Lecture #4 7.10.2008 ①

Main points of #3

- found volume element:

$$dV = \sqrt{g}\, d^3x$$

$$\sqrt{-g}\, d^4x \qquad g = \det g_{\mu\nu}$$

- defined covariant derivative:

$$d\vec{V} = V^{\alpha}_{\ ;\beta}\, \vec{e}_{\alpha}\, dx^{\beta}$$

difference of two vectors in points $\vec{x}$ and $\vec{x}+d\vec{x}$

$$V^{\alpha}_{\ ;\beta} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\gamma} V^{\gamma}$$

- defined $\Gamma^{\alpha}_{\beta\gamma}$ via derivatives of basis vectors and via metric
- wrote a pair of Maxwell equations in arbitrary coordinate systems

Today:

- a remark about transformations of $\Gamma^{\alpha}_{\beta\gamma}$
- antisymmetric tensor
- what do we know about gravity and what we want to do with it

Important remarks

x^α is <u>not</u> a vector for arbitrary coordinate system

dx^α is <u>a vector</u>

x_α does not exist: for coordinates we only use x with index <u>up</u>

$\varphi_{;\mu}$ exists and $\varphi_{;\mu} = \varphi_{,\mu}$

$\varphi^{;\mu}$ exists but $\varphi^{,\mu}$ does not!!

$$\varphi^{;\mu} = g^{\mu\nu} \varphi_{;\nu} = g^{\mu\nu} \varphi_{,\nu}$$

Tensorial nature of $\Gamma^{\alpha}_{\beta\gamma}$:

Transformation properties are complicated

In Cartesian coordinate system

all $\Gamma = 0$.

Some important tensors:

we defined ε_{ijkl} — completely antisymmetric tensor [or ε_{ijk}, in euclidean 3d space]

How does it look like in arbitrary coordinate system?

Consider $x^{\alpha} = x^{\alpha}(x'^{i})$

cartesian

arbitrary

We can write:

$$E^{\alpha\beta\gamma\delta} = \frac{\partial x^{\alpha}}{\partial x'^{i}} \frac{\partial x^{\beta}}{\partial x'^{j}} \frac{\partial x^{\gamma}}{\partial x'^{k}} \frac{\partial x^{\delta}}{\partial x'^{l}} \cdot \varepsilon^{ijkl}$$

$E^{\alpha\beta\gamma\delta}$ is antisymmetric: change $\alpha \to \beta$, then $i \leftrightarrow j \Rightarrow$

$$E^{\alpha\beta\gamma\delta} = C\, \varepsilon^{\alpha\beta\gamma\delta}$$

to find C: multiply by $\varepsilon_{\alpha\beta\gamma\delta}$

$$\underbrace{C\, \varepsilon_{\alpha\beta\gamma\delta}\, \varepsilon^{\alpha\beta\gamma\delta}}_{4!} = \underbrace{\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\alpha}{\partial x'^i} \cdots \frac{\partial x^\delta}{\partial x'^l}\, \varepsilon^{ijkl}}_{4!\, \det\left(\frac{\partial x^\alpha}{\partial x'^j}\right)}$$ (3)

Jacobian ;

$$C = \det \frac{\partial x^\alpha}{\partial x'^j}$$

then, $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = \eta_{ij}\, dx'^i dx'^j$

with $g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j} = \eta_{ij} \Rightarrow$

$$\det g \left[\det \frac{\partial x^\alpha}{\partial x'^i}\right]^2 = -1 \Rightarrow$$

$$\det \frac{\partial x^\alpha}{\partial x'^i} = \frac{1}{\sqrt{-\det g}} \Rightarrow \quad (\det g = g)$$

$$E^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g}}\, \underline{\varepsilon^{\alpha\beta\gamma\delta}}$$

Lower indeces : $E_{\alpha\beta\gamma\delta} = \sqrt{-g}\, \varepsilon_{\alpha\beta\gamma\delta}$

Some useful formulas; to prove at exers.

- $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

  in antisymmetric tensor the parts related to $\Gamma^{\mu}_{\nu\rho}$ disappear

- $$A^{\mu}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, A^\mu)}{\partial x^\mu}$$ divergence of vector

- $$\varphi^{;\mu}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial\left[\sqrt{-g}\, g^{\mu\nu} \frac{\partial\varphi}{\partial x^\nu}\right]}{\partial x^\mu}$$ Delambertian of Scalar

- if $F^{\mu\nu}$ – antisymmetric tensor, then

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}} \frac{\partial\left[\sqrt{-g}\, F^{\mu\nu}\right]}{\partial x^\nu}$$

What did people know about gravity before Einstein?

(i) Newtons law: (3rd N law)

$$\underset{m_1}{\circ} \xrightarrow{\quad r \quad} \underset{m_2}{\circ}$$

$$\vec{F} = -G_N \frac{m_1 \cdot m_2}{r^3} \cdot \vec{r}$$

atraction between two masses

(ii) The mass that appear in Newtons gravity law is the same as the mass appearing in the second Newtons law:

$$\vec{F} = m \cdot \vec{a}$$

„inertial" mass and gravitational mass
2nd law

are the same.

(equivalence principle)

From were this is coming from?

Different experiments

Galileo : speed of free fall does not depend on the mass

Non-trivial statement one could have

$$\vec{F} = m_i \vec{a}$$

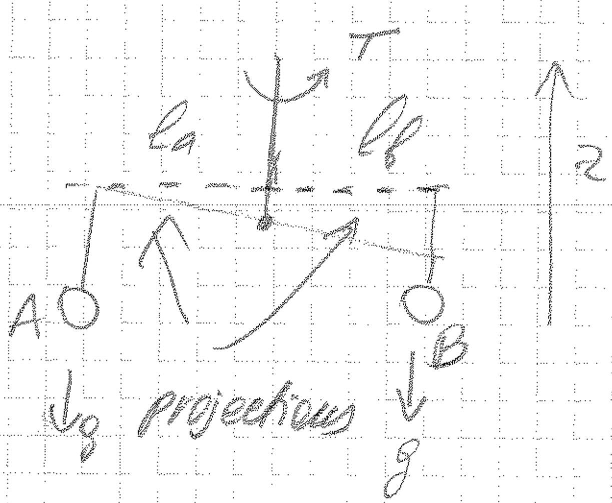
and gravity law as $\vec{F} = m_g \vec{g}$

acceleration of free fall, depending on positions of of other bodies.

Potentially, $\vec{a} = \left(\frac{m_g}{m_i}\right)\vec{g}$

Experimental idea: test this with different materials

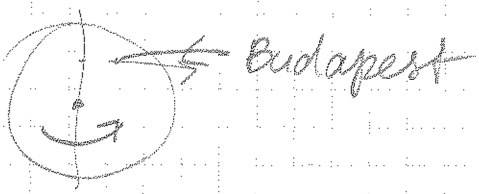
Etvesh experiments: $\frac{\delta a}{g} \leq 10^{-8}$
(1889)



also, g'_z, ← acceleration related to Earth rotation

Forces: (i) gravity

(ii) force related to earth rotation



equilibrium: $l_A \left[m_{gA} \cdot g - m_{iA} \cdot g_z' \right] =$

$$l_B \left[m_{gB} \cdot g - m_{iB} \cdot g_z' \right]$$

Forces in direction of center of the Earth

Rotation moment: horizontal component of acceleration

$$T = l_A \cdot m_{iA} \cdot g_s' - l_B \, m_{iB} \, g_s'$$

Since $l_B = l_A \dfrac{m_{gA} \cdot g - m_{iA} \cdot g_z'}{m_{gB} \cdot g - m_{iB} \, g_z'} \Rightarrow$

$$T \approx l_A \cdot g_s' \, m_{gA} \left[\frac{m_{iA}}{m_{gA}} - \frac{m_{iB}}{m_{gB}} \right],$$

which is valid for $g_s' << g$.

twisting of line was not observed!

Current constraints:

$$\frac{\Delta m_i}{m_g} \leq 2 \cdot 10^{-13}$$

Now, inverse square law

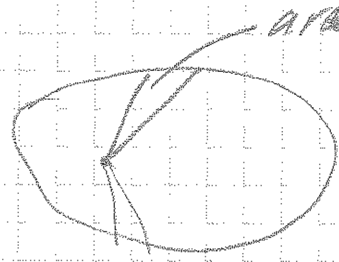
Astronomy: Keplers laws

1st: The orbit of every planet is an ellipse with the Sun at one of the foci

$AB + BC = \text{const}$

2nd law

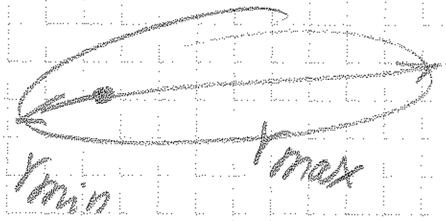
area 1 = area 2 for same Δt



3rd law

$T^2 \sim a^3$

orbital period



$$a = \frac{r_{min} + r_{max}}{2}$$

If $F \sim \frac{1}{r^2}$, one can reproduce

Newtons 3rd law

Problem we are facing:

$$\vec{F} = -G \frac{m_1 m_2 \vec{r}}{r^3}$$ — is not at least visibly Lorents invariant

should be a component of 4-vector, like

$$m \frac{d^2x^\mu}{ds^2}$$

Right hand side — is certainly not a 4-vector

Analogue: Coulomb's law:

$$\vec{F} = -\frac{e_1 e_2 \vec{r}}{r^3}$$

Replacement of Relativistic C. law — Maxwell electrodynamics

Many steps away from Coul. law:

no instantaneous action

— Existence of electric and magnetic fields, E & B (or potentials)

— non-trivial equations for E & B

— non-trivial expression for force

General structure

- (some diff. operator) [field] = source

$$\Box A^\mu = j^\mu$$

- Force $\sim F^{\mu\nu} u_\nu$: combination of fields and 4-velocity of the particle

For Electrodynamics:

fields A_μ - 4-vector

$F_{\mu\nu}$ - antisymmetric tensor

Source: j^μ 4-vector current

Questions to answer for gravity

- gravitational field. vector? tensor? scalar?
- Source of gravitational field: Scalar? vector? tensor?
- invariant equations for gravitational field
- infariant expression for the source?

(Failed) attempt to get relativistic theory of gravity

Coulomb law looks like (almost) N. law $\Rightarrow$

gravity – some new field G_μ,

mass – analogue of electric charge –

let us write the same equations as Maxwell ed with replacement

$A_\mu \Rightarrow G_\mu$; $e \Rightarrow m$, etc

Does not work. Why – in ed similar charges are repulsive, but gravity is always attractive!!

Remark: antiparticles like positron, etc also attract each other by gravitational force

(Can be found from analysis of $K\bar{K}^0$ mixing experiment, which occures in Earth gravitational field)

So: gravitational field as 4-vector does not work

Yet another try:

Gravitational field – scalar field φ

Then: how it interacts with photons

$$\Box\varphi = \underbrace{a F_{\mu\nu}F^{\mu\nu} + b\,\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}}_{\text{only these combinations are possible (Lorents- invariant)}}$$

Scalar Scalar

But: $F_{\mu\nu}F^{\mu\nu} = E^2 - B^2$

and $= 0$ for electomagnetic wave $(E = B)$

$\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = (\vec{E}\cdot\vec{B}) =$

0 for EM wave $\Rightarrow$

electromagnetic wave does not create gravitational field

Grav. field depends on the nature of substance – eq. principle

Does gravitational field influence wave propagation?

So, gravitational field is not
- scalar field like φ
- vector field like A_μ
- antisymmetric tensor field like $F_{\mu\nu}$

Scalar gravity —

$$\partial_\mu F^{\mu\nu} = \nabla^\nu \varphi$$

← gravitational field?

Does not work, since we can apply ∂_ν to this equation $\Rightarrow$

$\Box\varphi = 0$, but $\Box\varphi =$ source of gravity $\Rightarrow$ does not work

possibility

$$\partial_\mu F^{\mu\nu} = j^\nu + c\,\partial_\mu\left[F^{\mu\nu}\cdot\varphi\right] \approx F^{\mu\nu}\partial_\mu\varphi$$

← grav. potential

$\nu = 0$:

$$\operatorname{div} E = j^0 + c\,\vec{E}\cdot\vec{\nabla}\varphi$$

$\nu = i$

$$-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} = \vec{j} + c\left[\vec{E}\cdot\dot{\varphi} + \vec{B}\times\vec{\nabla}\varphi\right]$$

does not work in observations

A solid argument that scalar gravity does not work

Consider eq of motion

$$m\frac{d^2x^\mu}{ds^2} = \nabla^\mu\varphi \quad , \quad \Box\varphi = m\delta(\vec{x}-\vec{y})$$

For $\mu = 1,2,3$ this gives Newton law — OK

Take now $\mu = 0$. Then it tells that energy of particle does not change in static field. Or:

$$m\frac{du^\mu}{ds} = \nabla^\mu\varphi; \quad u_\mu m\frac{du^\mu}{ds} = u_\mu\nabla^\mu\varphi;$$

$$m\frac{d}{ds}u_\mu u^\mu = u_\mu\nabla^\mu\varphi$$

But $u^\mu u_\mu = 1 \Rightarrow u_\mu\nabla^\mu\varphi = 0$

or $\frac{dx^\mu}{ds}\frac{\partial\varphi}{\partial x^\mu} = 0 \Rightarrow \frac{\partial\varphi}{\partial s} = 0 \Rightarrow$

potential does not change if particle moves in it!!!