Solution 8

Exercise 1

a) Exponent α. The specific heat is:

$$C(t,B) = -T\frac{\partial^2 f(T,B)}{\partial T^2} = \frac{-T}{T_c^2}\frac{\partial^2 f(t,B)}{\partial t^2}$$

From the scaling relation of the free energy we obtain that

$$f(t,B) = \frac{f(\lambda^s t, \lambda^r B)}{\lambda^d} \tag{1}$$

and the specific heat can be calculated:

$$\begin{aligned} C(t,B) &= -\frac{t+1}{T_c}\frac{\partial^2 f(t,B)}{\partial t^2} \\ &= -\frac{t+1}{T_c}\lambda^{2s-d}\frac{\partial^2 f(\lambda^s t,\lambda^r B)}{\partial(\lambda^s t)^2} \\ &= \frac{t+1}{\lambda^s t+1}\lambda^{2s-d}C(\lambda^s t,\lambda^r B) \end{aligned}$$

Fixing $B=0$ (zero magnetic field) and the product $\lambda^s|t| = \varepsilon << 1$ constant (the homogeneity hypothesis is valid close to the critical point):

$$C(t,0) = \frac{t+1}{\varepsilon+1}\varepsilon^{\frac{2s-d}{s}}C(\varepsilon,0)|t|^{-\frac{2s-d}{s}} \sim |t|^{-\frac{2s-d}{s}}.$$

The multiplicative factors do not contribute to the critical exponent. So $\alpha = \frac{2s-d}{s}$.

b) Exponent β: The magnetization is:

$$m(t,B) = -\frac{\partial f(t,B)}{\partial B}$$

From (1):

$$m(t,B) = \lambda^{r-d}m(\lambda^s t,\lambda^r B)$$

Taking $\lambda = |t|^{\frac{-1}{s}}$ and $B=0$

$$m(t,0) = \lambda^{r-d}m(\pm 1,0) = |t|^{\frac{d-r}{s}}m(\pm 1,0)$$

So $\beta = \frac{d-r}{s}$.

c) Exponent γ: The magnetic susceptibility:

$$\chi(t,B) = -\left.\frac{\partial^2 f(t,B)}{\partial B^2}\right|_{B=0}$$

One finds:

$$\chi(t,B) = \lambda^{2r-d}\chi(\lambda^s t,\lambda^r B)$$

And taking again $\lambda = |t|^{\frac{-1}{s}}$ and $B=0$ we obtain the exponent $\gamma = \frac{2r-d}{s}$.

d) Exponent δ: In the scaling relation of the magnetization we pose $\lambda = |B|^{\frac{-1}{r}}$ and $t = 0$

$$m(0,B) = \lambda^{r-d} m(\pm 1, 0) = |t|^{\frac{d-r}{r}} m(\pm 1, 0)$$

So $\delta = \frac{r}{d-r}$.

The Rushbrooke and Griffin relations are immediately verified.

Exercise 2

We consider a scale change parameter λ for lengths and μ for times:

$$r \rightarrow \lambda r$$

$$t \rightarrow \mu t$$

The Newton equation $F = ma$ is invariant under scale change.

We have:

$$a(r,t) = \frac{\Delta r}{\Delta t^2} \rightarrow a(\lambda r, \mu t) = \frac{\lambda \Delta r}{\mu^2 \Delta t^2} = \frac{\lambda}{\mu^2} a(r,t)$$

$$F(r) = -\nabla_r U(r) \rightarrow F(\lambda r) = -\nabla_{\lambda r} U(\lambda r) = \frac{1}{\lambda} \nabla_r (\lambda^d U(r)) = \lambda^{d-1} F(r)$$

So we obtain:

$$\lambda^{d-1} F(r) = \frac{\lambda}{\mu^2} a(r,t) \Leftrightarrow \lambda^{d-2} = \mu^{-2}$$

Using the Kepler law we see that $\lambda^3 = \mu^2$. So:

$$d = -1$$

Which means:

$$U(\lambda r) = \frac{1}{\lambda} U(r)$$

Fixing $\lambda = \frac{1}{r}$, we obtain $U(r) \sim \frac{1}{r}$.

In the case of a constant force we have: $F(r) = F(\lambda r)$, so $d = 1$. From which $\lambda = \mu^2$ which implies $r \sim t^2$.