Quantum Field Theory

Set 9: solutions

Exercise 1

In addition to the usual transformation properties under the Poincaré group (that define scalars, vectors, spinors, ...), any given field can have non trivial transformation properties under the action of additional symmetries. In the present case we consider a pair of Lorentz scalars ϕ_1 and ϕ_2, which are assumed to transform in the representation $j = 1/2$ of an $SU(2)$ group. This *isospin* symmetry acts on fields as

$$\begin{aligned}
&\Phi(x) \longrightarrow \Phi'(x') = \mathcal{U}\Phi(x), \qquad \Phi^\dagger(x) \longrightarrow \Phi'^\dagger(x') = \Phi^\dagger(x)\mathcal{U}^\dagger, \\
&\phi_a(x) \longrightarrow \phi'_a(x') = \mathcal{U}_a{}^b\phi_b(x), \qquad \phi^{*a}(x) \longrightarrow \phi'^{*a}(x') = \phi^{*b}(x)(\mathcal{U}^\dagger)_b{}^a,
\end{aligned}$$

where $\mathcal{U}$ is a $SU(2)$ element and we have defined the complex conjugate fields with the index up to recall that they transform with the hermitian conjugate matrix (they are a row instead that a column). The Lagrangian density

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi - V\left(\Phi^\dagger\Phi\right)$$

is invariant because the two doublets Φ and $\Phi^\dagger$ are contracted in such a way to form an invariant under $SU(2)$ transformations (that is to say an object transforming in the $j = 0$ representation):

$$\begin{aligned}
&\Phi^\dagger\Phi \longrightarrow \Phi'^\dagger\Phi' = \Phi^\dagger\underbrace{\mathcal{U}^\dagger\mathcal{U}}_{1}\Phi, \\
&\partial_\mu\Phi^\dagger\partial^\mu\Phi \longrightarrow \partial_\mu\Phi'^\dagger\partial^\mu\Phi' = \partial_\mu(\Phi^\dagger\mathcal{U}^\dagger)\partial^\mu(\mathcal{U}\Phi) = \partial_\mu\Phi^\dagger\underbrace{\mathcal{U}^\dagger\mathcal{U}}_{1}\partial^\mu\Phi,
\end{aligned}$$

where we have used the definition of $SU(2)$ matrices. Since the invariance of the Lagrangian density has been proved for a generic potential being function only of the $SU(2)$ invariant combination $\Phi^\dagger\Phi$, one can go on and compute the Noether's current associated to this symmetry. Using the exponential representation of the $SU(2)$ matrices in terms of the generators (which we showed to be the Pauli matrices divided by two in the $j = 1/2$ representation)

$$\begin{aligned}
&\phi'_a(x') = \left(e^{i\alpha^i\frac{\sigma^i}{2}}\right)_a{}^b\phi_b(x) \simeq \phi_a(x) + i\alpha^i\left(\frac{\sigma^i}{2}\right)_a{}^b\phi_b \equiv \phi_a(x) + \Delta_{\phi a}{}^i(x)\alpha^i, \\
&\phi'^{*c}(x') = \phi^{*b}(x)\left(e^{-i\alpha^i\frac{\sigma^i}{2}}\right)_b{}^c \simeq \phi^{*c}(x) - i\alpha^i\phi^{*b}(x)\left(\frac{\sigma^i}{2}\right)_b{}^c \equiv \phi^{*c}(x) + \Delta_{\phi^*}{}^{ci}(x)\alpha^i,
\end{aligned}$$

where we have defined Δ_ϕ and Δ_{ϕ^*} as the variations of ϕ and ϕ^*, respectively. Note that in this case there is no variation of the position x, so the total variation of the fields coincides with the variation at fixed coordinate, hence the use of the name $\Delta(x)$ instead of $\varepsilon(x)$, see Solution8. The Noether's current then reads

$$\begin{aligned}
J^{i\,\mu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta_{\phi a}{}^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{*a})}\Delta_{\phi^*}{}^{ai} = i\partial^\mu\phi^{*a}\left(\frac{\sigma^i}{2}\right)_a{}^b\phi_b - i\partial^\mu\phi_c\phi^{*b}\left(\frac{\sigma^i}{2}\right)_b{}^c \\
&= \frac{i}{2}\partial^\mu\phi^{*a}(\sigma^i)_a{}^b\phi_b - \frac{i}{2}\phi^{*b}(\sigma^i)_b{}^c\partial^\mu\phi_c = \frac{i}{2}\partial^\mu\Phi^\dagger\sigma^i\Phi - \frac{i}{2}\Phi^\dagger\sigma^i\partial^\mu\Phi.
\end{aligned}$$

One has to notice the similarity of this expression with the Noether's current associated to the $U(1)$ symmetry: the previous current is its obvious generalization and the only difference is the presence of the generators between the fields in the appropriate representation. Another point to be stressed is the complete independence of the result here obtained from the explicit form of the potential V, since this doesn't contain derivative of the fields.

Let us now consider another set of three Lorentz scalar fields $\vec{A} = (A_1,\, A_2,\, A_3)$ and impose that under the action of $SU(2)$ they transform in the representation $j = 1$:

$$\begin{aligned}
&\Phi(x) \longrightarrow \Phi'(x') = \mathcal{U}\Phi(x), \qquad \Phi^\dagger(x) \longrightarrow \Phi'^\dagger(x') = \Phi^\dagger(x)\mathcal{U}^\dagger, \\
&\vec{A}(x) \longrightarrow \vec{A}(x') = \mathcal{R}[\mathcal{U}]^{(j=1)}\vec{A}.
\end{aligned}$$

Notice that the representation $j=1$ of $SU(2)$ consists of rotations acting on a three dimensional vector space (and coincides with $SO(3)$ rotations) therefore we have collectively denoted the fields A_i with a vector $\vec{A}$. The matrix $\mathcal{R}[\mathcal{U}]^{(j=1)}$ is the representative of $\mathcal{U}$ in the representation $j=1$, that is to say the element of $SO(3)$ associated to $\mathcal{U}$. We add to the previous Lagrangian density a kinetic term for the fields A and an interaction term:

$$\tilde{\mathcal{L}} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - V\left(\Phi^\dagger \Phi\right) + \frac{1}{2}\partial_\mu A^T \partial^\mu A + \lambda A_i \Phi^\dagger \sigma^i \Phi.$$

The last term of the r.h.s. is called an *interaction* term since it connects the equations of motion for the different fields:

$$\begin{aligned}
\phi_a : \qquad \Box \phi^{*a} &= -\frac{\partial V}{\partial \phi_a} + \lambda A_i \phi^{*c} (\sigma^i)_c{}^a, \\
\phi^{*c} : \qquad \Box \phi_c &= -\frac{\partial V}{\partial \phi^{*c}} + \lambda A_i (\sigma^i)_c{}^a \phi_a, \\
A_i : \qquad \Box A_i &= \lambda \Phi^\dagger \sigma^i \Phi.
\end{aligned}$$

One has now to check whether the part of the Lagrangian involving A is invariant under $SU(2)$ transformations:

$$\partial_\mu A^T \partial^\mu A \longrightarrow \partial_\mu A'^T \partial^\mu A' = \partial^\mu A^T \underbrace{(\mathcal{R}[\mathcal{U}]^{(j=1)})^T \mathcal{R}[\mathcal{U}]^{(j=1)}}_{1} \partial_\mu A = \partial_\mu A^T \partial^\mu A,$$

because $\mathcal{R}[\mathcal{U}]^{(j=1)}$ is a matrix of $SO(3)$. The interaction term is more subtle (from now on we will omit the notation $(j=1)$):

$$A'_i \Phi'^\dagger \sigma^i \Phi' = \mathcal{R}[\mathcal{U}]_i{}^j A_j \Phi^\dagger (\mathcal{U}^\dagger \sigma^i \mathcal{U}) \Phi.$$

In order to complete the proof of the invariance of $\mathcal{L}$ under $SU(2)$, one should recall the definition of *adjoint* representation of $SU(2)$, that is to say the representation acting on the algebra itself. In Solution5 we have shown that, given the space of the hermitian traceless 2×2 matrices M (which coincides with the Lie algebra of $SU(2)$), the action of an element $\mathcal{V}$ of $SU(2)$, defined by

$$M = x_i \sigma^i \longrightarrow \mathcal{V} M \mathcal{V}^\dagger = M' = x'_i \sigma^i,$$

induces on this space a representation which turns out to be the $j=1$ representation, since $x'_i = \mathcal{R}[\mathcal{V}]_i{}^j x_j$. In particular the Pauli matrices themselves transform as a triplet:

$$\sigma^k = \delta_i^k \sigma^i \longrightarrow \mathcal{V}(\delta_i^k \sigma^i)\mathcal{V}^\dagger = (\mathcal{R}[\mathcal{V}]_i{}^j \delta_j^k)\sigma^i = \mathcal{R}[\mathcal{V}]_i{}^k \sigma^i$$

In the case at hand we have $\mathcal{U}^\dagger \sigma^i \mathcal{U} = \mathcal{R}[\mathcal{U}^\dagger]_l{}^i \sigma^l$, and collecting all together we obtain:

$$A'_i \Phi'^\dagger \sigma^i \Phi' \longrightarrow \underbrace{\mathcal{R}[\mathcal{U}]_i{}^j \mathcal{R}[\mathcal{U}^\dagger]_l{}^i}_{\delta_l^j} A_j \Phi^\dagger \sigma^l \Phi = A_i \Phi^\dagger \sigma^i \Phi,$$

where we have used the fact that $(\mathcal{R}[\mathcal{U}^\dagger]^{(j=1)} \mathcal{R}[\mathcal{U}]^{(j=1)})_i{}^m = \mathcal{R}[\mathcal{U}^\dagger \mathcal{U}]_i{}^m = \mathcal{R}[1]_i{}^m = \delta_i{}^m$. This completes the proof of invariance under the $SU(2)$ symmetry of the Lagrangian density $\tilde{\mathcal{L}}$.

Exercise 2

We want to show the invariance under Lorentz transformations of the measure over momentum space $\frac{d^3k}{(2\pi)^3 2k_0}$, where $k_0 \equiv \omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2}$ is the energy associated to a given particle of mass m. There are two ways to achieve the result: the first consists in checking explicitly the invariance performing a Lorentz transformation on momenta; however we first prove it performing a manipulation. The measure can be rewritten as

$$\frac{d^3k}{(2\pi)^3 2k_0} = \frac{d^3k}{(2\pi)^3}\, dk_0\, \delta(k^2 - m^2)\theta(k_0) = \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k_0). \tag{1}$$

In order to convince oneself that this is true, one can consider a test function f of momenta and integrate it over k_0:

$$\int \frac{d^3k}{(2\pi)^3} dk_0 \delta(k_0^2 - |\vec{k}|^2 - m^2)\theta(k_0) f(\vec{k}, k_0) = \int \frac{d^3k}{(2\pi)^3} dk_0 \left(\frac{\delta(k_0 + \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} + \frac{\delta(k_0 - \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} \right) \theta(k_0) f(\vec{k}, k_0),$$

where we have used the well known relation for the δ function: given a function $g(x)$ which vanishes in the points $\{x_1, ..., x_n\}$, then

$$\delta(g(x)) = \sum_{i=1}^{n} \delta(x - x_i) \frac{1}{|g'(x_i)|}.$$

In the present case the equation $(k_0)^2 - |\vec{k}|^2 - m^2 = 0$ admits two opposite solutions, hence the two terms in the parenthesis, but the theta function gets rid of the second one since in that case $k_0 < 0$. Finally, integrating on k_0 one gets the initial measure:

$$\int \frac{d^3k}{(2\pi)^3} dk_0 \frac{\delta(k_0 - \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} \theta(k_0) f(\vec{k}, k_0) = \int \frac{d^3k}{(2\pi)^3 2\sqrt{|\vec{k}|^2 + m^2}} f(\vec{k}, \sqrt{|\vec{k}|^2 + m^2}).$$

Notice that the integrated function depends only on $\vec{k}$ (and the measure we are considering is defined on three-momenta). We have formally extended it to be a function of $\vec{k}$, k_0 but this is only a trick because the δ function forces the variables to be related by the *mass shell condition* $k^2 = m^2$.
The form of the measure we got allows us to show immediately the invariance under a Lorentz transformations.

- d^4k is invariant since the jacobian determinant of the change of variables is 1.
- $\delta(k^2 - m^2)$ is a function of the scalar $k^2 = k_\mu k^\mu$ and therefore it's itself invariant.
- The theta function is not a priori invariant under Lorentz transformations: it is so only if the sign of k'_0 is the same of that of k_0. Lorentz transformations in general don't preserve the sign of the 0-component of a four vector: for example if $v_\mu = (1, 0, 0, 2)$, then one can easily find a boost in the third direction that makes $v'_0 < 0$:

  $$v'_0 = \gamma v_0 - \beta\gamma v_3 = (1 - 2\beta)\gamma \implies \beta \geq \frac{1}{2}.$$

  However one has to recall that the mass-shell condition $k_0 = \sqrt{|\vec{k}|^2 + m^2}$ makes k_μ a timelike fourvector (a fourvector in which the 0-component is larger than the modulus of its spatial threevector), that is to say a vector which lies inside the future lightcone centered in the origin. Transformations of the orthochronous Lorentz group, defined by the condition ${\Lambda^0}_0 > 0$ (those non containing the time reversal) send the future lightcone in itself and therefore if the four vector k_μ have positive 0-component, the same will be for k'_μ.
  One can prove this explicitly. The transformed 0-component of k^μ is $k'^0 = {\Lambda^0}_0 k^0 + \sum_i {\Lambda^i}_0 k^i$, with $k^0 \geq \sqrt{\sum_i (k^i)^2}$, due to the mass-shell condition. The defining relation $\eta_{\mu\nu} {\Lambda^\mu}_\rho {\Lambda^\nu}_\sigma = \eta_{\rho\sigma}$ implies ${\Lambda^0}_0 > \sqrt{\sum_i ({\Lambda^i}_0)^2}$. Moreover, since $\sum_i {\Lambda^i}_0 k^i \geq -\sqrt{\sum_i (k^i)^2}\sqrt{\sum_i ({\Lambda^i}_0)^2}$, then:

  $$(k^0)' \geq {\Lambda^0}_0 k^0 - \sqrt{\sum_i (k^i)^2}\sqrt{\sum_i ({\Lambda^i}_0)^2} \geq \left({\Lambda^0}_0 - \sqrt{\sum_i ({\Lambda^i}_0)^2} \right) k^0 > 0$$

  implying that the sign of k'^0 is the same as the one of k^0.
  Therefore the theta function will be left invariant in the distribution (1).

One can also check it explicitly performing Lorentz transformations: clearly the measure $\frac{d^3k}{(2\pi)^3 2k_0}$ is invariant under space rotations since d^3k is so and k_0 is a scalar under $SO(3)$. Therefore only pure boosts are left to check. Consider then a boost in the direction $\vec{n}$ with rapidity η (recall that the rapidity is defined as $\eta = \tanh^{-1}(\beta)$): we decompose the spatial momentum $\vec{k}$ in its longitudinal (i.e. along $\vec{n}$) and transverse (i.e. orthogonal to $\vec{n}$) parts:

$\vec{k} = \vec{k}_T + \vec{k}_L$. Then the transformed quantities are:

$$\begin{aligned} k'_0 &= k_0 \cosh(\eta) + |\vec{k}_L| \sinh(\eta), \\ \vec{k}'_L &= k_0 \vec{n} \sinh(\eta) + \vec{k}_L \cosh(\eta), \\ \vec{k}'_T &= \vec{k}_T. \end{aligned}$$

Note that the direction of $\vec{k}_L$ is fixed to be parallel to $\vec{n}$, therefore one can remove the symbol of vector and consider only the modulus k_L. One can as well decompose the differential $d^3k \longrightarrow d^2k_T\, dk_L$; therefore the measure transforms as:

$$\begin{aligned} d^2k'_T &= d^2k_T, \\ dk'_L &= \frac{\partial k'_L}{\partial k_L} dk_L = \frac{\partial}{\partial k_L}(k_0 \sinh(\eta) + k_L \cosh(\eta)) dk_L = \frac{\partial}{\partial k_L}(\sqrt{k_L^2 + |\vec{k}_T|^2 + m^2}\ \sinh(\eta) + k_L \cosh(\eta)) dk_L \\ &= \left(\frac{k_L}{\sqrt{k_L^2 + |\vec{k}_T|^2 + m^2}} \sinh(\eta) + \cosh(\eta) \right) dk_L = \frac{1}{k_0} \left(k_L \sinh \eta + k_0 \cosh \eta \right) dk_L = \frac{k'_0}{k_0} dk_L. \end{aligned}$$

At the end the ratio $\frac{d^3k}{(2\pi)^3 2k_0}$ is invariant also under Lorentz boosts.

Finally one can consider the distribution $d^3k\delta^3(\vec{k})$. The fastest way to see that it is invariant is to see it as the result of an integration over k_0:

$$d^3k\delta^3(\vec{k}) = \int dk^0 d^3k\, \delta^4(k) = \int d^4k\, \delta^4(k) \tag{2}$$

The expression on the right-hand side is Lorentz invariant because $d^4k' = |J(\Lambda)| d^4k$ and $\delta^4(k') = \delta^4(k)|J(\Lambda)|^{-1}$, where $J(\Lambda)$ is the determinant of the Jacobian of the Lorentz transformation.
Finally, the distribution $(2\pi)^3 k^0 \delta^3(\vec{k})$ is Lorentz-invariant because it's obtained by dividing the two invariant distributions $d^3k\delta^3(\vec{k})$ and $\frac{d^3k}{(2\pi)^3 k^0}$

Exercise 3

To begin with, we derive formally the expression of the Noether's current for a general Lagrangian density $\mathcal{L}[\phi_a, \partial_\mu \phi_a]$ invariant under a Lie group defined by the following transformations

$$\begin{aligned} &x^\mu \longrightarrow x'^\mu = f^\mu(x) \simeq x^\mu - \epsilon_i^\mu \alpha^i, \\ &\phi_a(x) \longrightarrow \phi'_a(x') = D[\phi]_a(f^{-1}(x')) \simeq \phi_a(x) + \alpha^i \varepsilon[\phi]_{ai}(x) \simeq \phi_a(x') + \alpha^i \Delta[\phi]_{ai}(x'). \end{aligned}$$

The action written in terms of the transformed fields is

$$\begin{aligned} &\int_{\Omega'} \mathcal{L}[\phi'_a, \partial'_\mu \phi'_a](x')\, d^4x' \simeq \int_{\Omega'} \left(\mathcal{L}[\phi_a, \partial_\mu \phi_a](x') + \frac{\partial \mathcal{L}}{\partial \phi'_a}(x') \Delta_{ai}(x') \alpha^i + \frac{\partial \mathcal{L}}{\partial (\partial'_\mu \phi'_a)}(x')\, \partial'_\mu (\Delta_{ai}(x')\alpha^i) + O(\alpha^2) \right) d^4x' \\ &= \int_\Omega \left(\mathcal{L}[\phi_a, \partial_\mu \phi_a](x') + \frac{\partial \mathcal{L}}{\partial \phi'_a}(x') \Delta_{ai}(x') \alpha^i + \frac{\partial \mathcal{L}}{\partial (\partial'_\mu \phi'_a)}(x')\, \partial'_\mu (\Delta_{ai}(x')\alpha^i) + O(\alpha^2) \right) \left| \frac{\partial x}{\partial x'} \right|^{-1} d^4x \\ &= \int_\Omega \left(\mathcal{L}[\phi_a, \partial_\mu \phi_a](x) - \alpha^i \epsilon_i^\mu \partial_\mu \mathcal{L}[\phi_a, \partial_\mu \phi_a](x) + \frac{\partial \mathcal{L}}{\partial \phi_a}(x) \Delta_{ai}(x) \alpha^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)}(x)\, \partial_\mu (\Delta_{ai}(x)\alpha^i) + O(\alpha^2) \right) \left| \frac{\partial x'}{\partial x} \right| d^4x \\ &\simeq \int_\Omega \left(\mathcal{L}[\phi_a, \partial_\mu \phi_a](x) - \alpha^i \epsilon_i^\mu \partial_\mu \mathcal{L}[\phi_a, \partial_\mu \phi_a](x) + \frac{\partial \mathcal{L}}{\partial \phi_a}(x) \Delta_{ai}(x) \alpha^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)}(x)\, \partial_\mu (\Delta_{ai}(x)\alpha^i) \right) (1 - \partial_\mu(\epsilon_i^\mu \alpha^i))\, d^4x. \end{aligned}$$

Note that in last two steps we have traded derivatives w.r.t. primed fields with derivatives w.r.t. untransformed fields in terms that are linear in α, since the difference between ϕ and ϕ' is itself $O(\alpha)$, so the error we are doing is $O(\alpha^2)$. If the parameters α^i are constant, then, neglecting higher order terms, one gets

$$\int_{\Omega'} \mathcal{L}[\phi'_a, \partial'_\mu \phi'_a]\, d^4x' \simeq \int_\Omega \mathcal{L}[\phi_a, \partial_\mu \phi_a]\, d^4x + \alpha^i \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \phi_a}(x) \Delta_{ai}(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu \Delta_{ai}(x) - \partial_\mu \left(\mathcal{L}(x) \epsilon_i^\mu(x) \right) \right) d^4x.$$

The request that the transformations define a symmetry translates in the equation

$$\frac{\partial \mathcal{L}}{\partial \phi_a}(x)\Delta_{ai}(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu \Delta_{ai}(x) - \partial_\mu \left(\mathcal{L}(x)\epsilon_i^\mu(x)\right) = 0 \tag{3}$$

that, adding a subtracting the term $\partial_\mu \left(\partial\mathcal{L}/\partial(\partial_\mu \phi_a)\right) \Delta_{ai}(x)$, can be rewritten as

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \Delta_{ai} - \mathcal{L}\epsilon_i^\mu \right) + \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \Delta_{ai} = 0.$$

The quantity in first parentheses is the Noether current, and its divergence vanishes if the fields satisfy the equations of motion.

Let us now consider the following construction:

- Consider the same (global) symmetry defined at the beginning of the exercise.
- Define the following transformation:

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x^\mu - \epsilon_i^\mu(x)\beta^i(x), \\ \phi_a(x) &\longrightarrow \phi'_a(x') \simeq \phi_a(x') + \Delta[\phi]_{ai}(x')\beta^i(x'), \end{aligned}$$

  where Δ_{ai} and ϵ_i^μ are the ones given by the symmetry transformation at the beginning of the exercise. Note that the parameters β^i depend explicitly on x. In general this transformation will *not* define a symmetry of the theory, in the sense that if the theory is invariant under the global transformation, it is not said that it's invariant under its local version (while the converse is always true, since the local symmetry contains the global one as a particular case).

- Compute the variation of the action to first order in $\beta^i(x)$:

$$\begin{aligned} &\int_{\Omega'} \mathcal{L}[\phi'_a, \partial'_\mu \phi'_a] d^4x' - \int_\Omega \mathcal{L}[\phi_a, \partial_\mu \phi_a] d^4x \\ &\simeq \int_\Omega \left(-\beta^i \epsilon_i^\mu \partial_\mu \mathcal{L}[\phi_a, \partial_\mu \phi_a] - \mathcal{L}[\phi_a, \partial_\mu \phi_a] \partial_\mu(\epsilon_i^\mu \beta^i) + \frac{\partial \mathcal{L}}{\partial \phi_a} \Delta_{ai}\beta^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu(\Delta_{ai}\beta^i) \right)(x)\, d^4x \\ &= \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \phi_a} \Delta_{ai}\beta^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu(\Delta_{ai}\beta^i) \right)(x)\, d^4x \end{aligned}$$

  (the first two terms are a total derivative).
  Now plug in the condition for invariance under global transformations, Eq.(3):

$$\begin{aligned} &\int_{\Omega'} \mathcal{L}[\phi'_a, \partial'_\mu \phi'_a] d^4x' - \int_\Omega \mathcal{L}[\phi_a, \partial_\mu \phi_a] d^4x \\ &= \int_\Omega \left(-\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} (\partial_\mu \Delta_{ai})\beta^i + \partial_\mu(\mathcal{L}\epsilon_i^\mu)\beta^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu(\Delta_{ai}\beta^i) \right) d^4x \\ &= \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta_{ai} - \mathcal{L}\, \epsilon_i^\mu \right) \partial_\mu \beta^i \, d^4x \\ &= \int_\Omega j_i^\mu \partial_\mu \beta^i \, d^4x \end{aligned}$$

  where we made use of integration by parts in some steps. This suggests a simple way to find the Noether's current: when a Lagrangian has a global symmetry, one can simply consider the parameters of the transformation as local instead of global (that is, dependent on x instead of constant), and evaluate the change in the action (which is not 0 in general since the local transformation is not a symmetry): this change is proportional to the space-time derivative of the parameters (since for constant parameters it has to vanish), and the coefficient is the current associated to the global transformation.

We can check how this construction works in the explicit example of a $U(1)$ transformation applied to a free massive complex field with Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \, \partial^\mu \phi - m^2 \phi^\dagger \phi.$$

The transformation is simply:

$$\begin{aligned}
&x^\mu \longrightarrow x'^\mu = x^\mu, \\
&\phi(x) \longrightarrow \phi'(x) = \mathcal{U}(\alpha)\phi(x) \equiv e^{i\alpha}\phi(x), \\
&\phi^\dagger(x) \longrightarrow \phi'^\dagger(x) = \phi^\dagger \mathcal{U}^\dagger(\alpha) \equiv \phi(x)^\dagger e^{-i\alpha}.
\end{aligned}$$

Let us apply the procedure and consider the same transformation with parameter $\alpha \to \beta(x)$. In this case the symmetry is *internal* (i.e. it only changes fields, not the space-time), so one can consider the variation in the Lagrangian instead of the variation of the action (i.e. the measure remains the same):

$$\mathcal{L}[\phi'_a, \partial'_\mu \phi'_a] - \mathcal{L}[\phi_a, \partial_\mu \phi_a] = \phi^\dagger(\partial_\mu \mathcal{U}^\dagger)\mathcal{U}(\partial^\mu \phi) + \phi^\dagger(\partial_\mu \mathcal{U}^\dagger)(\partial^\mu \mathcal{U})\phi + (\partial_\mu \phi^\dagger)\mathcal{U}^\dagger(\partial^\mu \mathcal{U})\phi.$$

Considering that $\partial_\mu \mathcal{U} = i\mathcal{U}\partial_\mu \beta$, and $\partial_\mu \mathcal{U}^\dagger = -i\mathcal{U}^\dagger \partial_\mu \beta$, then, at first order in β, one gets

$$\mathcal{L}[\phi'_a, \partial'_\mu \phi'_a] - \mathcal{L}[\phi_a, \partial_\mu \phi_a] = -i(\partial_\mu \beta) \; \phi^\dagger \partial^\mu \phi + i(\partial_\mu \beta) \; \partial^\mu \phi^\dagger \phi = \partial_\mu \beta J^\mu,$$

hence

$$J^\mu = i(\partial^\mu \phi^\dagger \phi - \phi^\dagger \partial^\mu \phi).$$

One can recognize the usual $U(1)$ Noether's current for a complex scalar field.

Exercise 4

Let's first introduce some notation for delta functions, used also in next exercise.

$$\begin{aligned}
&\int d^3x \; e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{k}), \\
&\int d^3k \; e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{x}), \\
&\int d^3x \; \delta^3(\vec{x}) = \int d^3k \; \delta^3(\vec{k}) = 1.
\end{aligned}$$

When $\vec{k}$ can assume only discrete values $\vec{k}_n = 2\pi\vec{n}/L$, the Dirac delta becomes a Kronecker delta, since it has to give 1 when summed, *not* when integrated, and the integral on momenta becomes a sum over numbers. Basically the discrete case can be deduced from the continuous one making the following formal replacements.

$$\begin{aligned}
\delta^3(\vec{k}) \quad &\longrightarrow \quad \left(\frac{L}{2\pi}\right)^3 \delta^3_{\vec{n},\vec{0}}, \\
\int d^3k \quad &\longrightarrow \quad \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{n}\in\mathbb{Z}^3},
\end{aligned}$$

and in particular

$$\int d^3x \; e^{i\vec{k}\cdot\vec{x}} \quad \longrightarrow \quad \int d^3x \; e^{i\vec{k}_n\cdot\vec{x}} = (2\pi)^3 \left(\frac{L}{2\pi}\right)^3 \delta^3_{\vec{n},\vec{0}}.$$

Now one can compute explicitly the required expression:

$$\begin{aligned}
\int d^3x \sum_{\vec{n}\in\mathbb{Z}^3} \frac{1}{L^{3/2}} \phi_n(t) \partial_i \, e^{i\frac{2\pi}{L}\vec{n}\cdot\vec{x}} \sum_{\vec{m}\in\mathbb{Z}^3} \frac{1}{L^{3/2}} \phi_m(t) \partial_i \, e^{i\frac{2\pi}{L}\vec{m}\cdot\vec{x}} &= \frac{1}{L^3}\left(\frac{i2\pi}{L}\right)^2 \sum_{\vec{n},\vec{m}\in\mathbb{Z}^3} \vec{n}\cdot\vec{m} \; \phi_n(t)\phi_m(t) \int d^3x \, e^{i\frac{2\pi}{L}(\vec{m}+\vec{n})\cdot\vec{x}} \\
&= \frac{1}{L^3}\left(\frac{i2\pi}{L}\right)^2 \sum_{\vec{n},\vec{m}\in\mathbb{Z}^3} \vec{n}\cdot\vec{m} \; \phi_n(t)\phi_m(t) L^3 \delta^3_{\vec{n}+\vec{m},\vec{0}} = \left(\frac{2\pi}{L}\right)^2 \sum_{\vec{n}\in\mathbb{Z}^3} |\vec{n}|^2 |\phi_n(t)|^2.
\end{aligned}$$

where we used the fact that $\phi_{-n}(t) = \phi^*_n(t)$ since $\phi(\vec{x}, t)$ is real.

Exercise 5

The first commutator is

$$
\begin{aligned}
[H, a(\vec{p})] &= \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left[a^\dagger(\vec{k})a(\vec{k}), a(\vec{p})\right] = \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left(a^\dagger(\vec{k})\left[a(\vec{k}), a(\vec{p})\right] + \left[a^\dagger(\vec{k}), a(\vec{p})\right]a(\vec{k})\right) \\
&= \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left(0 - (2\pi)^3\delta^3(\vec{p}-\vec{k})a(\vec{k})\right) = -\omega(\vec{p})a(\vec{p}).
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
\left[H, a^\dagger(\vec{p})\right] &= \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left[a^\dagger(\vec{k})a(\vec{k}), a^\dagger(\vec{p})\right] = \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left(a^\dagger(\vec{k})\left[a(\vec{k}), a^\dagger(\vec{p})\right] + \left[a^\dagger(\vec{k}), a^\dagger(\vec{p})\right]a(\vec{k})\right) \\
&= \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left(a^\dagger(\vec{k})(2\pi)^3\delta^3(\vec{p}-\vec{k}) + 0\right) = +\omega(\vec{p})a^\dagger(\vec{p}).
\end{aligned}
$$