Solution 9

a) The partition function looks like:

\[ Z = \sum_{\sigma_1, \sigma_3, \ldots} \left( \sum_{\sigma_2} \exp \left( K \sigma_1 \sigma_2 + \frac{1}{2} H (\sigma_1 + \sigma_2) \right) \exp \left( K \sigma_2 \sigma_3 + \frac{1}{2} H (\sigma_2 + \sigma_3) \right) \right) \]

\[ \sum_{\sigma_4} \exp \left( K \sigma_3 \sigma_4 + \frac{1}{2} H (\sigma_3 + \sigma_4) \right) \exp \left( K \sigma_4 \sigma_5 + \frac{1}{2} H (\sigma_4 + \sigma_5) \right) \ldots \]

We pose:

\[ C \exp \left( K' \sigma_1 \sigma_3 + \frac{1}{2} H' (\sigma_1 + \sigma_3) \right) = \]

\[ = \sum_{\sigma_2} \exp \left( K \sigma_1 \sigma_2 + \frac{1}{2} H (\sigma_1 + \sigma_2) \right) \exp \left( K \sigma_2 \sigma_3 + \frac{1}{2} H (\sigma_2 + \sigma_3) \right) = \]

\[ = \exp \left( \frac{1}{2} H (\sigma_1 + \sigma_3) \right) \sum_{\sigma_2} \exp \left( K \sigma_2 (\sigma_1 + \sigma_3) + H \sigma_2 \right) = \]

\[ = \exp \left( \frac{1}{2} H (\sigma_1 + \sigma_3) \right) 2 \cosh (K (\sigma_1 + \sigma_3) + H) \]

(1)

b) We evaluate the equation (1) at \( \sigma_1 = \sigma_3 = 1, \sigma_1 = -\sigma_3 = 1 \) and \( \sigma_1 = \sigma_3 = -1 \).

\[ \left\{ \begin{array}{l}
\exp (2K) \cosh (2K + H) = Ce^{K' + H'} \\
2 \cosh (H) = Ce^{-K'} \\
\exp (-2K) \cosh (2K - H) = Ce^{K' - H'}
\end{array} \right. \]

After some manipulation we obtain:

\[ \left\{ \begin{array}{l}
H' = H + \frac{1}{2} \ln \left( \frac{\cosh (2K + H)}{\cosh (2K - H)} \right) \\
K' = \frac{1}{4} \ln \left( \frac{\cosh (2K + H) \cosh (2K - H)}{\cosh^2 H} \right) \\
C^2 = 4 \cosh (H) \left( \cosh (2K + H) \cosh (2K - H) \right)^{\frac{1}{2}}
\end{array} \right. \]

c) If \( K^* = 0 \), it is clear that \( K^{*'} = 0 \) and \( H' = H \forall H \).

If \( H^* = 0 \) the second equation gives

\[ e^{2K} = \frac{e^{2K} + e^{-2K}}{2} \]

which is satisfied if \( K = 0 \), which is a particular case of the previous solution.

The solution \( K = \infty \) corresponds to the solution at \( \beta = \infty \) that is \( T = 0 \), which is the ferromagnetic point of the Ising model in one dimension, so we expect to find it as a fixed point. In order to see this it is worth to make the substitution: \( x = e^{-K} \) and \( y = e^{-H} \) (and of course \( x' = e^{-K'} \))
and \( y' = e^{-H'} \) so the infinite value \( K = \infty \) corresponds more properly to \( x = 0 \). In particular, substituting in the first equation of point b) we find the renormalization relations:

\[
\begin{align*}
    x' &= \frac{(1+y^2)^2 x^4}{(1-x^4 y^2)(y^2+x^4)} \\
y'^2 &= \frac{y^2(y^2+x^4)}{1+x^4 y^2} \\
C &= \frac{1+y^2}{y} \left( \frac{(1+x^4 y^2)(y^2+x^4)}{(1+y^2)^2} \right)^{1/4}
\end{align*}
\]

The case \( K^* = 0 \) corresponds to \( x^* = 1 \) and the equations are satisfied \( \forall y \). The case \( H^* = 0 \) corresponds to \( y^* = 1 \) and the equations are satisfied if \( x = 0 \) (ferromagnetic) or \( x = 1 \) (paramagnetic).

d) For small \( H \):

\[
\begin{align*}
    H' &= H + \frac{1}{2} \ln \left( \frac{\cosh(2K + H)}{\cosh(2K - H)} \right) \\
    &= H + \frac{1}{2} \ln \left( \frac{\cosh(2K) + \sinh(2K)H}{\cosh(2K) - \sinh(2K)H} \right) \\
    &= H + \frac{1}{2} \ln \left( \frac{1 + \tanh(2K)H}{1 - \tanh(2K)H} \right) \\
    &= H + H \tanh(2K)
\end{align*}
\]

and:

\[
\begin{align*}
    K' &= \frac{1}{4} \ln \left( \frac{\cosh(2K + H) \cosh(2K - H)}{\cosh^2 H} \right) \\
    &= \frac{1}{4} \ln \left( (\cosh(2K) + \sinh(2K)H)(\cosh(2K) - \sinh(2K)H) \right) \\
    &= \frac{1}{4} \ln \left( (\cosh^2(2K)) = \frac{1}{2} \ln \left( (\cosh(2K)) \right) \right) \\
    \leftrightarrow \tanh(K') &= \frac{\cosh^2(2K) - \cosh^2(2K)}{\cosh^2(2K) + \cosh^2(2K)} \\
    &= \frac{\cosh(2K) - 1}{\cosh(2K) + 1} \\
    &= \tanh^2(K)
\end{align*}
\]

e) Substituting \( v = \tanh(K) \) the ferromagnetic case \( K^* = \infty \) corresponds to \( v^* = 1 \) and the obtained equations (for small \( H \)) are:

\[
\begin{align*}
    v' &= v^2 \\
    H' &= H \left( 1 + \frac{2v}{1+v^2} \right)
\end{align*}
\]

The matrix of first derivatives is

\[
\begin{pmatrix}
    2v & 0 \\
    2H \frac{1-v^2}{(1+v^2)^2} & 1 + \frac{2v}{1+v^2}
\end{pmatrix}
\]
To linearize the transformation around \( (v^* = 1, H^* = 0) \) means to approximate the transformation at the first order in the Taylor expansion: we just need to evaluate the first derivative matrix at \( (v^* = 1, H^* = 0) \) which gives:

\[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]

An equilibrium point is stable if all the eigenvalues of the linearized application have absolute value less than 1. This matrix has two eigenvalues \( \lambda_1 = \lambda_2 = 2 \). So the point is not stable. Physically this means that as soon as the temperature is different than zero the system is no more ferromagnetic (so no transition phase).

Around \( v^* = 0 \) and small \( H \), the linearized transformation is:

\[
\begin{pmatrix}
0 & 0 \\
2H & 1
\end{pmatrix}
\]

We conclude that the variety is stable along the \( v \) axes because the eigenvalue is 0. So as soon as \( T \neq 0 \) the system tends to the point \( v = 0 \). But we can say nothing about \( H \), because the eigenvalue is 1: this means that it behaves as a power law and not as an exponential low.