Quantum Field Theory

Set 6: solutions

Exercise 1

A particle with spin j is an object that under rotations transforms as a state of the representation j of the group $SU(2)$. If one chooses a spatial direction, the 3rd one for example, the representation j can be defined considering the possible eigenvectors of the generator of rotation in this direction, τ^3. These eigenvectors form a basis $\mathcal{B}$ of the $2j+1$ dimensional vector space where the group is represented:

$$\mathcal{B} = \{|j,m\rangle,\ m = -j,\ -j+1, ..., j-1,\ j\}\,.$$

The action of the generators on this vector space is given by

$$\begin{aligned}
\tau^3|j,m\rangle &= m|j,m\rangle,\\
\tau^{\pm}|j,m\rangle &= \frac{1}{\sqrt{2}}\sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1\rangle,\\
\sum_{i=1}^{3}(\tau^i)^2|j,m\rangle &= j(j+1)|j,m\rangle.
\end{aligned}$$

Let us specialize to the $j = 1/2$ representation. The vector space in this case is 2-dimensional and a basis consists simply of the two states

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle,\quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle,$$

and the generators are represented by the three matrices

$$\tau^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},\qquad \tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\qquad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider now two copies of the previous representation, corresponding for example to two distinct particles with spin 1/2 each. If one wants to consider the spin of the bound state formed by these two particles one needs to introduce the notion of *Tensor Product* of two vector spaces. Given two vector spaces V_1, V_2, with vectors $|v_1\rangle$, $|v_2\rangle$, the tensor product of the two is the set of all possible pairs:

$$V_T = \{|v_1\rangle \otimes |v_2\rangle\,,\ \text{where } v_i \in V_i\}\,.$$

Moreover the sum of any two elements is defined starting from the sums defined in the original vector spaces:

$$(|v_1\rangle + |w_1\rangle) \otimes |v_2\rangle = |v_1\rangle \otimes |v_2\rangle + |w_1\rangle \otimes |v_2\rangle, \qquad |v_1\rangle \otimes (|v_2\rangle + |w_2\rangle) = |v_1\rangle \otimes |v_2\rangle + |v_1\rangle \otimes |w_2\rangle.$$

In addition it can be shown that a basis of the tensor product of two vector spaces is given by all the possible pairs obtained by taking one element from the basis of the first vector space and one element from the basis of the second vector space.
In the case in exam, we are considering the tensor product of two 2-dimensional vector spaces on which act the representation $j = 1/2$ of $SU(2)$. A complete basis for the tensor product space is given by the set

$$\mathcal{B}_V = \left\{\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle,\ \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle,\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle,\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle\ \right\}.$$

The tensor product is hence 4-dimensional. One can use the following notation for short:

$$\mathcal{B}_V = \{|\uparrow\uparrow\rangle\,,\ |\uparrow\downarrow\rangle\,,\ |\downarrow\uparrow\rangle\,,\ |\downarrow\downarrow\rangle\}\,.$$

The representation acting on the tensor product space is called tensor product representation, and it is easy to show that indeed it is a true representation of the group (even if, in general, it is not reducible). Denoting by $D^1(g)$ and $D^2(g)$ two representations of the same element g of a given group $\mathcal{G}$, acting on vector spaces V_1 and V_2, the tensor product representation $D^1(g) \otimes D^2(g) \equiv D^{1\otimes 2}(g)$, has the following properties:

$$\begin{aligned} D^{1\otimes 2}(g_a)D^{1\otimes 2}(g_b) &\equiv D^1(g_a) \otimes D^2(g_a)D^1(g_b) \otimes D^2(g_b) = D^1(g_a)D^1(g_b) \otimes D^2(g_a)D^2(g_b) \\ &= D^1(g_a \circ g_b) \otimes D^2(g_a \circ g_b) = D^{1\otimes 2}(g_a \circ g_b), \\ D^{1\otimes 2}(e) &= D^1(e) \otimes D^2(e) = 1_{V_1} \otimes 1_{V_2} = 1_V, \end{aligned}$$

where V is the tensor product space $V = V_1 \otimes V_2$.
In passing from first to second line it has been employed the fact that D^1 and D^2 act on *different* vector spaces, thus they commute (note that this is true even if D^1 and D^2 are two copies of *the same* representation). The system above shows that the tensor product representation is a representation of $\mathcal{G}$.
It is also possible to build explicitly the generators of the group in the tensor product representation. Denoting as $(t_1^a)_{ij}$ and $(t_2^a)_{xy}$ the generators in representations D^1 and D^2 respectively, one can write down the expression for an element near to the identity in representation $D^{1\otimes 2}$ as

$$\begin{aligned} \left[D^1(\alpha)\right]_{ij}\left[D^2(\alpha)\right]_{xy} &= \left[D^{1\otimes 2}(\alpha)\right]_{ijxy} = [\delta_{ij} + i\alpha^a(t_1^a)_{ij}][\delta_{xy} + i\alpha^a(t_2^a)_{xy}] + O(a^2) \\ &= \delta_{ij}\delta_{xy} + i\alpha^a\left[(t_1^a)_{ij}\delta_{xy} + \delta_{ij}(t_2^a)_{xy}\right] + O(\alpha^2), \end{aligned}$$

which can be written in tensor product notation as

$$\begin{aligned} D^1(\alpha) \otimes D^2(\alpha) &= D^{1\otimes 2}(\alpha) = [1_{V_1} + i\alpha^a t_1^a] \otimes [1_{V_2} + i\alpha^a t_2^a] + O(a^2) \\ &= 1_V + i\alpha^a\left[t_1^a \otimes 1_{V_2} + 1_{V_1} \otimes t_2^a\right] + O(\alpha^2). \end{aligned}$$

The operators in squared parentheses are the generators in the tensor product representation. In the particular case analyzed in this exercise one has thus

$$\begin{aligned} \tau_V^3 &= \tau^3 \otimes 1 + 1 \otimes \tau^3, \\ \tau_V^+ &= \tau^+ \otimes 1 + 1 \otimes \tau^+, \\ \tau_V^- &= \tau^- \otimes 1 + 1 \otimes \tau^-. \end{aligned}$$

One can verify that the elements of the basis of the tensor product space are still eigenvectors of the generator τ_V^3:

$$\begin{aligned} \tau_V^3 \left|\uparrow\uparrow\right\rangle &= \tau^3 \left|\uparrow\right\rangle \otimes 1 \left|\uparrow\right\rangle + 1 \left|\uparrow\right\rangle \otimes \tau^3 \left|\uparrow\right\rangle = \frac{1}{2}\left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle + \left|\uparrow\right\rangle \otimes \frac{1}{2}\left|\uparrow\right\rangle = \left|\uparrow\uparrow\right\rangle, \\ \tau_V^3 \left|\uparrow\downarrow\right\rangle &= \tau^3 \left|\uparrow\right\rangle \otimes 1 \left|\downarrow\right\rangle + 1 \left|\uparrow\right\rangle \otimes \tau^3 \left|\downarrow\right\rangle = \frac{1}{2}\left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle + \left|\uparrow\right\rangle \otimes \frac{-1}{2}\left|\downarrow\right\rangle = 0, \\ \tau_V^3 \left|\downarrow\uparrow\right\rangle &= \tau^3 \left|\downarrow\right\rangle \otimes 1 \left|\uparrow\right\rangle + 1 \left|\downarrow\right\rangle \otimes \tau^3 \left|\uparrow\right\rangle = \frac{-1}{2}\left|\downarrow\right\rangle \otimes \left|\uparrow\right\rangle + \left|\downarrow\right\rangle \otimes \frac{1}{2}\left|\uparrow\right\rangle = 0, \\ \tau_V^3 \left|\downarrow\downarrow\right\rangle &= \tau^3 \left|\downarrow\right\rangle \otimes 1 \left|\downarrow\right\rangle + 1 \left|\downarrow\right\rangle \otimes \tau^3 \left|\downarrow\right\rangle = \frac{-1}{2}\left|\downarrow\right\rangle \otimes \left|\downarrow\right\rangle + \left|\downarrow\right\rangle \otimes \frac{-1}{2}\left|\downarrow\right\rangle = -\left|\downarrow\downarrow\right\rangle. \end{aligned}$$

Hence the tensor product space contains eigenstates of τ_V^3 relative to the eigenvalues $1, 0, 0, -1$. The representations we have started with were by construction two irreducible representation of the Algebra of $SU(2)$, while in general their tensor product is not an irreducible representation. However it is always possible to decompose it in direct sum of irreducible representations $D^{1\otimes 2} \equiv D^1 \otimes D^2 = D_a \oplus D_b$.
Let's now construct explicitly these two representations. In order to do so, one first considers the basis $\mathcal{B}_V$ and takes its element with the largest eigenvalue of τ_V^3, in this case $\left|\uparrow\uparrow\right\rangle$; this state is called the *highest weight* state in the tensor product representation. The action of the raising operator on this state is

$$\tau_V^+ \left|\uparrow\uparrow\right\rangle = \tau^+ \left|\uparrow\right\rangle \otimes 1 \left|\uparrow\right\rangle + 1 \left|\uparrow\right\rangle \otimes \tau^+ \left|\uparrow\right\rangle = 0.$$

Since $\tau_V^3 \left|\uparrow\uparrow\right\rangle \equiv M \left|\uparrow\uparrow\right\rangle = \left|\uparrow\uparrow\right\rangle$, one can write this state in notation $\left|J, M\right\rangle$ as

$$\left|\uparrow\uparrow\right\rangle \equiv \left|J = 1, M = 1\right\rangle,$$

where the fact that for this state $J = M$ is due to the definition of highest weight state (remember Set5, where the j labeling an irreducible representation was a shorthand notation for m_{max}). Thus we have just noticed that in the tensor product of two representations $j = \frac{1}{2}$ of $SU(2)$ there is a representation $J = 1$. To build the remaining part of the basis of this representation it is sufficient to apply $2J = 2$ times the lowering operator τ_V^-, and use the explicit knowledge about the action of τ^- on the states of the representations $j = \frac{1}{2}$.

$$\begin{aligned}
\tau_V^- \left|\uparrow\uparrow\right\rangle &= \tau^- \left|\uparrow\right\rangle \otimes 1 \left|\uparrow\right\rangle + 1 \left|\uparrow\right\rangle \otimes \tau^- \left|\uparrow\right\rangle \\
&= \frac{1}{\sqrt{2}} \sqrt{1/2\,(1/2+1) - 1/2\,(1/2-1)} \left|\downarrow\right\rangle \otimes \left|\uparrow\right\rangle + \left|\uparrow\right\rangle \otimes \frac{1}{\sqrt{2}} \sqrt{1/2\,(1/2+1) - 1/2\,(1/2-1)} \left|\downarrow\right\rangle \\
&= \frac{1}{\sqrt{2}} \left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) \\
\equiv \tau_V^- \left|1,1\right\rangle &= \frac{1}{\sqrt{2}} \sqrt{1\,(1+1) - 1\,(1-1)} \left|1,0\right\rangle = \left|1,0\right\rangle ,
\end{aligned}$$

$$\begin{aligned}
\tau_V^- \frac{1}{\sqrt{2}} \left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) &= \frac{1}{\sqrt{2}} (\tau^- \left|\uparrow\right\rangle \otimes 1 \left|\downarrow\right\rangle + 1 \left|\downarrow\right\rangle \otimes \tau^- \left|\uparrow\right\rangle) \\
&= \frac{1}{2} \sqrt{1/2\,(1/2+1) - 1/2\,(1/2-1)} \left|\downarrow\right\rangle \otimes \left|\downarrow\right\rangle + \left|\downarrow\right\rangle \otimes \frac{1}{2} \sqrt{1/2\,(1/2+1) - 1/2\,(1/2-1)} \left|\downarrow\right\rangle \\
&= \left|\downarrow\downarrow\right\rangle \\
\equiv \tau_V^- \left|1,0\right\rangle &= \frac{1}{\sqrt{2}} \sqrt{1\,(1+1) - 0\,(0-1)} \left|1,-1\right\rangle = \left|1,-1\right\rangle , \\
\tau_V^- \left|\downarrow\downarrow\right\rangle = 0. &
\end{aligned}$$

Notice that the three elements in this basis, namely $\left|\uparrow\uparrow\right\rangle \equiv \left|1,1\right\rangle$, $\frac{1}{\sqrt{2}} \left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) \equiv \left|1,0\right\rangle$ and $\left|\downarrow\downarrow\right\rangle \equiv \left|1,-1\right\rangle$, have the same symmetry properties under permutations of the two spins, i.e. the raising and lowering operators don't change the symmetry properties of the states they act on; the representation $J = 1$ of SU(2) is a symmetric representation.

Note that this is a general statement: the highest weight representation (i.e. the representation containing the highest weight state) in the decomposition of a tensor product of n identical representations of $SU(2)$ is always symmetric under permutations of the particles of the component representations. This is so because the highest weight state is always of the form $\left|j_a, j_a\right\rangle \otimes \left|j_a, j_a\right\rangle \otimes ... \otimes \left|j_a, j_a\right\rangle$ and the raising/lowering operators don't modify the symmetry of the states.

Since the vector space on which the representation $\frac{1}{2} \otimes \frac{1}{2}$ acts is 4-dimensional, and we have found that one of the irreducible representations in which it decomposes is 3-dimensional, then only another irreducible 1-dimensional representation of $SU(2)$ can appear in the direct sum, and this is in fact the representation with $J = 0$ (and consequently $M = 0$). A basis for this representation is build by considering the state with $M = 0$ in the representation with $J = 1$ and finding a linear combination of the states $\left|\uparrow\downarrow\right\rangle$ and $\left|\downarrow\uparrow\right\rangle$ (the ones with $M = 0$) orthogonal to $\left|1,0\right\rangle$:

$$\begin{aligned}
0 &= \Big(\left\langle\downarrow\right| \otimes \left\langle\uparrow\right| + \left\langle\uparrow\right| \otimes \left\langle\downarrow\right| \Big) \Big(A \left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle + B \left|\downarrow\right\rangle \otimes \left|\uparrow\right\rangle \Big) \\
&= A \underbrace{\left\langle\downarrow \mid \uparrow\right\rangle \left\langle\uparrow \mid \downarrow\right\rangle}_{=0} + A \left\langle\uparrow \mid \uparrow\right\rangle \left\langle\downarrow \mid \downarrow\right\rangle + B \left\langle\downarrow \mid \downarrow\right\rangle \left\langle\uparrow \mid \uparrow\right\rangle + B \underbrace{\left\langle\uparrow \mid \downarrow\right\rangle \left\langle\downarrow \mid \uparrow\right\rangle}_{=0} \\
&= A + B.
\end{aligned}$$

Therefore the state belonging to the $J = 0$ representation is the antisymmetric combination $\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle$, and a prefactor of $\frac{1}{\sqrt{2}}$ ensures its correct normalization: $\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle) \equiv \left|0,0\right\rangle$.

The advantage of performing such a decomposition is that now it is simple to write the action of the algebra on this vector space: organizing the basis as follows

$$\mathcal{B}_V = \left\{ \left|\uparrow\uparrow\right\rangle , \ \frac{\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle}{\sqrt{2}} , \ \left|\downarrow\downarrow\right\rangle ; \quad \frac{\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle}{\sqrt{2}} \right\} ,$$

and calling

$$\left|\uparrow\uparrow\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \frac{\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle}{\sqrt{2}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \quad \left|\downarrow\downarrow\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad \frac{\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ,$$

any vector $v \in V$ will be written as

$$v = \left(\frac{v_{J=1}}{v_{J=0}} \right),$$

where $v_{J=1}$ is a three dimensional vector while $v_{J=0}$ is one dimensional. Moreover the generators will have the simple form

$$\tau_V^i = \left(\begin{array}{ccc|c} & & & 0 \\ & \tau_{J=1}^i & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & \tau_{J=0}^i \end{array} \right),$$

and the same will be for the representative of the group elements, so the representation matrices are in block-diagonal form. This proves that the tensor product representation $\frac{1}{2} \otimes \frac{1}{2}$ is fully decomposed into the direct sum $1 \oplus 0$, and thus the vector space V is as well decomposed into a direct sum of a 3-dimensional and a 1-dimensional invariant subspaces, $V = V_{J=1} \oplus V_{J=0}$, spanned respectively by the first three and by the fourth element of $\mathcal{B}_V$.

Let's summarize the steps to be followed in order to decompose a tensor product representation.

1) Build a basis for the tensor product space with all the possible combinations of vectors of the bases of the 'component' spaces (the spaces on which the 'component' representations act).

2) Find the highest weight state in the basis: this is always possible because the action of τ_V^3 on the tensor product space is known in terms of the action of the τ^3 on the 'component' spaces. The representation containing the highest weight state (whose eigenvalue of τ_V^3, called *weight*, is M) has $J = M$.

3) Build the representation J by acting $2J$ times with the lowering operator τ_V^- on the highest weight state.

4) Set aside the subspace associated to the spin $J = M$ representation. Build, with the states in the basis of the tensor product space, the combinations orthogonal to the basis of the spin $J = M$ representation. Find among these combinations the state with weight $M - 1$: this is the highest weight state of the representation $J = M - 1$.

5) Reiterate the procedure from point 2) until all the states are assigned to irreducible representations.

Strongly recommended reading: *H. Georgi, Lie Algebras in Particle Physics*, chapter 3.

Exercise 2

Lorentz transformations are defined as the linear transformations acting on the spacetime coordinates that leave invariant the spacetime distance

$$s^2 = c^2 t^2 - \vec{x} \cdot \vec{x} = x^\mu x^\nu \eta_{\mu\nu}.$$

If one applies such a transformation to the four-vector x^μ, namely $x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$, and imposes this to leave invariant the above defined distance one gets the constraint

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}, \qquad \text{or} \qquad \Lambda^T \eta \Lambda = \eta.$$

This equation defines a relation between the set of 4×4 real matrices that identifies a group called

$$SO(1,3) = \left\{ \Lambda \in GL(4, \mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta \right\},$$

where $\eta = \text{diag}(1, -1, -1 - 1)$. Geometrically the Lorentz group corresponds to the set of transformations that preserve the generalized scalar product defined by the matrix η.
In order to identify the Lie algebra associated to the Lorentz group one can consider the infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + w^\mu{}_\nu$ and plug it inside the constraint:

$$\begin{aligned} &\left(\delta^\mu{}_\rho + w^\mu{}_\rho\right) \eta_{\mu\nu} \left(\delta^\nu{}_\sigma + w^\nu{}_\sigma\right) = \eta_{\rho\sigma} + w^\mu{}_\rho \eta_{\mu\sigma} + \eta_{\nu\rho} w^\nu{}_\sigma + O(w^2) = \eta_{\rho\sigma} \\ &\Longrightarrow \; w_{\rho\sigma} = -w_{\sigma\rho}, \end{aligned}$$

therefore the algebra consist of the antisymmetric 4×4 real matrices and thus it has dimension $\frac{4 \times 3}{2} = 6$. One would like to write the general element of the algebra as a linear combination of generators $w^\mu{}_\nu = -i w^a (J^a)^\mu{}_\nu / 2$.

In order to write a compact expression for the generators it's useful to make use of a different notation: instead of a single index $a = 1, 2, ..., 6$ one can use a pair of indices $\alpha, \beta = 0, 1, 2, 3$ and make the following identification:

a	$\alpha\,\beta$
1	01
2	02
3	03
4	12
5	13
6	23

with $\qquad T^{\alpha\beta} = -T^{\beta\alpha}.$

In this way the pairs of *spacetime indices* label exactly six generators. Now one is able to write a complete basis for the Lie algebra:

$$\mathcal{B}: \ \ (\mathcal{J}^{\mu\nu})^{\rho}{}_{\sigma} = i\left(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma}\right),$$

where as already explained the indices inside the parenthesis label the six generators while the other two are the proper indices of the matrix. Just to make an example, the matrix $\mathcal{J}^{01}$ is of the form

$$(\mathcal{J}^{01})^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the generators $(\mathcal{J}^{\mu\nu})^{\rho}{}_{\sigma}$ are not antisymmetric matrices: only the $(\mathcal{J}^{\mu\nu})^{\rho\sigma} \equiv i\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}\right)$ are. Now that one has an explicit form for the generators it becomes possible to compute the commutation relations and read out the structure constants:

$$\begin{aligned} \left(\left[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}\right]\right)^{\gamma}{}_{\rho} &= (\mathcal{J}^{\mu\nu})^{\gamma}{}_{\sigma}(\mathcal{J}^{\alpha\beta})^{\sigma}{}_{\rho} - (\mathcal{J}^{\alpha\beta})^{\gamma}{}_{\sigma}(\mathcal{J}^{\mu\nu})^{\sigma}{}_{\rho} \\ &= -\left(\eta^{\mu\gamma}\delta^{\nu}_{\sigma} - \eta^{\nu\gamma}\delta^{\mu}_{\sigma}\right)\left(\eta^{\alpha\sigma}\delta^{\beta}_{\rho} - \eta^{\beta\sigma}\delta^{\alpha}_{\rho}\right) + \begin{pmatrix} \mu \leftrightarrow \alpha \\ \nu \leftrightarrow \beta \end{pmatrix}. \end{aligned}$$

The result of the commutator has be a matrix with indices $(\ )^{\gamma}{}_{\rho}$, therefore we try to reproduce this combination in the r.h.s of the above expression:

$$\begin{aligned} \left(\left[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}\right]\right)^{\gamma}{}_{\rho} = &-\Big(\underbrace{\eta^{\mu\gamma}\eta^{\alpha\nu}\delta^{\beta}_{\rho}}_{1} - \underbrace{\eta^{\nu\beta}\eta^{\mu\gamma}\delta^{\alpha}_{\rho}}_{2} - \underbrace{\eta^{\mu\alpha}\eta^{\nu\gamma}\delta^{\beta}_{\rho}}_{3} + \underbrace{\eta^{\nu\gamma}\eta^{\beta\mu}\delta^{\alpha}_{\rho}}_{4}\Big) \\ &+ \Big(\underbrace{\eta^{\alpha\gamma}\eta^{\mu\beta}\delta^{\nu}_{\rho}}_{4} - \underbrace{\eta^{\nu\beta}\eta^{\alpha\gamma}\delta^{\mu}_{\rho}}_{2} - \underbrace{\eta^{\mu\alpha}\eta^{\beta\gamma}\delta^{\nu}_{\rho}}_{3} + \underbrace{\eta^{\beta\gamma}\eta^{\nu\alpha}\delta^{\mu}_{\rho}}_{1}\Big) \\ &= i\left(\eta^{\nu\alpha}(\mathcal{J}^{\mu\beta})^{\gamma}{}_{\rho} - \eta^{\nu\beta}(\mathcal{J}^{\mu\alpha})^{\gamma}{}_{\rho} - \eta^{\mu\alpha}(\mathcal{J}^{\nu\beta})^{\gamma}{}_{\rho} + \eta^{\mu\beta}(\mathcal{J}^{\nu\alpha})^{\gamma}{}_{\rho}\right). \end{aligned}$$

Summarizing

$$\left[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}\right] = i\left(\eta^{\nu\alpha}\mathcal{J}^{\mu\beta} - \eta^{\nu\beta}\mathcal{J}^{\mu\alpha} - \eta^{\mu\alpha}\mathcal{J}^{\nu\beta} + \eta^{\mu\beta}\mathcal{J}^{\nu\alpha}\right).$$

Let us come back to the usual notation in which generators are labelled by a single index and define

$$\begin{aligned} &J^i = \frac{1}{2}\epsilon^{ijk}\mathcal{J}^{jk}, \qquad \mathcal{J}^{jk} = \epsilon^{jki}J^i, \\ &K^i = \mathcal{J}^{i0}, \\ &i,\ j,\ k = 1, 2, 3 \qquad \text{and} \qquad \epsilon^{123} = 1. \end{aligned}$$

Note that J^i, K^i are still 4×4 matrices. One can rewrite the commutation relation in terms of the new quantities

$$\begin{aligned} \left[J^i, J^j\right] &= \frac{1}{4}\epsilon^{iab}\epsilon^{jcd}\left[\mathcal{J}^{ab}, \mathcal{J}^{cd}\right] = \frac{i}{4}\epsilon^{iab}\epsilon^{jcd}\left(\eta^{bc}\mathcal{J}^{ad} - \eta^{bd}\mathcal{J}^{ac} - \eta^{ac}\mathcal{J}^{bd} + \eta^{ad}\mathcal{J}^{bc}\right) \\ &= -\frac{i}{4}\epsilon^{iab}\epsilon^{jcd}\left(\delta^{bc}\mathcal{J}^{ad} - \delta^{bd}\mathcal{J}^{ac} - \delta^{ac}\mathcal{J}^{bd} + \delta^{ad}\mathcal{J}^{bc}\right) \\ &= -\frac{i}{4}\left(\epsilon^{iab}\epsilon^{jbd}\epsilon^{adk} - \epsilon^{iab}\epsilon^{jcb}\epsilon^{ack} - \epsilon^{iab}\epsilon^{jad}\epsilon^{bdk} + \epsilon^{iab}\epsilon^{jca}\epsilon^{bck}\right)J^k \\ &= i\left(\delta^{ij}\delta^{ad} - \delta^{id}\delta^{aj}\right)\epsilon^{adk}J^k = -i\epsilon^{jik}J^k, \\ \left[J^i, J^j\right] &= i\epsilon^{ijk}J^k. \end{aligned}$$

One immediately recognizes the algebra of $SU(2)$: the above generators form a *subalgebra* of the Lorentz Algebra. Indeed the Lorentz group contains the spatial rotations as a *subgroup*. The other commutation relations read

$$\begin{aligned}
\left[J^i, K^j\right] &= \frac{1}{2}\epsilon^{ika}\left[\mathcal{J}^{ka}, \mathcal{J}^{j0}\right] = \frac{i}{2}\epsilon^{ika}\left(\eta^{aj}\mathcal{J}^{k0} - \eta^{a0}\mathcal{J}^{kj} - \eta^{kj}\mathcal{J}^{a0} + \eta^{k0}\mathcal{J}^{aj}\right) \\
&= -\frac{i}{2}\epsilon^{ika}\left(\delta^{aj}\mathcal{J}^{k0} - \delta^{kj}\mathcal{J}^{a0}\right) = i\epsilon^{ijk}K^k, \\
\left[K^i, K^j\right] &= \left[\mathcal{J}^{i0}, \mathcal{J}^{j0}\right] = i\left(\eta^{0j}\mathcal{J}^{i0} - \eta^{00}\mathcal{J}^{ij} - \eta^{ij}\mathcal{J}^{00} + \eta^{i0}\mathcal{J}^{0j}\right) = -i\mathcal{J}^{ij} = -i\epsilon^{ijk}J^k.
\end{aligned}$$

It's important to underline the commutation rules of the generators of *boosts* K^i with those of rotations J^i: it states that the generators of boosts transform under rotation as a vector, that is to say according to the representation $J = 1$ of $SU(2)$. This becomes evident if one considers the adjoin representation of the Lorentz group acting on its algebra.
The fact that the commutator of two K's is a J rather than another K can be guessed considering the parity transformations (i.e. transformation under reflection of spatial coordinates) of these vectors. The angular momentum J is invariant under parity (indeed it is the vector product of position and momentum, two polar vectors), while the boost generator K changes sign reflecting the coordinates: thus a product of two K's cannot be a proportional to a K, and since the algebra has to close, it cannot but be some linear combination of J's.

Exercise 3

The implementation of a group on functions presents some subtleties. Let us review in general how the representation must be implemented. Consider a group $\mathcal{G}$ that acts on spacetime coordinates as follows:

$$\mathcal{G}: \; x \xrightarrow{g_1} g_1(x) \xrightarrow{g_2} g_2(g_1(x)) = g_2 \circ g_1(x) \equiv g_3(x).$$

The action of $\mathcal{G}$ on the functions of spacetime coordinates is defined through the action of the inverse element on coordinates:

$$\mathcal{D}_{g_1}: \; \phi(x) \xrightarrow{g_1} \phi'(g_1(x)) \equiv \phi(x) \implies \phi'(x) = \mathcal{D}_{g_1}[\phi](x) = \phi(g_1^{-1}(x)).$$

The correct implementation of the composition of transformations on the space of functions is thus the following:

$$\mathcal{D}_{g_3}: \; \phi(x) \xrightarrow{g_3} \phi'(g_3(x)) \equiv \phi(x) \implies \phi'(x) = \mathcal{D}_{g_3}[\phi](x) = \phi((g_2 \circ g_1)^{-1}(x)).$$

Let us see how this applies to a transformation of the Poincaré group. The action on coordinates is defined as

$$\begin{gathered}
\mathcal{P}: \; x^\mu \xrightarrow{(\Lambda_1, a_1)} (\Lambda_1)^\mu{}_\nu x^\nu + a^\mu \equiv P_{(\Lambda_1, a_1)}(x) \xrightarrow{(\Lambda_2, a_2)} (\Lambda_2)^\mu{}_\nu (\Lambda_1)^\nu{}_\rho x^\rho + (\Lambda_2)^\mu{}_\nu a_1^\nu + a_2^\mu \\
P_{(\Lambda_2, a_2)} \circ P_{(\Lambda_1, a_1)}(x) \equiv P_{(\Lambda_3, a_3)}(x).
\end{gathered}$$

Thus the composition of the transformation $(\Lambda_2,\, a_2)$ and $(\Lambda_1,\, a_1)$ gives a third transformation with parameters $(\Lambda_3, a_3) = (\Lambda_2\Lambda_1,\, \Lambda_2 a_1 + a_2)$.
We now verify that the implementation on functions presented in the text reproduces this composition rule (we suppress spacetime indices for shortness):

$$\mathcal{D}_{(\Lambda_3, a_3)}: \; \phi(x) \xrightarrow{(\Lambda_3, a_3)} \phi'(\Lambda_3 x + a_3) \equiv \phi(x) \implies \phi'(x) = \mathcal{D}_{(\Lambda_3, a_3)}[\phi](x) = \phi(\Lambda_1^{-1}\Lambda_2^{-1}(x - \Lambda_2 a_1 - a_2))$$

and

$$\begin{aligned}
&\mathcal{D}_{(\Lambda_2, a_2)}\mathcal{D}_{(\Lambda_1, a_1)}: \; \phi(x) \rightarrow \phi'(P_{(\Lambda_2, a_2)} \circ P_{(\Lambda_1, a_1)}(x)) \equiv \phi(x) \\
&\implies \phi'(x) = \mathcal{D}_{(\Lambda_2, a_2)}\mathcal{D}_{(\Lambda_1, a_1)}[\phi](x) = \phi(P^{-1}_{(\Lambda_1, a_1)} \circ P^{-1}_{(\Lambda_2, a_2)}(x)) = \phi(\Lambda_1^{-1}(\Lambda_2^{-1}(x - a_2) - a_1)).
\end{aligned}$$

Thus, with the rule $\phi'(x) = \phi(\Lambda^{-1}(x - a))$ one gets that the composition of transformations is respected, i.e. acting on functions with $\mathcal{D}_{(\Lambda_3, a_3)}$ or with $\mathcal{D}_{(\Lambda_2, a_2)}\mathcal{D}_{(\Lambda_1, a_1)}$ is the same, as it is acting on fourvectors with $P_{(\Lambda_3, a_3)}$ or with $P_{(\Lambda_2, a_2)} \circ P_{(\Lambda_1, a_1)}$. Moreover, since the identity e corresponds to parameters $(\Lambda = 1_4, a = 0)$, then

$$\mathcal{D}_{(1_4, 0)}: \; \phi(x) \rightarrow \phi'(1_4 x + 0) = \phi'(x) = \phi(x),$$

and so the identity has the correct representation (it does not change the functional form of ϕ), proving that indeed the transformations presented in the text of the exercise define the action of the Poincaré group on functions.