Solution 9

a) The partition function looks like:

$$Z = \sum_{\sigma_1,\sigma_3,\ldots} \quad \left[\sum_{\sigma_2} \exp\left(K\sigma_1\sigma_2 + \tfrac{1}{2}H(\sigma_1+\sigma_2)\right) \exp\left(K\sigma_2\sigma_3 + \tfrac{1}{2}H(\sigma_2+\sigma_3)\right)\right.$$
$$\left.\sum_{\sigma_4} \exp\left(K\sigma_3\sigma_4 + \tfrac{1}{2}H(\sigma_3+\sigma_4)\right) \exp\left(K\sigma_4\sigma_5 + \tfrac{1}{2}H(\sigma_4+\sigma_5)\right)\ldots\right]$$

We pose:

$$\begin{aligned}
& C\exp\left(K'\sigma_1\sigma_3 + \frac{1}{2}H'(\sigma_1+\sigma_3)\right) = \\
= \; & \sum_{\sigma_2} \exp\left(K\sigma_1\sigma_2 + \frac{1}{2}H(\sigma_1+\sigma_2)\right) \exp\left(K\sigma_2\sigma_3 + \frac{1}{2}H(\sigma_2+\sigma_3)\right) = \\
= \; & \exp\left(\frac{1}{2}H(\sigma_1+\sigma_3)\right) \sum_{\sigma_2} \exp\left(K\sigma_2(\sigma_1+\sigma_3) + H\sigma_2\right) = \\
= \; & \exp\left(\frac{1}{2}H(\sigma_1+\sigma_3)\right) 2\cosh\left(K(\sigma_1+\sigma_3) + H\right) \qquad (1)
\end{aligned}$$

b) We evaluate the equation (1) at $\sigma_1 = \sigma_3 = 1$, $\sigma_1 = -\sigma_3 = 1$ and $\sigma_1 = \sigma_3 = -1$.

$$\begin{cases} e^H 2\cosh(2K+H) = Ce^{K'+H'} \\ 2\cosh(H) = Ce^{-K'} \\ e^{-H} 2\cosh(2K-H) = Ce^{K'-H'} \end{cases}$$

After some manipulation we obtain:

$$\begin{cases} H' = H + \frac{1}{2}\ln\left(\frac{\cosh(2K+H)}{\cosh(2K-H)}\right) \\ K' = \frac{1}{4}\ln\left(\frac{\cosh(2K+H)\cosh(2K-H)}{\cosh^2 H}\right) \\ C^2 = 4\cosh(H)\left(\cosh(2K+H)\cosh(2K-H)\right)^{\frac{1}{2}} \end{cases}$$

c) If $K^* = 0$, it is clear that $K^{*'} = 0$ and $H' = H \; \forall H$.

If $H^* = 0$ the second equation gives

$$e^{2K} = \frac{e^{2K} + e^{-2K}}{2}$$

which is satisfied if $K = 0$, which is a particular case of the previous solution.

The solution $K = \infty$ corresponds to the solution at $\beta = \infty$ that is $T = 0$, which is the ferromagnetic point of the Ising model in one dimension, so we expect to find it as a fixed point. In order to see this it is worth to make the substitution: $x = e^{-K}$ and $y = e^{-H}$ (and of course $x' = e^{-K'}$

and $y' = e^{-H'}$) so the infinite value $K = \infty$ corresponds more properly to $x = 0$. In particular, substituting in the first equation of point b) we find the renormalization relations:

$$\begin{cases} x'^4 = \frac{(1+y^2)^2 x^4}{(1+x^4y^2)(y^2+x^4)} \\ y'^2 = \frac{y^2(y^2+x^4)}{1+x^4y^2} \\ C = \frac{1+y^2}{y}\left(\frac{(1+x^4y^2)(y^2+x^4)}{(1+y^2)^2x^4}\right)^{1/4} \end{cases}$$

The case $K^* = 0$ corresponds to $x^* = 1$ and the equations are satisfied $\forall y$. The case $H^* = 0$ corresponds to $y^* = 1$ and the equations are satisfied if $x = 0$ (ferromagnetic) or $x = 1$ (paramagnetic).

d) For small H:

$$\begin{aligned} H' &= H + \frac{1}{2}\ln\left(\frac{\cosh(2K+H)}{\cosh(2K-H)}\right) \\ &= H + \frac{1}{2}\ln\left(\frac{\cosh(2K)+\sinh(2K)H}{\cosh(2K)-\sinh(2K)H}\right) \\ &= H + \frac{1}{2}\ln\left(\frac{1+\tanh(2K)H}{1-\tanh(2K)H}\right) \\ &= H + H\tanh(2K) \end{aligned} \tag{2}$$

and:

$$\begin{aligned} K' &= \frac{1}{4}\ln\left(\frac{\cosh(2K+H)\cosh(2K-H)}{\cosh^2 H}\right) \\ &= \frac{1}{4}\ln\left((\cosh(2K)+\sinh(2K)H)(\cosh(2K)-\sinh(2K)H)\right) \\ &= \frac{1}{4}\ln\left((\cosh^2(2K)\right) = \frac{1}{2}\ln\left((\cosh(2K)\right) \\ \Leftrightarrow \tanh(K') &= \frac{\cosh^{\frac{1}{2}}(2K) - \cosh^{-\frac{1}{2}}(2K)}{\cosh^{\frac{1}{2}}(2K) + \cosh^{-\frac{1}{2}}(2K)} \\ &= \frac{\cosh(2K)-1}{\cosh(2K)+1} \\ &= \tanh^2(K) \end{aligned} \tag{3}$$

e) Substituting $v = tanh(K)$ the ferromagnetic case $K^* = \infty$ corresponds to $v^* = 1$ and the obtained equations (for small H) are:

$$\begin{cases} v' = v^2 \\ H' = H\left(1 + \frac{2v}{1+v^2}\right) \end{cases}$$

The matrix of first derivatives is

$$\begin{pmatrix} 2v & 0 \\ 2H\frac{1-v^2}{(1+v^2)^2} & 1+\frac{2v}{1+v^2} \end{pmatrix}$$

To linearize the transformation around $(v^* = 1, H^* = 0)$ means to approximate the transformation at the first order in the Taylor expansion: we just need to evaluate the first derivative matrix at $(v^* = 1, H^* = 0)$ whhich gives:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

An equilibrium point is stable if all the eigenvalues of the linearized application have absolute value less than 1. This matrix has two eigenvalues $\lambda_1 = \lambda_2 = 2$. So the point is not stable. Physically this means that as soon as the temperature is different than zero the system is no more ferromagnetic (so no transition phase).

Around $v^* = 0$ and small H, the linearized transformation is:

$$\begin{pmatrix} 0 & 0 \\ 2H & 1 \end{pmatrix}$$

We conclude that the variety is stable along the v axes because the eigenvalue is 0. So as soon as $T \neq 0$ the system tends to the point $v = 0$. But we can say nothing about H, because the eigenvalue is 1: this means that it behaves as a power law and not as an exponential low.