L}(x+\delta x, \dot{x}+\delta \dot{x}) \, dt =$$

$$= \int_0^T \mathcal{L}(x, \dot{x}) \, dt + \int_0^T \left\{ \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x} \right\} \, dt$$

$$= \int_0^T \delta x \left[ \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right] \, dt$$

+ boundary term \text{ equal to zero}
What is the action for particle in gravitational field?

guess: $\int ds'$ interval (for proper time)

the only lorentz invariant expression we can have, which describes the particle.

Assume this is true and find equations of motion:

$$S = \int ds = \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^n}{dt}} dt$$

$t$: any variable fixing the trajectory $x^\mu = x^\mu(t)$

$$\Rightarrow x^\mu(t) \mapsto$$

$$\delta S = \int \frac{1}{2} \left( \frac{1}{\sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^n}{dt}}} \right) dt$$

$$\left[ \frac{\delta g_{\mu\nu}}{\delta x^a} \frac{dx^a}{dt} \frac{dx^\mu}{dt} + 2 \frac{\delta g_{\mu\nu}}{\delta x^a} \frac{dx^a}{dt} \frac{dx^n}{dt} \right]$$
\[ \int \left[ \frac{1}{2} \frac{dt}{ds} - \frac{d\xi}{ds} \cdot \frac{\partial_{\mu} \partial_{\nu}}{\partial \xi^\mu} \frac{d\xi^\nu}{ds} \cdot \left( \frac{\partial_{\alpha}}{\partial \xi^\mu} \frac{d\xi^\nu}{ds} - \frac{\partial_{\beta}}{\partial \xi^\mu} \frac{d\xi^\nu}{ds} \right) \right] \, ds \frac{1}{ds} \]

\[ = \int ds \left\{ \frac{1}{2} \frac{\partial_{\mu} \partial_{\nu}}{\partial \xi^\mu} \frac{d\xi^\nu}{ds} \frac{d\xi^\sigma}{ds} \right\} \]

\[ + \beta_{\mu} \frac{d\xi^\mu}{ds} \frac{d\xi^\nu}{ds} \right\} = \]

\[ = \int ds \cdot 8x^a \left\{ \frac{1}{2} \frac{\partial_{\mu} \partial_{\nu}}{\partial \xi^\mu} \frac{d\xi^\nu}{ds} \frac{d\xi^\sigma}{ds} \right\} \]

\[ - \delta_{\mu \nu} \frac{d^2 \xi^\mu}{ds^2} \]

\[ - \frac{d\xi^\mu}{ds} \frac{\partial_{\nu}}{\partial \xi^\rho} \frac{\partial_{\rho}}{\partial \xi^\mu} \frac{d\xi^\nu}{ds} \right\} \]

\[ - \gamma_{\mu \nu} \frac{d\xi^\mu}{ds} \frac{d\xi^\nu}{ds} \frac{d\xi^p}{ds} \frac{d\xi^q}{ds} \]

can be rewritten as (show!)

\[ -g_{\mu \nu} \Gamma^\gamma_{\mu \nu} \frac{d\xi^p}{ds} \frac{d\xi^q}{ds} \]
So that of of motion is indeed
\[ \frac{d^2x}{dt^2} + \rho \frac{dx}{dt} \cdot \frac{dx}{ds} \cdot \frac{ds}{dt} = 0 \]  
\((*)\)

Physical sense: particle moves in gravitational field (free fall) in such a way that the proper time is minimal (or maximal).

We can say that it moves along shortest trajectory (or longest).

Shortest = minimal free time.

This type of trajectory is called geodesic.