Wick 3

We want to show that

$$
\begin{aligned}
&: \phi_{a_1} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \phi_{b_m} : \quad = \\
&\quad : \phi_{a_1} \cdots \phi_{a_n} \phi_{b_1} \cdots \phi_{b_m} : + \sum_{a,b} : \phi_{a_1} \underbrace{\cdots \phi_{a_n} \phi_{b_1} \cdots} \phi_{b_m} : \quad ,
\end{aligned} \tag{1}
$$

where the sum runs over all the contractions (meaning also multiple contractions) between some ϕ_a's and some ϕ_b's (note that no contractions between ϕ_a's and ϕ_a's or ϕ_b's and ϕ_b's appear).
Let's show this by induction.
We consider as the step 0 the one in which $n = m = 1$ (for either $n = 0$ or $m = 0$ the statement is trivially true):

$$
: \phi_a :: \phi_b : \quad = \quad \phi_a^+ \phi_b^+ + \phi_a^+ \phi_b^- + \phi_a^- \phi_b^+ + \phi_a^- \phi_b^- \quad = \quad : \phi_a \phi_b : + : \underbrace{\phi_a \phi_b} : .
$$

We recall the following properties

$$
\begin{aligned}
\underbrace{\phi_a^+ \phi_b} &= 0, \\
\underbrace{\phi_a^- \phi_b} &= \underbrace{\phi_a \phi_b}.
\end{aligned}
$$

To complete the proof, we now suppose the theorem to hold for $n-1+m$ fields, and want to induce its validity for $n+m$. So we take as true

$$
\begin{aligned}
&: \phi_{a_2} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \phi_{b_m} : \quad = \\
&: \phi_{a_2} \cdots \phi_{a_n} \phi_{b_1} \cdots \phi_{b_m} : + \sum_{a_{>1},b} : \phi_{a_2} \underbrace{\cdots \phi_{a_n} \phi_{b_1} \cdots} \phi_{b_m} : \quad .
\end{aligned}
$$

Now

$$
\begin{aligned}
: \phi_{a_1} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \phi_{b_m} : \quad &= \quad \phi_{a_1}^+ : \phi_{a_2} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \phi_{b_m} : + : \phi_{a_2} \cdots \phi_{a_n} : \phi_{a_1}^- : \phi_{b_1} \cdots \phi_{b_m} : \\
&= \quad \phi_{a_1}^+ : \phi_{a_2} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \phi_{b_m} : + : \phi_{a_2} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \phi_{b_m} : \phi_{a_1}^- + \\
&\quad \sum_{a_1,b_i} : \phi_{a_2} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \underbrace{\phi_{a_1} \phi_{b_i}} \cdots \phi_{b_m} : \quad .
\end{aligned}
$$

Using the relation for $n+m$ fields one rewrites this as

$$
\begin{aligned}
: \phi_{a_1} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \phi_{b_m} : \quad &= \quad \phi_{a_1}^+ : \phi_{a_2} \cdots \phi_{a_n} \phi_{b_1} \cdots \phi_{b_m} : + \sum_{a_{>1},b} \phi_{a_1}^+ : \phi_{a_2} \underbrace{\cdots \phi_{a_n} \phi_{b_1}} \cdots \phi_{b_m} : + \\
&\quad : \phi_{a_2} \cdots \phi_{a_n} \phi_{b_1} \cdots \phi_{b_m} : \phi_{a_1}^- + \sum_{a_{>1},b} : \phi_{a_2} \underbrace{\cdots \phi_{a_n} \phi_{b_1}} \cdots \phi_{b_m} : \phi_{a_1}^- + \\
&\quad \sum_{a_1,b_i} : \phi_{a_2} \cdots \phi_{a_n} :: \phi_{b_1} \cdots \underbrace{\phi_{a_1} \phi_{b_i}} \cdots \phi_{b_m} :
\end{aligned}
$$

The fifth term can be split in two contributions:

$$\sum_{a_1,b_i} :\phi_{a_2}\cdots\phi_{a_n}::\phi_{b_1}\cdots\underbracket{\phi_{a_1}\phi_{b_i}}\cdots\phi_{b_m}: \quad =$$

$$\sum_{a_1,b_i} :\phi_{a_2}\cdots\phi_{a_n}\phi_{b_1}\cdots\underbracket{\phi_{a_1}\phi_{b_i}}\cdots\phi_{b_m}: + \sum_{a_1,b_i}\sum_{a_{>1},b} :\phi_{a_2}\underbracket{\cdots\phi_{a_n}\phi_{b_1}\cdots}\cdots\underbracket{\phi_{a_1}\phi_{b_i}}\cdots\phi_{b_m}: \quad ,$$

so that the overall expression is

$$\begin{aligned} :\phi_{a_1}\cdots\phi_{a_n}::\phi_{b_1}\cdots\phi_{b_m}: \quad = \quad & \phi^+_{a_1}:\phi_{a_2}\cdots\phi_{a_n}\phi_{b_1}\cdots\phi_{b_m}: + \sum_{a_{>1},b}\phi^+_{a_1}:\phi_{a_2}\underbracket{\cdots\phi_{a_n}\phi_{b_1}}\cdots\phi_{b_m}: + \\ & :\phi_{a_2}\cdots\phi_{a_n}\phi_{b_1}\cdots\phi_{b_m}:\phi^-_{a_1} + \sum_{a_{>1},b}:\phi_{a_2}\underbracket{\cdots\phi_{a_n}\phi_{b_1}}\cdots\phi_{b_m}:\phi^-_{a_1} + \\ & \sum_{a_1,b_i}:\phi_{a_2}\cdots\phi_{a_n}\phi_{b_1}\cdots\underbracket{\phi_{a_1}\phi_{b_i}}\cdots\phi_{b_m}: + \\ & \sum_{a_1,b_i}\sum_{a_{>1},b}:\phi_{a_2}\underbracket{\cdots\phi_{a_n}\phi_{b_1}\cdots}\cdots\underbracket{\phi_{a_1}\phi_{b_i}}\cdots\phi_{b_m}: \quad . \end{aligned}$$

The sum of the first and the third term is $:\phi_{a_1}\cdots\phi_{b_n}:$. The sum of the second and fourth term is $\sum_{a,b}:\phi_{a_1}\underbracket{\cdots\phi_{a_n}\phi_{b_1}\cdots}\phi_{b_m}:$, where a_1 is not involved. The sum of the last two terms is $\sum_{a,b}:\phi_{a_1}\underbracket{\cdots\phi_{a_n}\phi_{b_1}\cdots}\phi_{b_m}:$, where a_1 is involved. Thus the sum of the second, fourth, fifth and sixth term is $\sum_{a,b}:\phi_{a_1}\underbracket{\cdots\phi_{a_n}\phi_{b_1}\cdots}\phi_{b_m}:$ as defined in equation (1).

Since step 0 is true and step $n-1+m$ implies step $n+m$, then the induction is complete, and the theorem (1) is proved.