Lecture # 2 23.09.2008

Main points of # 1

- system of units $\hbar = c = k_B = 1$ — everything in GeV.
- symmetries of dynamical equations = symmetries of space-time
- reminded you definition of metric $\eta_{\mu\nu}$, $\eta^{\mu\nu}$, 4-vectors, Lorentz transformation

Today – finish the reminder
constant acceleration

Euclidean space and SR] in arbitrary coordinate systems
- coordinate transform, vectors, tensors
- metric

- tensor quantity with covariant indexes $\alpha\beta\ldots$ and contravariant indexes $\mu\nu\ldots$ which transfors as

$$T^{\mu\nu\ldots}_{\alpha\beta\ldots} \Rightarrow T'^{\mu\nu\ldots}_{\alpha\beta\ldots} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} \ldots \Lambda_{\alpha}{}^{\alpha'} \Lambda_{\beta}{}^{\beta'} \ldots$$

$$T^{\mu'\nu'\ldots}_{\alpha'\beta'\ldots}$$

Note: $\Lambda^{\mu}{}_{\nu} \neq \Lambda_{\nu}{}^{\mu}$

$$\Lambda_{\nu}{}^{\mu} = \eta_{\nu\alpha} \eta^{\mu\beta} \Lambda^{\alpha}{}_{\beta}$$

- important invariant [does not change if we change a reference frame] (pseudo) tensor

$$\varepsilon_{\mu\nu\rho\sigma} = \pm 1, \quad \text{for } \varepsilon_{0123} = 1$$

- scalar product is invariant:

$$A_{\mu} B^{\mu} \Leftarrow \text{does not change}$$

- $\frac{\partial}{\partial x^{\mu}} = \partial_{\mu} = \nabla_{\mu}$: transform as covariant vector

- Electric and magnetic fields:

  form antisymmetric tensor,

  $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

  where A_μ is 4-vector potential

  $A^0 = \varphi$ - scalar potential

- Maxwell equations can be written as

  $$\partial_\mu F^{\mu\nu} = j^\nu$$

  $$\varepsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0,$$

  where j^ν - 4-vector of current,

  $$j^0 = \rho; \quad j^i = \vec{j}$$

- interval, $ds^2 = dx_\mu dx^\mu$ - invariant

- 4-velocity: $\frac{dx^\mu}{ds} = u^\mu$ $\quad (x^\mu = (t, \vec{x}(t)))$

- 4-acceleration: $\frac{d^2x^\mu}{ds^2}$

- for moving particle $ds^2 = dt^2 - v^2 dt^2 =$
  $= dt^2(1 - v^2)$

- Newton equation:

  $$m \frac{d^2x^\mu}{ds^2} = q F^{\mu\nu} u_\nu$$

- 4- momentum,

$$p^\mu = m u^\mu$$

- Lorentz transformations are parametrised by 6 numbers:

  3 boosts ($\vec{v}$)

  and 3 rotations

- Poincare transformations:

$$x'^\mu \to x^\mu = \Lambda^\mu{}_\nu x^\nu + a^\nu$$

translation

$\mu = 0,1,2,3$ (Greek letters)

$i,j,k \ldots 123$ (Latin indexes)

importance of interval:

$$ds^2 = dx_\mu dx^\mu$$

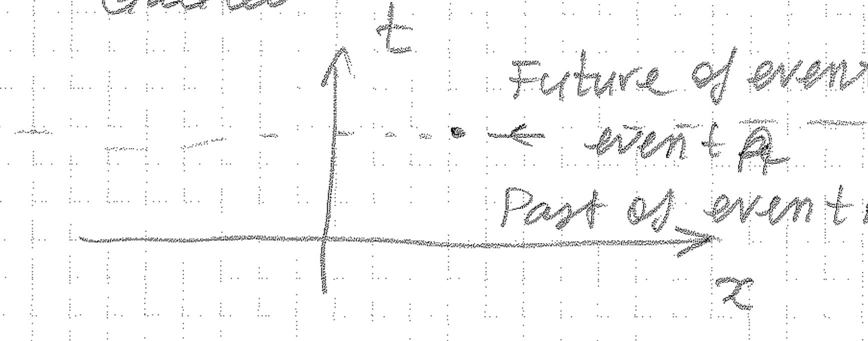
ds^2 can be positive, zero, and negative

$ds^2 > 0$: time-like: if $ds^2 > 0$,
two events could be in causal contact

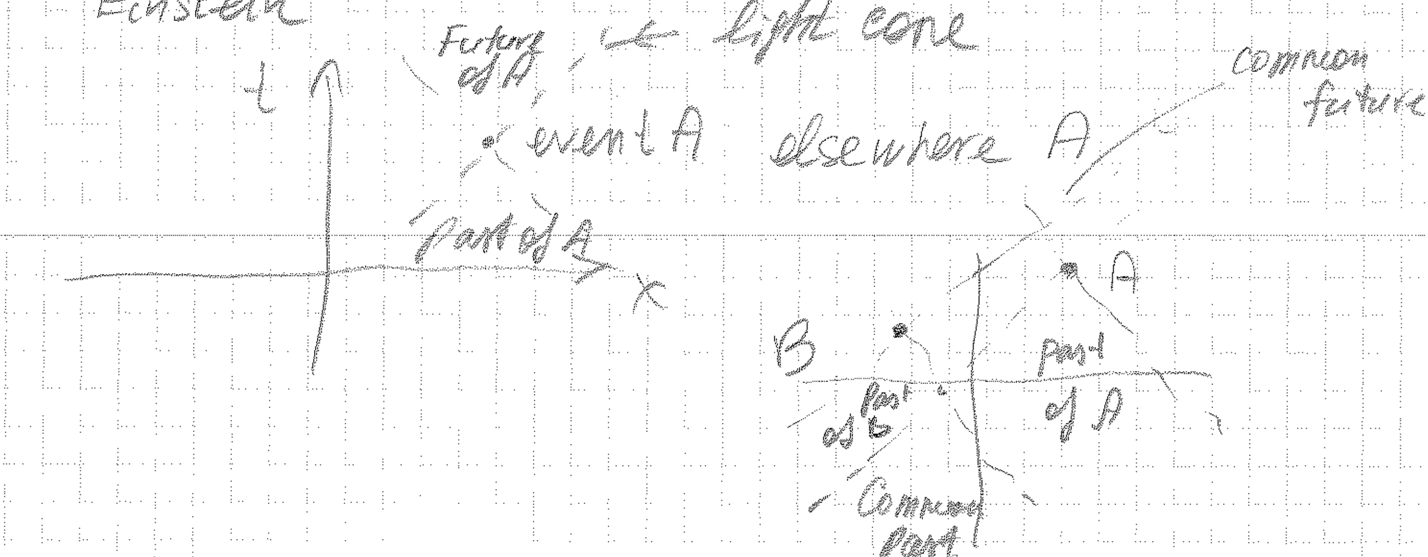
$ds^2 = 0$: light-cone.

$ds^2 < 0$: spacelike - no reason-consequence relation between events

Galileo



Einstein



We will see later that acceleration plays an important role for theory of gravity. Again, reminder:

in ED lectures we considered constant acceleration:

constant,
E ↙ electric field

$$\xrightarrow{\hspace{3cm}}$$

$t=0$: $x=x_0$

and solved for coordinates, velocity:

$$\frac{d}{dt}\left[\frac{\vec{v}}{\sqrt{1-v^2}}\right] = \frac{eE}{m} = a$$

$$v = \frac{at}{\sqrt{1+a^2t^2}} \quad , \to c \text{ for } t \to \infty$$

$$x = x_0 + \frac{1}{a}\left[\sqrt{1+a^2t^2} - 1\right] \text{: trajectory of a particle}$$

Suppose that we live in this accelerated system of coordinates (cosmic ship, considered last time in exers)

"Proper time": Clock rate at accelerated "coordinate system:

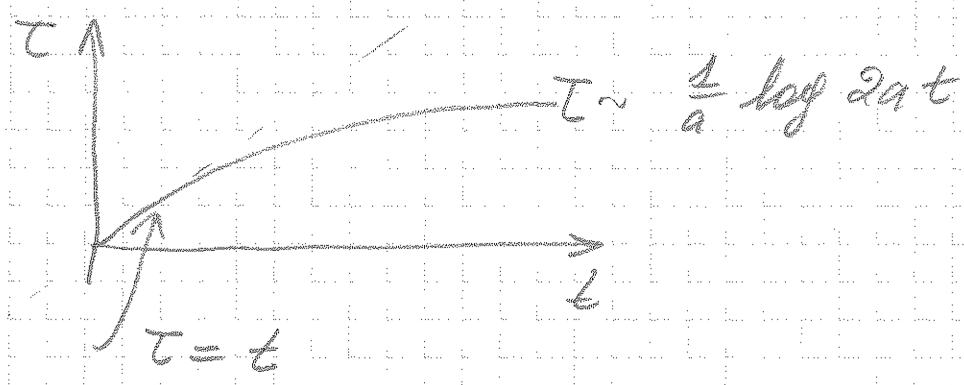
$ds^2 = dt^2 - dx^2$, $dx = v\,dt \Rightarrow$

$ds^2 = dt^2(1-v^2)$ – interval in laboratory frame. In accelerated coord. system we do not move $\Rightarrow$ $ds^2 = d\tau^2 \Rightarrow$

$$d\tau = dt\sqrt{1-v^2} = dt\sqrt{1-\frac{a^2t^2}{1+a^2t^2}} = \frac{dt}{\sqrt{1+a^2t^2}}$$

Relation of proper time to time t:

$$\tau = \frac{1}{a}\log\left[at + \sqrt{1+a^2t^2}\right]$$



$\tau < t$. proper time goes slowly

Mathematical observation: Relations between coordinates and time is not linear:

$$\left.\begin{aligned} x' &= x + \frac{1}{a}\left[\sqrt{1+a^2t^2} - 1\right] \\ t' = \tau &= \frac{1}{a}\log\left[at + \sqrt{1+a^2t^2}\right] \end{aligned}\right]$$

Mathematical problem how to deal with arbitrary coordinate transformations

Special relativity in arbitrary coordinate systems

In 3d $\overbrace{\ }^{\text{or } 2d}$ for many problems the use of cartesian coordinates x, y, z $\overbrace{\ }^{\text{or } (x,y)}$ is not always convenient.

examples: spherical symmetry $\Rightarrow$

$z = r\cos\theta$, $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$

cilyndrical symmetry,

$z' = z$

$x = \rho\cos\varphi$, $y = \rho\sin\varphi$, etc.

So, we will consider tensor analysis for arbitrary coordinate system

Let

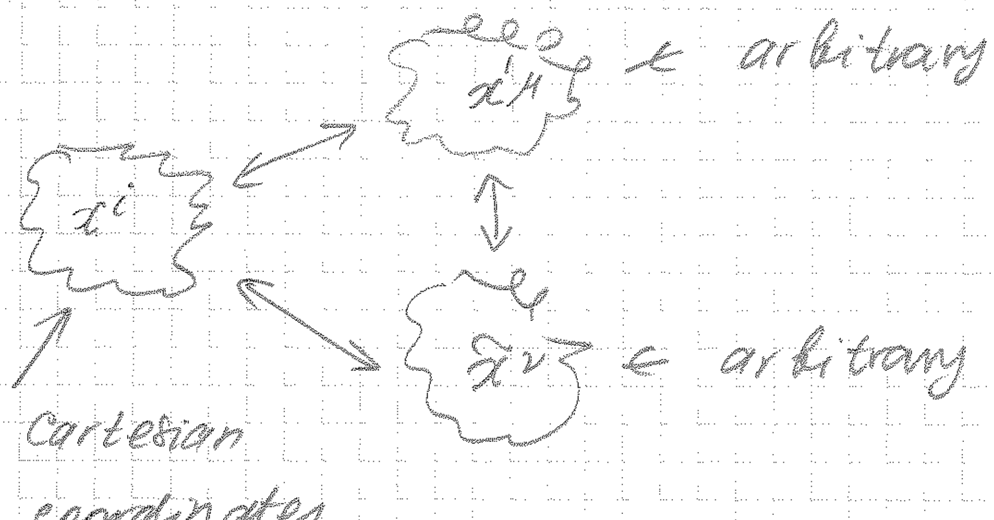
$x^i = f^i(x')$ $\quad x : x^0, x^1, x^2, x^3$

f^i - some functions

> notations:
> Greek letters, $\mu, \nu, \rho \ldots$ — arbitrary coordinates
> Latin letters i, j, k - cartesian coordinates

$$dx^i = \frac{\partial x^i}{\partial x'^\mu} dx'^\mu$$

$J = \dfrac{\partial x^i}{\partial x'^\mu}$ — Jacobian of transformation



$$x^i = x^i(x'^{\mu}) \qquad dx^i = \frac{\partial x^i}{\partial x'^{\mu}} dx'^{\mu}$$

$$x^i = x^i(\tilde{x}^{\mu}) \qquad dx^i = \frac{\partial x^i}{\partial \tilde{x}^{\mu}} d\tilde{x}^{\mu}$$

direct connection:

$$x^{\mu} = x^{\mu}(\tilde{x}^{\nu}) \Rightarrow$$

$$dx^{\mu} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}} d\tilde{x}^{\nu}$$

transformation for arbitrary coordinate systems one to one correspondence $\Rightarrow$ $\det J \neq 0$.

Contravariant (4) – vector:
the same transformation as dx^μ:

$$A^\mu = \frac{\partial x^\mu}{\partial x'^\nu} A'^\nu$$

Let φ: scalar function $\Rightarrow$

$$\frac{\partial \varphi}{\partial x^\mu} = \frac{\partial \varphi}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\mu}$$

Covariant 4- vector:

$$A_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A'_\nu$$

Practically the same definition as for cartesian coordinates and for Lorentz transformations.

tensors: similar definitions:

$$A^{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} A'^{\rho\sigma}$$

etc

Scalar product:

$$A_\mu B^\mu \to \frac{\partial x'^\alpha}{\partial x^\mu} A'_\alpha \frac{\partial x^\mu}{\partial x'^\beta} B'^\beta =$$

$$= \frac{\partial x'^\alpha}{\partial x'^\beta} A'_\alpha B'^\beta = A'_\alpha B'^\alpha \text{ - invariant}$$

Metric structure [distances for Euclidean space]

let x^0, x^1, x^2, x^3 are orthogonal coordinates

$$ds^2 = \eta_{ij}\, dx^i dx^j$$

let x' are arbitrary coordinates $\Rightarrow$

$$ds^2 = \eta_{ij} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} dx'^\alpha dx'^\beta =$$

$$= g_{\alpha\beta}\, dx'^\alpha dx'^\beta$$

where $g_{\alpha\beta} = \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} \eta_{ij}$

is „metric" in arbitrary coordinates