Solution 4

a)

$$\begin{aligned} Z &= \sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\dots\sum_{\sigma_N=\pm1}\prod_{<ij>}\exp(\beta J\sigma_i\sigma_j) \\ &= (\cosh L)^M \sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\dots\sum_{\sigma_N=\pm1}\prod_{<ij>}(1+\tanh(L)\sigma_i\sigma_j) \end{aligned}$$

b) We develope in powers of $u=\tanh(L)$:

$$Z = (\cosh L)^M \sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\dots\sum_{\sigma_N=\pm1}\left(1+u\sum_{<ij>}\sigma_i\sigma_j+u^2\sum_{<ij>}\sum_{<kl>}(\sigma_i\sigma_j)(\sigma_k\sigma_l)+\dots\right)$$

where each pair of links $<ij>$ appears only once in the terms $(\sigma_i\sigma_j)(\sigma_k\sigma_l)\dots$ We see that the sum within the coefficients of u^n runs on all the trails we can have on the lattice with n different links (the trails are not necessarily connected).

Let's imagine the case in which one σ_i appears only one time, which means a trial with one free end. Taking the sum $\sum_{\sigma_i=\pm1}\sigma_i=0$ we see that the contribution of such trails is null. It follows that only the trails without free ends are meaningful in the partition function. These closed trails are polygones (we point out that a trail can be composed by several polygones, if each point has an even number of links). For each closed trail there are 2^N possible spin configurations and each configuration counts for 1 (because $\sigma_i^{2n}=1$ independently to the spin values). Finally we can write the partition function as:

$$Z = 2^N(\cosh L)^M(1+\sum_{n=1}^{\infty}\Omega_n u^n)$$

with Ω_n the number of closed trails with n links such that each point has an even number of links.

We reduced in this way the sum over all the configurations to the sum over the polygones we can draw on the lattice. This formalism has the advantage to be valid on any geometry (hexagonal lattice, random graph...). In particular we notice that the terms in u and u^2 are always zero. We also notice that on the square lattice only the terms with even n contribute.

c) At high temperature we see that $L\longrightarrow 0$, that is $u\longrightarrow 0$. In this case the partition function results:

$$Z = 2^N(\cosh L)^M$$

d) In the case of one dimension lattice we don't have any loop and $N = M$. So all the Ω_n are zero:

$$Z = 2^N (\cosh L)^N$$

e) In the case with periodic boundary conditions the only contribution to the sum is from the term u^N and there is only one ($\Omega_N = 1$) closed trial connecting all the sites, so we obtain:

$$Z = 2^N (\cosh L)^N [1 + u^N] = (2\cosh L)^N + (2\sinh L)^N$$

which is a regular function of L, so there is no phase transition.