We discuss here the Thomson scattering of a point particle by an incoming electromagnetic monochromatic wave. The radiation fields are given by:

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0 c}\frac{1}{R}\left(\mathbf{n}\wedge(\mathbf{n}\wedge\dot{\boldsymbol{\beta}})\right), \qquad \mathbf{B} = \frac{e}{4\pi\epsilon_0 c^2}\frac{1}{R}\left[\mathbf{n}\wedge\left((\mathbf{n}\wedge\dot{\boldsymbol{\beta}})\wedge\mathbf{n}\right)\right], \tag{1}$$

where R is the distance between the observer and the particle. The Poynting vector is therefore given by:

$$\mathbf{S} = \frac{e^2}{16\pi^2\epsilon_0 c}|\mathbf{n}\wedge(\mathbf{n}\wedge\dot{\boldsymbol{\beta}})|^2\frac{\mathbf{n}}{R^2}, \tag{2}$$

so the emitted power per unit solid angle is:

$$\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2\epsilon_0 c}|\mathbf{n}\wedge(\mathbf{n}\wedge\dot{\boldsymbol{\beta}})|^2. \tag{3}$$

The incoming electric field of the incident wave can be written as:

$$\begin{aligned}\mathbf{E} = \;& \mathbf{E_0}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \text{c.c.} = \boldsymbol{\epsilon}E_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \text{c.c.} = \\ & 2(\text{Re}\boldsymbol{\epsilon})E_0\cos(\mathbf{k}\cdot\mathbf{x}-\omega t) - 2(\text{Im}\boldsymbol{\epsilon})E_0\sin(\mathbf{k}\cdot\mathbf{x}-\omega t),\end{aligned} \tag{4}$$

where E_0 is now a real constant and $\boldsymbol{\epsilon}$ satisfies the conditions:

$$\boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon} = 1, \qquad \boldsymbol{\epsilon}\cdot\mathbf{k} = 0. \tag{5}$$

The equation of motion for the free particle is therefore:

$$m\dot{\mathbf{v}} = e(\boldsymbol{\epsilon}E_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \text{c.c.}), \tag{6}$$

from which one can derive $\dot{\boldsymbol{\beta}}$. The incoming energy flux can be derived from the Poynting vector, $S_{in} = 2\epsilon_0 c|\mathbf{E_0}|^2 = 2\epsilon_0 c E_0^2$, from which we find the cross section as:

$$\frac{d\sigma}{d\Omega} = \frac{1}{S_{in}}\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2\epsilon_0 c}\frac{1}{2\epsilon_0 c E_0^2}\left(\frac{e}{mc}\right)^2|\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{E})|^2. \tag{7}$$

The quantity $|\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{E})|^2$ must be averaged over time and over the initial polarization of the electromagnetic wave. Averaging over time, we get:

$$\begin{aligned}|\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{E})|^2 = \;& |\mathbf{n}\wedge\left[\mathbf{n}\wedge(\boldsymbol{\epsilon}E_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \boldsymbol{\epsilon}^* e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]|^2 = \\ & 2\left(\mathbf{n}\wedge(\mathbf{n}\wedge\boldsymbol{\epsilon}E_0)\right)\cdot\left(\mathbf{n}\wedge(\mathbf{n}\wedge\boldsymbol{\epsilon}^*E_0)\right) = 2E_0^2|\mathbf{n}\wedge(\mathbf{n}\wedge\boldsymbol{\epsilon})|^2.\end{aligned} \tag{8}$$

We can rewrite the last result using

$$[\mathbf{n}\wedge(\mathbf{n}\wedge\boldsymbol{\epsilon})]_i = \epsilon_{ijk}n_j\epsilon_{klm}n_l\epsilon_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})n_j n_l\epsilon_m = (n\cdot\epsilon)n_i - \epsilon_i = (\delta_{ij}-n_i n_j)\epsilon_j. \tag{9}$$

Averaging over polarizations, we then find:

$$<|\mathbf{n}\wedge(\mathbf{n}\wedge\boldsymbol{\epsilon})|^2> = <(\delta_{ij}-n_i n_j)\epsilon_j(\delta_{ik}-n_i n_k)\epsilon_k^*> = (\delta_{ij}-n_i n_j)(\delta_{ik}-n_i n_k)<\epsilon_j\epsilon_k^*>. \tag{10}$$

Now, $\boldsymbol{\epsilon}$ lives in the plane orthogonal to the wave vector of the electromagnetic wave; assuming the wave is propagating in the $\mathbf{z}$ direction, $\mathbf{z} = (0, 0, 1)$, $\boldsymbol{\epsilon}$ must be a vector in the (x, y) plane. If the incoming wave is not polarized, then $< \epsilon_j \epsilon_k^* >$ must be the most symmetric tensor in the (x, y) plane, so we can write:

$$< \epsilon_j \epsilon_k^* >= a(\delta_{jk} - z_j z_k), \tag{11}$$

where a is a constant that can be obtained by noticing that $\delta_{ij} \epsilon_i \epsilon_j^* = 1$. So we have:

$$1 = \delta_{ij} < \epsilon_i \epsilon_j^* >= a(3 - 1) = 2a, \tag{12}$$

so that $a = 1/2$. We have finally that the average over polarizations can be written as:

$$< \epsilon_j \epsilon_k^* >= \frac{1}{2}(\delta_{jk} - z_j z_k). \tag{13}$$

We can now compute the average over time and polarization of the quantity in Eq. (9):

$$\begin{aligned} < |\mathbf{n} \wedge (\mathbf{n} \wedge \boldsymbol{\epsilon})|^2 > &= (\delta_{ij} - n_i n_j)(\delta_{ik} - n_i n_k)\frac{1}{2}(\delta_{jk} - z_j z_k) = \frac{1}{2}(\delta_{jk} - n_j n_k)(\delta_{jk} - z_j z_k) \\ &= \frac{1}{2}(3 - 1 - 1 + (\mathbf{n} \cdot \mathbf{z})^2) = \frac{1}{2}(1 + \cos^2 \theta). \end{aligned} \tag{14}$$

Putting together the results so far obtained in Eq. (7), we find:

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{16\pi^2 c^4 (\epsilon_0 m)^2} \frac{1}{2}(1 + \cos^2 \theta). \tag{15}$$

The total cross section is easily found from the previous expression:

$$\sigma_{tot} = \int d\Omega \frac{d\sigma}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{e^2}{mc^2}\right)^2 \frac{8\pi}{3}. \tag{16}$$

In the case of the electron, the quantity

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2} = 2.82 \cdot 10^{-13} cm \tag{17}$$

is called classical electron radius.