Quantum Field Theory

Set 7: solutions

Exercise 1

Given a Lorentz transformation $\Lambda : x \longrightarrow x' = \Lambda x$, the transformation of a scalar field at fixed coordinate x is $\phi(x) \longrightarrow \phi'(x)$. Since the field is scalar, it satisfies $\phi'(x') = \phi(x)$, or $\phi'(x) = \phi(\Lambda^{-1}x)$, which is the representation of the Lorentz transformation on functions we have already met in Set6. We can now expand for infinitesimal transformation:

$$\delta_0\phi(x) \equiv \phi'(x) - \phi(x) = \phi(\Lambda^{-1}x) - \phi(x) \simeq \phi(x - \omega x) - \phi(x) \simeq -\omega^{\mu}{}_{\nu} x^{\nu}\partial_{\mu}\phi(x) = \frac{1}{2}\omega_{\mu\nu}(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\phi(x).$$

This variation has to be identified with the infinitesimal action of the Lorentz generators on scalar fields, namely

$$\delta_0\phi(x) = \exp\left[-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right]\phi(x) - \phi(x) \simeq -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi(x),$$

which proves that

$$L^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$$

is the representation of Lorentz generators on scalar fields. (Notice that the generators in this representation have been denoted as $L^{\mu\nu}$, not as $\mathcal{J}^{\mu\nu}$, since $\mathcal{J}^{\mu\nu}$ are the generators in the defining representation). It's now straightforward to obtain the Lorentz algebra:

$$\begin{aligned}
[L^{\mu\nu}, L^{\rho\sigma}] &= i^2\left[(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}), (x^{\rho}\partial^{\sigma} - x^{\sigma}\partial^{\rho})\right] \\
&= -\left[(x^{\mu}\eta^{\nu\rho}\partial^{\sigma} - x^{\rho}\eta^{\mu\sigma}\partial^{\nu}) + (x^{\nu}\eta^{\mu\sigma}\partial^{\rho} - x^{\sigma}\eta^{\nu\rho}\partial^{\mu}) - (x^{\nu}\eta^{\mu\rho}\partial^{\sigma} - x^{\rho}\eta^{\nu\sigma}\partial^{\mu}) - (x^{\mu}\eta^{\nu\sigma}\partial^{\rho} - x^{\sigma}\eta^{\mu\rho}\partial^{\nu})\right] \\
&= -\left[\eta^{\nu\rho}(x^{\mu}\partial^{\sigma} - x^{\sigma}\partial^{\mu}) + \eta^{\mu\sigma}(x^{\nu}\partial^{\rho} - x^{\rho}\partial^{\nu}) + \eta^{\nu\sigma}(x^{\rho}\partial^{\mu} - x^{\mu}\partial^{\rho}) + \eta^{\mu\rho}(x^{\sigma}\partial^{\nu} - x^{\nu}\partial^{\sigma})\right] \\
&= i\left[\eta^{\nu\rho}L^{\mu\sigma} + \eta^{\mu\sigma}L^{\nu\rho} + \eta^{\nu\sigma}L^{\rho\mu} + \eta^{\mu\rho}L^{\sigma\nu}\right].
\end{aligned}$$

Consider now tranlations $x \longrightarrow x' = x + a$. The transformation of a scalar field at fixed coordinate is $\phi(x) \longrightarrow \phi'(x)$, and since $\phi'(x') \equiv \phi'(x + a) = \phi(x)$, then $\phi'(x) = \phi(x - a)$. Identifying

$$\phi'(x) = \exp[ia^{\mu}P_{\mu}]\phi(x) = \phi(x - a) = \exp[-a^{\mu}\partial_{\mu}]\phi(x),$$

one immediately finds that the representation of the generators of translations on fields is given by

$$P_{\mu} = i\partial_{\mu}.$$

Using this explicit representation, the commutators are

$$\begin{aligned}
[P^{\mu}, P^{\nu}] &= i^2\left[\partial^{\mu}, \partial^{\nu}\right] = 0, \\
[P^{\mu}, L^{\rho\sigma}] &= i^2\left[\partial^{\mu}, x^{\rho}\partial^{\sigma} - x^{\sigma}\partial^{\rho}\right] = -(\eta^{\mu\rho}\partial^{\sigma} - \eta^{\mu\sigma}\partial^{\rho}) = i(\eta^{\mu\rho}P^{\sigma} - \eta^{\mu\sigma}P^{\rho}).
\end{aligned}$$

These relations are general since the structure constants of course do not depend on the representation used to compute the commutators. The above relations define thus the Poincaré algebra in *any* representation. Summarizing, the commutation relations between Poincaré generators are:

$$\begin{aligned}
[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= i\left[\eta^{\nu\rho}\mathcal{J}^{\mu\sigma} + \eta^{\mu\sigma}\mathcal{J}^{\nu\rho} + \eta^{\nu\sigma}J^{\rho\mu} + \eta^{\mu\rho}\mathcal{J}^{\sigma\nu}\right], \\
[P^{\mu}, \mathcal{J}^{\rho\sigma}] &= i(\eta^{\mu\rho}P^{\sigma} - \eta^{\mu\sigma}P^{\rho}), \\
[P^{\mu}, P^{\nu}] &= 0.
\end{aligned}$$

Note. To characterize the Poincaré representation on fields we have considered the variation *at fixed coordinate* x, namely $\delta_0\phi(x)$, not the variation $\delta\phi(x) \equiv \phi'(x') - \phi(x) = 0$: this is because the transformation $x \longrightarrow x'$ simply corresponds to a change of reference frame, i.e. to expressing the position of a given point P, with coordinate x according to observer O, in terms of the coordinates x' of observer O'. In doing so, the point P is kept fixed, so studying the variation $\delta\phi(x)$ corresponds to studying how a single degree of freedom (the value of the field at fixed point P) changes under change of parametrization. The basis for this representation is thus one dimensional, and since $\delta\phi(x) = 0$ the generators in this representation are zero: this is called scalar representation. Conversely, considering the variation keeping fixed the coordinate x, not the point P, means that we are comparing the field at different points, i.e. the point P which is called x by observer O, and the point P' which is called x by observer O'; in this case the base space is the set of functions $\phi(P)$, with P varying in space-time, thus this gives the infinite dimensional representation of the Poincaré group on fields, which is what we were looking for.

A recommended reading on the Lorentz representation on scalar fields is: *M. Maggiore, A Modern Introduction to Quantum Field Theory*, chapters 2.6.1 and 2.7.1.

Exercise 2

Every irreducible finite-dimensional representation of the Lorentz group is defined by a couple (j_+, j_-) where j_+ and j_- label the irreducible representation of the two commuting $SU(2)$ subgroups of $SO(3,1) \sim SU(2)_+ \times SU(2)_-$ generated by

$$\mathbf{J}_\pm = \frac{\mathbf{J} \pm i\mathbf{K}}{2}. \tag{1}$$

Notice that given representations $D_{j_\pm}$ of $SU(2)$ on vector spaces $V_\pm$, the (j_+, j_-) representation act on the vector space $V_+ \times V_-$ as the direct sum representation of D_{j_+} and D_{j_-}. Both $\mathbf{J}_\pm$ are in particular defined on $V_1 \times V_2$, as the generators of such representations.

Consider now the $(1/2, 0)$ representation. In this case V_+ is 2-dimensional and V_- is 1-dimensional, with D_{j_-} being the trivial representation. We can thus forget about V_- in the cartesian product $V_+ \times V_-$ and simply write

$$\mathbf{J}_+ = \frac{\vec{\sigma}}{2}, \qquad \mathbf{J}_- = \vec{0}_{2\times 2} \tag{2}$$

where $\vec{\sigma}$ is the vector of the three Pauli matrices, which furnish the spin-1/2 representation of $SU(2)$. From this the form of $\mathbf{J}$ and $\mathbf{K}$ in the $(1/2, 0)$ follows from eq 1 and 3

$$\mathbf{J} = \frac{\vec{\sigma}}{2}, \qquad \mathbf{K} = -i\frac{\vec{\sigma}}{2}. \tag{3}$$

Given a set of parameters $\vec{\alpha}$ for rotations and another one $\vec{\beta}$ for boosts the explicit form of the elements of the group representation is

$$D_{(1/2,0)}(\vec{\alpha}, \vec{\beta}) = e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\alpha} - i\vec{\beta})}. \tag{4}$$

$D_{(1/2,0)}$ acts on 2-dimensional complex vector with an index a such as

$$s_a \to [D_{(1/2,0)}(\vec{\alpha}, \vec{\beta})]_a{}^b s_b. \tag{5}$$

Notice in particular that $D_{(1/2,0)}(\vec{\alpha}, \vec{\beta})$ is an invertible linear transformation with unit determinant, so that it belongs to $SL(2,\mathbb{C})$ (the universal covering group of $SO(3,1)$).

With a similar reasoning one obtains the explicit form of the $(0, 1/2)$ representation, which differs in the sign of the boost generator $\mathbf{K}$. Since $D_{(1/2,0)}$ and $D_{(0,1/2)}$ are not equivalent, the latter representation acts on a different set of indices. If we denote such indices as dotted one we have

$$s_{\dot{a}} \to [D_{(0,1/2)}(\vec{\alpha}, \vec{\beta})]_{\dot{a}}{}^{\dot{b}} s_{\dot{b}}. \tag{6}$$

The $(1/2, 1/2)$ representation is now readily constructed. Indeed, it is the direct product of the previous representations: $(1/2, 0) \times (0, 1/2) = (1/2 \times 0, 0 \times 1/2) = (1/2, 1/2)$. One introduces an object with 2 kinds of indices, a dotted one, transforming in the $(0, 1/2)$ representation and an un-dotted one transforming under the $(1/2, 0)$:

$$v_{a\dot{a}} \to [D_{(1/2,0)}(\vec{\alpha}, \vec{\beta})]_a{}^b [D_{(0,1/2)}(\vec{\alpha}, \vec{\beta})]_{\dot{a}}{}^{\dot{b}} v_{b\dot{b}}. \tag{7}$$

The object v is defined by 4 complex (8 real) parameters.
Note that the representation $D_{(j_+,j_-)}$ is not unitary. This is consistent with the fact that the Lorentz group, being non-compact, does not admit finite-dimensional unitary representations (but it does admit infinite-dimensional unitary representations, which are required to represent physical states).

Exercise 3

The explicit expression for $\mathcal{J}^{10}$ is

$$\mathcal{J}^{10} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where in the right hand side every entry is understood to be a 2×2 block. In particular, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is one of the Pauli matrices, satisfying $\sigma_1^2 = 1_2$ (this can be shown by explicit computation or using in general the anticommutation relation $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$). It is now possible to write the Lorentz transformation as a Taylor expansion in η:

$$\begin{aligned} \Lambda &= 1_4 - i\eta\mathcal{J}^{10} + \frac{(-i\eta)^2}{2!}(\mathcal{J}^{10})^2 + \cdots \\ &= \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} - \eta \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{(-i\eta)^2}{2!} \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix} + \cdots \\ &\equiv \begin{pmatrix} \lambda & 0 \\ 0 & 1_2 \end{pmatrix}, \end{aligned}$$

with

$$\lambda = 1_2 - \eta\sigma_1 + \frac{\eta^2}{2!}1_2 - \frac{\eta^3}{3!}\sigma_1 + \cdots \,.$$

Separating the terms proportional to 1_2 from the ones proportional to σ_1, and remembering the Taylor series $\cosh(x) = 1 + x^2/2! + \cdots$ and $\sinh(x) = x + x^3/3! + \cdots$, then one can rewrite λ as

$$\lambda = \cosh(\eta)1_2 - \sinh(\eta)\sigma_1 = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) \\ -\sinh(\eta) & \cosh(\eta) \end{pmatrix},$$

which proves what required in the text. Moreover, given the standard form of a boost along x,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

one can immediately identify γ and β in terms of the rapidity as

$$\beta = \tanh(\eta), \qquad \gamma = \cosh(\eta).$$

For what concerns the composition of boosts one can write explicitely

$$\Lambda\Lambda' = \begin{pmatrix} \lambda & 0 \\ 0 & 1_2 \end{pmatrix} \begin{pmatrix} \lambda' & 0 \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} \lambda\lambda' & 0 \\ 0 & 1_2 \end{pmatrix},$$

where

$$\begin{aligned} \lambda\lambda' &= \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) \\ -\sinh(\eta) & \cosh(\eta) \end{pmatrix} \begin{pmatrix} \cosh(\eta') & -\sinh(\eta') \\ -\sinh(\eta') & \cosh(\eta') \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\eta)\cosh(\eta') + \sinh(\eta)\sinh(\eta') & -\cosh(\eta)\sinh(\eta') - \sinh(\eta)\cosh(\eta') \\ -\cosh(\eta)\sinh(\eta') - \sinh(\eta)\cosh(\eta') & \cosh(\eta)\cosh(\eta') + \sinh(\eta)\sinh(\eta') \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\eta+\eta') & -\sinh(\eta+\eta') \\ -\sinh(\eta+\eta') & \cosh(\eta+\eta') \end{pmatrix}, \end{aligned}$$

and thus the composition of two boosts along x is a boost along x, its rapidity being the sum of rapidities of the two single boosts. These computations have been performed for the particular direction x, but can of course be extended to the other axes or to linear combination of them.

Note that the composition of rapidities could be proved without recursion to computations by considering that the total transformation is $\Lambda\Lambda' = \exp[-i\eta\mathcal{J}^{10}]\exp[-i\eta'\mathcal{J}^{10}] = \exp[-i(\eta+\eta')\mathcal{J}^{10}]$, which actually confirms the power and elegance of the exponential mapping.

Exercise 4

Consider a field $\phi_a(x)$ which belongs to a given representation $\mathcal{D}$ of the Poincaré group. A transformation acting on coordinates as

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu,$$

where Λ is an element of the Lorentz group and a^μ is a spacetime translation, induces a transformation on the field defined as follows:

$$\phi_a(x) \longrightarrow \phi'_a(x') = \mathcal{D}(\Lambda)_a{}^b\,\phi_b(x) \Longrightarrow \phi'_a(x) = \mathcal{D}(\Lambda)_a{}^b\,\phi_b(\Lambda^{-1}(x-a)),$$

where the matrix $\mathcal{D}(\Lambda)_a{}^b$ is the representative of the Lorentz transformation in the representation ϕ_a belongs to and acts on the index a only. For example:

- In the scalar representation the Lorentz group is trivially represented and $\mathcal{D}(\Lambda)_a{}^b = 1$ for all Λ. Therefore

$$\phi'(x) = \phi(\Lambda^{-1}(x-a)).$$

- In the vector representation the field transforms like the coordinate four vector and therefore the representation of the group element is Λ itself:

$$\phi'^\rho(x) = \Lambda^\rho{}_\sigma\,\phi^\sigma(\Lambda^{-1}(x-a)).$$

- In the spinorial representation, that we haven't met yet, the matrix $\mathcal{D}(\Lambda)_a{}^b$ is a 2×2 matrix and coincides with an $SU(2)$ matrix.

Let's consider a general action

$$\mathcal{S} = \int dt d^3x\ \mathcal{L}[\phi](x),$$

and consider the transformation acting on coordinates and fields:

$$\begin{aligned}
&x \longrightarrow x' \equiv f(x), \qquad x = f^{-1}(x'),\\
&\phi(x) \longrightarrow \phi'(x') = \mathcal{D}[\phi](x) = \mathcal{D}[\phi](f^{-1}(x')), \qquad \phi(x) = \mathcal{D}^{-1}[\phi'](f(x)).
\end{aligned}$$

One can then implement the transformation on x as the usual change of coordinates in an integral, and in addition express the field ϕ as a function of the transformed one:

$$\mathcal{S} = \int d^4x\ \mathcal{L}[\phi](x) = \int d^4x' |J| \mathcal{L}[\mathcal{D}^{-1}[\phi']](f^{-1}x') \equiv \int d^4x' \mathcal{L}'[\phi'](x').$$

A group of transformation is said to be a symmetry of a theory if the equations of motion have the same structure in terms of transformed quantities. A sufficient condition for this to happen is that the dependence on ϕ' of the functional $\mathcal{L}'$ be exactly the same dependence on ϕ of $\mathcal{L}$. The form of the Lagrangian has to be the same once we express it in terms of transformed field, i.e. $\mathcal{L} = \mathcal{L}'$ as a function, or

$$\text{if} \quad \mathcal{S} \equiv \int d^4x\ \mathcal{L}[\phi](x) = \int d^4x' \mathcal{L}[\phi'](x') \quad \Longrightarrow \quad \text{symmetry.}$$

Notice that in the right hand side of last equation the function is $\mathcal{L}$, not $\mathcal{L}'$ as in the previous equation (if it were $\mathcal{L}'$ then there would be no symmetry, but simply a trivial renaming of quantities). If this is the case the Euler-Lagrange equation of motion will have the same structure in terms of the transformed fields and therefore the dynamics will be unchanged.

One can verify that this is the case for Poincaré transformations in scalar field theory and electromagnetism. The Lagrangian density for a real scalar field reads

$$\mathcal{L}[\phi](x) = \frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - V[\phi](x).$$

In order to check the invariance of the above Lagrangian density one can write the action in terms of the transformed fields and coordinates and see if the functional form of the Lagrangian is the same as in terms of the untransformed quantities:

$$
\begin{aligned}
\mathcal{S} = \int d^4x \mathcal{L}[\phi](x) &= \int d^4x \left(\frac{1}{2} \partial_\mu \phi(x) \, \partial_\rho \phi(x) \eta^{\mu\rho} - V[\phi](x) \right) \\
&= \int d^4x' |J| \left(\frac{1}{2} \partial'_\alpha \phi'(x') \, \partial'_\beta \phi'(x') \eta^{\mu\rho} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\rho - V[\phi'](x') \right) \\
&= \int d^4x' \left(\frac{1}{2} \partial'_\alpha \phi'(x') \, \partial'_\beta \phi'(x') \eta^{\alpha\beta} - V[\phi'](x') \right) \\
&= \int d^4x' \mathcal{L}[\phi'](x').
\end{aligned}
$$

In writing last equations we have used the following properties:

$$
\begin{aligned}
&\phi'(x') = \phi(x), \\
&\partial_\mu \phi(x) = \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu \phi'(x') = \Lambda^\nu{}_\mu \partial'_\nu \phi'(x'), \\
&|J| = \det\left(\frac{\partial x^\alpha}{\partial x'^\beta} \right) = \det\left(\Lambda_\beta{}^\alpha \right) = 1 \qquad \text{by definition of } SO(1,3), \\
&\Lambda^\alpha{}_\mu \Lambda^\beta{}_\rho \eta^{\mu\rho} = \eta^{\alpha\beta}.
\end{aligned}
$$

Therefore the functional dependence of the Lagrangian upon the quantities ϕ and x^μ is the same as the one upon the transformed quantities, i.e. the Lagrangian remains the *same* function after the transformation. The Poincaré group is hence a symmetry of this theory.

One can repeat the argument for the Lagrangian of the electromagnetic field

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.
$$

At variance with the scalar field, $A_\nu(x)$ transforms in the vector representation of the Lorentz group, therefore:

$$
\begin{aligned}
A_\rho(x) &= \Lambda^\mu{}_\rho A'_\mu(x'), \\
\partial_\mu A_\nu(x) &= \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \partial'_\rho A'_\sigma(x'), \\
F_{\mu\nu}(x) &= \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu F'_{\rho\sigma}(x').
\end{aligned}
$$

As before, the Lagrangian has the same functional dependence upon the primed quantities as upon the untransformed ones and this implies that the Poincaré group is indeed a symmetry of the theory:

$$
\mathcal{S} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \right) = \int d^4x' \left(-\frac{1}{4} F'_{\mu\nu}(x') F'^{\mu\nu}(x') \right).
$$

A note on Lorentz transformations

The Lorentz Group is defined by the matrices satisfying the relation

$$
\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta^{\mu\rho} = \eta^{\nu\sigma},
$$

and normally one defines the transformation of a vector with lower index (covariant vector) as $v_\mu \longrightarrow \Lambda_\mu{}^\nu \, v_\nu$. However, introducing the vector (contravariant vector) with upper index as $v^\alpha = \eta^{\alpha\beta} v_\beta$, one obtains the transformation law for this vector as $v^\mu \longrightarrow \Lambda^\mu{}_\nu \, v^\nu$, where by definition

$$
\Lambda^\mu{}_\nu \equiv \eta^{\mu\rho} \eta_{\nu\sigma} \Lambda_\rho{}^\sigma,
$$

and it can easily shown that it defines a Lorentz transformation as well:

$$
\Lambda^\mu{}_\nu \Lambda^\alpha{}_\beta \eta^{\nu\beta} = \eta^{\mu\alpha}.
$$

This equation together with the first of this section can be used to express the form of the inverse of a Lorentz transformation:

$$
\begin{aligned}
&(\Lambda^{-1})_{\nu}{}^{\alpha}\eta^{\nu\sigma} = (\Lambda^{-1})_{\nu}{}^{\alpha}\Lambda_{\mu}{}^{\nu}\Lambda_{\rho}{}^{\sigma}\eta^{\mu\rho} = \delta_{\mu}{}^{\alpha}\Lambda_{\rho}{}^{\sigma}\eta^{\mu\rho}\\
&\Longrightarrow (\Lambda^{-1})_{\nu}{}^{\alpha} = \Lambda^{\alpha}{}_{\nu},
\end{aligned}
$$

and

$$
(\Lambda^{-1})^{\gamma}{}_{\mu}\eta^{\mu\alpha} = (\Lambda^{-1})^{\gamma}{}_{\mu}\Lambda^{\mu}{}_{\nu}\Lambda^{\alpha}{}_{\beta}\eta^{\nu\beta} = \delta^{\gamma}{}_{\nu}\Lambda^{\alpha}{}_{\beta}\eta^{\nu\beta} \tag{8}
$$

$$
\Longrightarrow (\Lambda^{-1})^{\gamma}{}_{\mu} = \Lambda_{\mu}{}^{\gamma}. \tag{9}
$$

Using a matrix notation defining $\Lambda^{\mu}{}_{\nu}x^{\nu} = \Lambda \cdot x$, (where $\Lambda^{\mu}{}_{\nu}$ is the element of the μth row and νth column) the statement that Λ is an element of the Lorentz group

$$
\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta_{\mu\nu} = \eta_{\alpha\beta} \tag{10}
$$

can be phrased as

$$
\Lambda^{T} \cdot \eta \cdot \Lambda = \eta \tag{11}
$$

where $\eta = \mathrm{diag}(1, -1, -1, -1)$. Multiplying both sides of eq. 11 by Λ^{-1} on the right and η on the left one gets

$$
\Lambda^{-1} = \eta \cdot \Lambda^{T} \cdot \eta \tag{12}
$$

where $\eta \cdot \eta = 1$ has been used.