Lecture # 3 30.09.2008

Main points of # 2

arbitrary coordinate systems:

$$x^\alpha = x^\alpha(x'^\mu) \qquad J = \frac{\partial x^\alpha}{\partial x'^\beta}; \quad \det J \neq 0$$

contravariant vector: transforms as dx^α

covariant vector: transforms as $\frac{\partial \varphi}{\partial x^\alpha}$

metric: $g_{\alpha\beta} = \eta_{ij} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta}$,

where x^i are cartesian coordinates

Today:
- volume elements
- covariant derivatives
- Christoffel symbols

volume element:

$$dx\,dy\,dz = \left|\frac{D(x,y,z)}{D(x',y',z')}\right| dx'dy'dz'$$

$$g_{\alpha\beta} = \eta_{ij}\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta} \Rightarrow$$

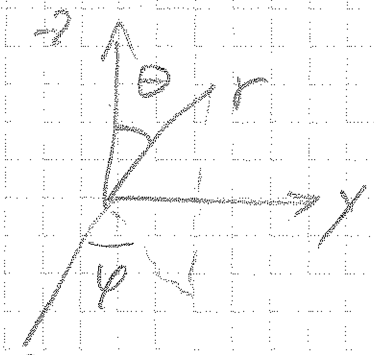
$$\det g = \left(\det \frac{\partial x^i}{\partial x'^\alpha}\right)^2 \Rightarrow \left|\frac{D(x,y,z)}{D(x',y',z')}\right| = \sqrt{g}$$

So, $dV = \sqrt{g}\,dx'dy'dz'$

For 4d: $dx\,dy\,dz\,dt = \sqrt{-g}\,dx'dy'dz'dt'$

examples: spherical coordinates:

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$



for cilyndrical coordinates:

$$dl^2 = dz^2 + d\rho^2 + \rho^2 d\varphi^2$$

spherical coordinates: $g_{\mu\nu} = \mathrm{diag}\begin{array}{ccc} r & \theta & \varphi \\ \end{array}\left[1, r^2, r^2\sin^2\theta\right]$

cilyndrical coordinats: $g_{\mu\nu} = \mathrm{diag}\left[1, 1, \rho^2\right]$ $\begin{array}{ccc} z & \rho & \theta \end{array}$

Volume elements:

spherical coordinates : $\sqrt{g} = r^2 \sin^2\theta$

Cylindrical —"— $\sqrt{g} = \rho$

$dV = r^2 \sin^2\theta \, dr \, d\theta \, d\varphi$

$dV = dz \, \rho \, d\rho \, d\varphi$

Basis vectors and Christoffel symbols

Lets consider for simplicity 3-d space

Cartesian coordinates $\Rightarrow$ Cartesian basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, e_i, i: $1 = x$; $2 = y$; $3 = z$

any vector: $\vec{V} = V^x \vec{e}_x + V^y \vec{e}_y + V^z \vec{e}_z$

arbitrary coordinates:

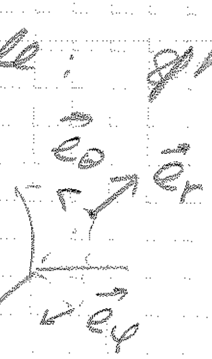
$$\xi, \eta, \rho : \quad \begin{array}{l} x = x(\xi, \eta, \rho) \\ y = y(\xi, \eta, \rho) \\ z = z(\xi, \eta, \rho) \end{array} \qquad \xi, \eta, \rho = x^\alpha$$

greek

basis: $\vec{e}_\xi = \dfrac{\partial x}{\partial \xi} \vec{e}_x + \dfrac{\partial y}{\partial \xi} \vec{e}_y + \dfrac{\partial z}{\partial \xi} \vec{e}_z$

etc

example: spherical coordinates:



metric tensor:

$$g_{\alpha\beta} = (\vec{e}_\alpha, \vec{e}_\beta)$$ ← ordinary scalar product

$(\alpha, \beta) = (\xi, \eta, \rho)$

Derivatives of vectors:

Cartesian coordinates: let V^i is a vector field, $\vec{V} = e_i V^i$ (basis and components are cartesian)

Then $\frac{\partial V^i}{\partial x^j}$ is a tensor

in arbitrary coordinates:

$$\vec{V} = \vec{e}_\xi V^\xi + \vec{e}_\eta V^\eta + \vec{e}_\rho V^\rho;$$

$$\frac{\partial \vec{V}}{\partial \xi} = \vec{e}_\alpha \frac{\partial V^\alpha}{\partial \xi} + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial \xi}$$

or, $$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}$$

Since $\vec{e}_\alpha$ form a basis, we can write

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} \vec{e}_\gamma$$

Γ : Christoffel symbols

So,

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \Gamma^\gamma_{\alpha\beta}\vec{e}_\gamma =$$

change $\alpha \to \gamma$

vector

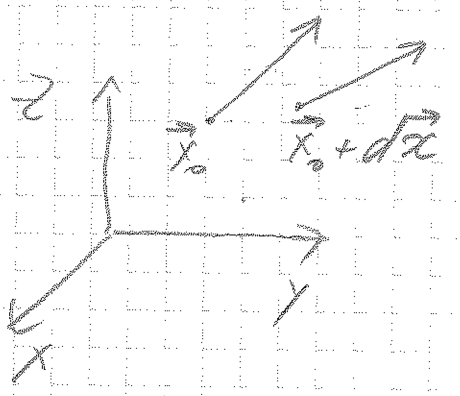
$$= \left(\frac{\partial V^\gamma}{\partial x^\beta} + V^\alpha \Gamma^\gamma_{\alpha\beta}\right)\vec{e}_\gamma$$

Covariant derivative : <u>is tensor</u>

$$V^\alpha_{;\beta} = \frac{\partial V^\alpha}{\partial x^\beta} + V^\gamma \Gamma^\alpha_{\gamma\beta} \quad ; \quad DV^\alpha = V^\alpha_{;\beta}\, dx^\beta$$

Geometric interpretation of covariant derivative: parallel transport of a vector

Euclidean space, cartesian coordinates

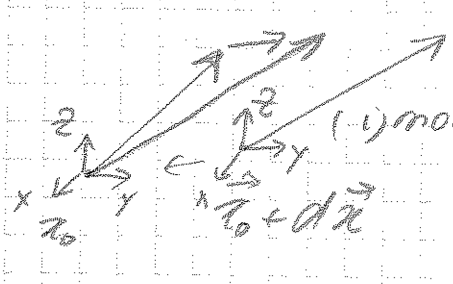


take a vector field $\vec{V}(x, y, z)$

and consider close points $\vec{x}_0$, $\vec{x}_0+d\vec{x}$

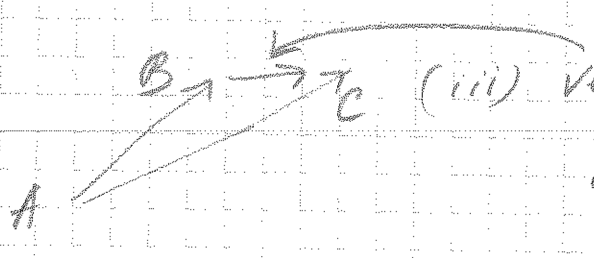
difference between $\vec{V}(\vec{x}_0)$ and $V(\vec{x}_0+d\vec{x})$, $d\vec{V}$ is a vector

How we can geometrically construct this vector:



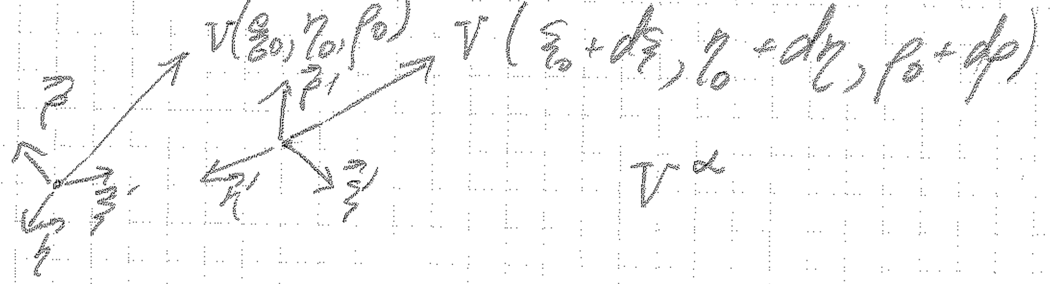
(i) move this vector parallel to itself to point x_0

(ii) construct triangle



(iii) vector $\vec{BC}$ is the result.

Suppose now that we are doing this with the use of arbitrary coordinate system



Parallel (in cartesian sence !!) transport to point (ξ_0, η_0, ρ_0):

$$\vec{e}_\alpha (x^\beta + dx^\beta)\, V^\alpha (x^\gamma + dx^\gamma)$$

Change of vector:

$$D\vec{V} = \vec{e}_\alpha (x + dx)\, V^\alpha (x + dx) - e_\alpha (x)\, V^\alpha (x)$$

$$= \left[\frac{\partial V^\alpha}{\partial x^\beta} + V^\gamma \Gamma^\alpha_{\beta\gamma} \right] \vec{e}_\alpha\, dx^\beta$$

Relation between covariant and contravariant vectors

To any V^α we put in correspondence

$$V_\alpha \equiv g_{\alpha\beta} V^\beta$$

transformation, $x^\mu = x^\mu(x')$:

Check that transformation properties are ok:

$$g_{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g'_{\mu\nu}$$

$$V^\beta = \frac{\partial x^\beta}{\partial x'^\rho} V'^\rho \Rightarrow$$

$$V_\alpha = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g'_{\mu\nu} \frac{\partial x^\beta}{\partial x'^\rho} V'^\rho =$$

$$= \frac{\partial x'^\mu}{\partial x^\alpha} g'_{\mu\rho} V'^\rho = \frac{\partial x'^\mu}{\partial x^\alpha} V'_\mu$$

QED

Inverse relation:

$V^\beta = g^{\beta\alpha} V_\alpha$, where, <u>by definition</u>

$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\beta \leftarrow$ Kronecker symbol

Covariant derivative of scalar:

difference of ordinary derivative and covariant derivative of a vector comes about since basis is changing $\frac{\partial \vec{e}_\alpha}{\partial x^\beta} \neq 0 \Rightarrow$ so,

for scalar $\varphi_{;\alpha} = \varphi_{,\alpha}$ $\qquad \varphi_{,\alpha} = \frac{\partial \varphi}{\partial x^\alpha}$

To find covariant derivative of covariant vector:

$$\underbrace{D[V_\alpha U^\alpha]}_{\text{scalar}} = \left(\frac{\partial V_\alpha}{\partial x^\beta} U^\alpha + \frac{\partial U^\alpha}{\partial x^\beta} V_\alpha\right) dx^\beta - DV_\alpha U^\alpha + V_\alpha D U^\alpha$$

$$= DV_\alpha U^\alpha + V_\alpha \left[\frac{\partial U^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta} U^\gamma\right] dx^\beta \Rightarrow$$

$$DV_\alpha = \frac{\partial V_\alpha}{\partial x^\beta} dx^\beta - V_\varkappa \Gamma^\varkappa_{\beta\alpha} dx^\beta \Rightarrow$$

$$V_{\alpha;\beta} = \frac{\partial V_\alpha}{\partial x^\beta} - \Gamma^\varkappa_{\alpha\beta} V_\varkappa$$

Christoffel symbols and the metric

We defined $V^{\alpha}{}_{;\beta}$ ← tensor

Then $V_{\alpha;\beta} = g_{\alpha\mu} V^{\mu}{}_{;\beta}$

↗ tensor

Consistency condition: $V_{\alpha;\beta} = \left(g_{\alpha\mu} V^{\mu}\right)_{;\beta}$

$= g_{\alpha\mu;\beta} V^{\mu} + g_{\alpha\mu} V^{\mu}{}_{;\beta} = g_{\alpha\mu} V^{\mu}{}_{;\beta} \Rightarrow$

$g_{\alpha\mu;\beta} = 0$ Important result. covariant derivative of the metric tensor is zero! (since vector V was arbitrary).

This allows us to find directly Chr. symbols via metric

general definition

$$\text{(+)} \quad g_{\alpha\mu;\beta} \overset{\downarrow}{=} \frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} - \Gamma^{x}_{\alpha\beta} g_{\mu x} - \Gamma^{x}_{\mu\beta} g_{\alpha x} = 0$$

$$\text{(+)} \quad g_{\alpha\beta;\mu} = \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}} - \Gamma^{x}_{\alpha\mu} g_{x\beta} - \Gamma^{x}_{\beta\mu} g_{x\alpha} = 0 \quad (*)$$

$$\text{(−)} \quad g_{\beta\mu;\alpha} = \frac{\partial g_{\beta\mu}}{\partial x^{\alpha}} - \Gamma^{x}_{\beta\alpha} g_{\mu x} - \Gamma^{x}_{\mu\alpha} g_{x\beta} = 0$$

To find Γ from here, we need an extra piece of information — relation between Γ^{x} … Γ^{x}

For this end consider in cartesian coordinates $\varphi_{,\alpha,\beta}$ this is a <u>symmetric</u> tensor: $\varphi_{,\alpha,\beta} = \varphi_{,\beta\alpha}$ $\Rightarrow$

it will be symmetric in any basis $\Rightarrow$

$$\varphi_{;\alpha;\beta} = \varphi_{;\beta;\alpha} \Rightarrow$$

$$\varphi_{;\alpha;\beta} = \varphi_{;\beta;\alpha} \Rightarrow$$

$$\varphi_{,\alpha,\beta} - \Gamma^{\varkappa}_{\alpha\beta}\varphi_{,\varkappa} = \varphi_{,\beta,\alpha} - \Gamma^{\varkappa}_{\beta\alpha}\varphi_{,\varkappa}$$

since this is valid for any φ $\Rightarrow$

$\Gamma^{\varkappa}_{\alpha\beta} = \Gamma^{\varkappa}_{\beta\alpha}$ – symmetry between lower indexes $\Rightarrow$ from (*)

we find:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \overbrace{\left(\Gamma^{\varkappa}_{\alpha\beta} - \Gamma^{\varkappa}_{\beta\alpha}\right)}^{=0} g_{\mu\varkappa}$$

$$+ \overbrace{\left(\Gamma^{\varkappa}_{\mu\alpha} - \Gamma^{\varkappa}_{\mu\alpha}\right)}^{=0} g_{\varkappa\beta}$$

$$+ \underbrace{\left(\Gamma^{\varkappa}_{\mu\beta} + \Gamma^{\varkappa}_{\beta\mu}\right)}_{2\Gamma^{\varkappa}_{\mu\beta}} g_{\varkappa\alpha}$$

$\Downarrow$

$$\Gamma^{\varkappa}_{\mu\beta} = \frac{1}{2} g^{\varkappa\alpha}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right)$$

This formalism allows us to rewrite Maxwell equations and relativistic Newton equations in arbitrary coordinate systems:

Steps: (i) going from cartesian coord. system to arbitrary introduce new coordinates, covariant and contravariant vectors and tensors.

(ii) replace ordinary derivatives by covariant derivatives

Result:

$$\partial_\mu F^{\mu\nu} = j^\nu \Rightarrow F^{\mu\nu}{}_{;\mu} = j^\nu$$

$$\varepsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0 \Rightarrow \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma;\nu} = 0$$

Maxwell

$$m \frac{du^\mu}{ds} = q F^{\mu\nu} u_\nu \quad \text{- Newton.}$$

See below the definition of $\varepsilon_{\mu\nu\rho\sigma}$