Quantum Field Theory

Set 4: solutions

Exercise 1

Using the Euler-Lagrange equation we can extract the equation of motion for the field ϕ:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial L}{\partial \phi} \quad \Longrightarrow \quad \Box \phi = -J - \frac{\lambda}{2}\phi^2.$$

Plugging in the ansatz $\phi = \phi_0 + \lambda\phi_1$ we get

$$\Box\phi_0 + J + \lambda \left(\Box\phi_1 + \frac{1}{2}\phi_0^2 \right) + O(\lambda^2) = 0$$

$$\Longrightarrow \quad \begin{cases} \Box\phi_0 = -J \\ \Box\phi_1 = -\frac{1}{2}\phi_0^2 \\ \ldots \end{cases}$$

Since we treat λ as a small parameter we can consider only the first order corrections and neglect the higher powers of λ. In general the exact solution for ϕ would contain an infinite series of terms: $\phi = \phi_0 + \lambda\phi_1 + \ldots.$.
Let us start solving the zeroth order equation. The standard procedure consists in finding first the Green function (which is defined as the solution of the equation $\Box\mathcal{G}(t, \vec{x}) = \delta(t)\delta^3(\vec{x})$) and then take the convolution with the general source J. We can easily solve the equation in Fourier space: let us define

$$\mathcal{G}(t, \vec{x}) = \int \frac{dw\, d^3k}{(2\pi)^4}\, \widetilde{\mathcal{G}}(w, \vec{k})\, e^{-iwt + i\vec{x}\cdot\vec{k}},$$

$$\delta(t)\delta^3(\vec{x}) = \int \frac{dw\, d^3k}{(2\pi)^4}\, e^{-iwt + i\vec{x}\cdot\vec{k}}.$$

Thus the equation $\Box\mathcal{G}(t, \vec{x}) = \delta(t)\delta^3(\vec{x})$ becomes

$$-(w^2 - |\vec{k}|^2)\widetilde{\mathcal{G}}(w, \vec{k}) = 1.$$

At this point the differential equation has become an algebraic equation we can invert and then perform the inverse Fourier transform:

$$\widetilde{\mathcal{G}}(w, \vec{k}) = -\frac{1}{w^2 - |\vec{k}|^2}\,, \quad \mathcal{G}(t, \vec{x}) = -\int \frac{dw\, d^3k}{(2\pi)^4}\, \frac{e^{-iwt + i\vec{x}\cdot\vec{k}}}{w^2 - |\vec{k}|^2}.$$

As suggested in the text we add to the denominator a regulator $\pm i\varepsilon$ in order to avoid the pole on the real axis when we integrate over w. The right choice turns out to be $+i\varepsilon$ since it describes a retarded solution (causal solution). Hence

$$\mathcal{G}(t, \vec{x}) = -\int \frac{dw\, d^3k}{(2\pi)^4}\, \frac{e^{-iwt + i\vec{x}\cdot\vec{k}}}{(w + i\varepsilon)^2 - |\vec{k}|^2}.$$

In order to integrate over w we can consider the complex w plane and close the integral in the upper half-plane if $\Re(t) < 0$ or in the lower half-plane if $\Re(t) > 0$. In the former case the integral gives zero since the function is analytic inside the contour. Instead in the latter case the contour encircles the poles at $w = \pm|\vec{k}| - i\varepsilon$ and we get the residues:

$$\mathcal{G}(t, \vec{x}) = \begin{cases} 0 & \text{if} \quad t < 0, \\ 2\pi i \int \frac{d^3k}{(2\pi)^4} \left(\frac{e^{-ikt}}{2k} + \frac{e^{ikt}}{-2k} \right) e^{i\vec{k}\cdot\vec{x}} & \text{if} \quad t > 0. \end{cases}$$

Therefore

$$\begin{aligned}\mathcal{G}(t,\vec{x}) &= \theta(t)\int \frac{d\cos\varphi dk}{(2\pi)^2} k\sin(kt)e^{ikx\cos\varphi}\\ &= \frac{1}{2\pi^2 x}\theta(t)\int_0^\infty dk\,\sin(kt)\,\sin(kx),\end{aligned}$$

where $k \equiv |\vec{k}|$ and $x \equiv |\vec{x}|$. Using the expression $\sin(\alpha) = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$ and changing variable one gets

$$\begin{aligned}\mathcal{G}(t,\vec{x}) &= \frac{1}{4\pi x}\theta(t)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left(e^{ik(t-x)} - e^{-ik(t+x)}\right)\\ &= \frac{1}{4\pi x}\theta(t)\delta(t-x).\end{aligned}$$

In writing the last equality we used the fact that t is positive, so that $t \neq -x$. The above solution describes a spherical shell expanding from the time $t = 0$. At this point we can take the convolution of the green function with the source $J(t,\vec{x})$ to get the final solution for $\phi_0(t,\vec{x})$:

$$\begin{aligned}\mathcal{G}(x-z) &= \frac{1}{4\pi|\vec{x}-\vec{z}|}\theta(x_0-z_0)\delta(x_0-z_0-|\vec{x}-\vec{z}|),\\ \phi_0(t,\vec{x}) &= -\int dz_0\, d^3z\,\mathcal{G}(x-z)J(z_0,\vec{z}).\end{aligned}$$

If we consider a pointlike source constant in time which is switched on at the time $t = 0$ and switched off at time $t = \tau$ we can write $J(t,\vec{x}) = J\theta(t)\theta(\tau - t)\delta^3(\vec{x})$. Hence

$$\begin{aligned}\phi_0(t,\vec{x}) &= -\int dz_0\, d^3z \frac{J}{4\pi|\vec{x}-\vec{z}|}\theta(t-z_0)\delta(t-z_0-|\vec{x}-\vec{z}|)\theta(z_0)\theta(\tau-z_0)\delta^3(\vec{z})\\ &= -\frac{J}{4\pi|\vec{x}|}\theta(t-|\vec{x}|)\theta(\tau-t+|\vec{x}|).\end{aligned}$$

The above solution describes a shell with amplitude decreasing like $|\vec{x}|^{-1}$ but with constant width. The radial width of the shell is indeed $t > |\vec{x}| > t - \tau$. Now we can solve the first order equation: indeed the found solution now plays the role of a source for ϕ_1. Again

$$\begin{aligned}\phi_1(t,\vec{x}) &= -\frac{1}{2}\int dz_0\, d^3z\,\mathcal{G}(x-z)\phi_0^2(z)\\ &= -\frac{1}{2}\int dz_0\, d^3z \frac{1}{4\pi|\vec{x}-\vec{z}|}\theta(t-z_0)\delta(t-z_0-|\vec{x}-\vec{z}|)\frac{J^2}{16\pi^2|\vec{z}|^2}\theta(z_0-|\vec{z}|)\theta(\tau-z_0+|\vec{z}|).\end{aligned}$$

It has to be stressed that the Green function $\mathcal{G}$ is *the same* as before, since it is associated to the operator $\Box$, and it doesn't depend of course on the form of the source J.

Exercise 2

In order to define a group $\mathcal{G}$, a set of transformations $\{g_i\}$ has to satisfy the following requirements:

- An operation $\circ$ must be defined on the set $\{g_i\}$. In the present case this operation consists in the usual composition of the transformations.
- For each g_1, $g_2 \in \mathcal{G}$: $g_1 \circ g_2 \equiv g_3 \in \mathcal{G}$. Moreover the product must me associative: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$. For the collinear group these relations are indeed satisfied:

$$\begin{aligned} U(a_1,b_1)\circ U(a_2,b_2)x &= U(a_1,b_1)\,(a_2x+b_2) = (a_1a_2)x + (a_1b_2+b_1)\\ \Longrightarrow \quad U(a_1,b_1)\circ U(a_2,b_2) &= U(a_1a_2, a_1b_2+b_1),\end{aligned}$$

$$(U(a_1,b_1)\circ U(a_2,b_2))\circ U(a_3,b_3)x = a_1a_2a_3x + a_1a_2b_3 + a_1b_2 + b_1 = U(a_1,b_1)\circ(U(a_2,b_2)\circ U(a_3,b_3))x.$$

- The set $\{g_i\}$ must contain the neutral element e, such that, for each $g \in \mathcal{G}$, $e \circ g = g \circ e = g$. In the present case the neutral element corresponds to the identity transformation

$$e = U(1,0) : \, x \longrightarrow x.$$

- For each $g \in \mathcal{G}$ there must exist a $g^{-1} \in \mathcal{G}$, the *inverse element*, such that $g^{-1} \circ g = g \circ g^{-1} = e$. In order to find such an element one can denote $U^{-1}(a,b) \equiv U(c,d)$ and impose

$$U(a,b)U(c,d) = U(ac, ad+b) = U(1,0) \quad \Longrightarrow \quad \begin{cases} c = 1/a \\ d = -b/a. \end{cases}$$

Thus

$$U^{-1}(a,b) = U(1/a, -b/a).$$

The collinear group is an example of a class of groups called *Lie Groups*. These particular groups have the property that their elements can be labelled by a continuous set of parameters:

$$g = g_{a_1 \ldots a_n} \longrightarrow (a_1, ..., a_n) \, \in \, R^n.$$

This parametrization must be differentiable and makes the Lie group a manifold. An important notion in the Lie group theory is the notion of *Lie Algebra*. In general an algebra $\mathcal{A}$ is a vector space together with a binary operation $[,] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. At the present stage one does not need the exact definition of such an object. The important thing to have in mind is how a Lie algebra is related to a Lie group. Since the latter represents a manifold one can consider the tangent space to the manifold in a given point. The tangent space is the vector space generated by tangent vectors t^a. Standard theorems in differential geometry state that given a point g_0, one can reach the points in the neighborhood applying the exponential map $e^{i\alpha_a t^a}$ to g_0. For example if one chooses the identity as starting point, all the elements connected to the identity can be written as

$$g_{a_1 \ldots a_n} = e^{i(a_1 t^1 + \ldots + a_n t^n)}.$$

This expression gives an explicit meaning to the coordinates a_i. Moreover in the tangent space of a Lie group, an operation is naturally defined to satisfy all the requirements of an algebra: the commutator of two vectors (which are called *generators*), which is defined as

$$[t^a, t^b] = t^a t^b - t^b t^a.$$

Coming back to the collinear group, one defines the generators of the group as

$$\mathcal{D}(1+\epsilon) = e^{i\epsilon D}, \qquad \mathcal{T}(b) = e^{ibT}.$$

In this way the parameters ($\epsilon = 0, b = 0$) pick up the identity. One can verify that the commutators of two generators belong to the vector space spanned by them:

$$[T,T] = 0, \qquad [D,D] = 0, \qquad [D,T] = \alpha D + \beta T.$$

The first two are trivial. To obtain the latter, first consider that

$$\mathcal{D}(1+\epsilon)\mathcal{T}(b)\mathcal{D}(1+\epsilon)^{-1} = \mathcal{T}((1+\epsilon)b),$$

which can be easily proved by evaluating the action of both sides on a generic element x using the definitions for $\mathcal{D}$ and $\mathcal{T}$ given in the text.
Expanding now for small b, $\mathcal{T}(b) = e^{ibT} \simeq 1 + ibT + O(b^2)$, and taking the linear term in b gives

$$\mathcal{D}(1+\epsilon)T\mathcal{D}(1+\epsilon)^{-1} = (1+\epsilon)T.$$

Expanding for small ϵ, one has

$$(1+i\epsilon D)T(1-i\epsilon D) = T + i\epsilon[D,T] + O(\epsilon^2) = T + \epsilon T,$$

and therefore $[D,T] = -iT$.
Consider the space of the infinitely differentiable functions of one variable. One can define the action of the collinear group on that space as

$$\mathcal{D}(a)f(x) = f(x/a), \qquad \mathcal{T}(b)f(x) = f(x-b).$$

Note an important feature of a representation $\mathcal{R}$ of a group $\mathcal{G}$ on functions:

$$U(a,b) : x \to ax + b \qquad \text{then} \qquad \mathcal{R}(U)f(x) = f(U^{-1}(a,b)x) = f\left(\frac{x-b}{a}\right).$$

It's important to consider the inverse transformation to reproduce the product order:

$$\mathcal{R}(U_2 \circ U_1)f(x) \equiv \mathcal{R}(U_3)f(x) = f(U_3^{-1}x) = f((U_2 \circ U_1)^{-1}x) = f(U_1^{-1} \circ U_2^{-1}x).$$

It's possible to give an explicit form to the generators D and T in this space. Indeed for infinitesimal transformation one has

$$\mathcal{T}(b)f(x) = (1 + ibT + O(b^2))f(x) = f(x-b) = f(x) - bf'(x) + O(b^2),$$

and matching the linear term one gets $T = i\partial_x$, which is the standard representation of the generator of translations. Similarly

$$\mathcal{D}(1+\epsilon)f(x) = (1 + i\epsilon D + O(\epsilon^2))f(x) = f(x/(1+\epsilon)) \sim f((1-\epsilon)x) = f(x) - \epsilon x f'(x) + O(\epsilon^2),$$

and therefore $D = ix\partial_x$. Note how the algebra is correctly realized in this space:

$$[D,T] = ix\partial_x(i\partial_x) - i\partial_x(ix\partial_x) = -x\partial_x^2 + x\partial_x^2 + \partial_x = -iT.$$

Exercise 3

The following groups are the most common groups one can deal with in theoretical physics.

- $$U(N) \equiv \left\{U \in GL(N,\mathbb{C}) \middle| UU^\dagger = U^\dagger U = 1_N\right\}$$

  This is the group of $N \times N$ complex unitary matrices. Clearly the inverse corresponds to the hermitian conjugate. One can consider the associated algebra $u(N)$ and take a complete basis T^a of this vector space. Here T^a represents a basis of generators and the label a runs from 1 to dim(algebra). In order to identify the structure of the algebra one can make use of the exponential map to write a generic element U of the group in terms of the generator T^a and some coordinate α^a:

  $$U_\alpha = e^{i\alpha^a T^a} \simeq 1_N + i\alpha^a T^a + O(\alpha^2).$$

  The unitarity of U implies that

  $$1_N = U_\alpha U_\alpha^\dagger \simeq \left(1_N + i\alpha^a T^a\right)\left(1_N - i\alpha^a (T^a)^\dagger\right) \simeq 1_N + i\alpha^a T^a - i\alpha^a (T^a)^\dagger.$$

  Therefore the generators are all the matrices that satisfy $T = T^\dagger$, that is to say the hermitian $N \times N$ matrices. One can easily compute the dimension of this vector space counting the number of independent parameters appearing in a generic hermitian matrix.

  $$T_{ij} = (T^\dagger)_{ij} = (T_{ji})^* \implies \begin{cases} \text{Elements on the diagonal are real: } N \text{ components.} \\ \text{Elements symmetric w.r.t the diagonal are} \\ \text{complex conjugate: } N(N-1) \text{ components.} \end{cases}$$

  In the end the dimension of the algebra (equal to the dimension of the vector space of complex hermitian matrices) is N^2. A complete set of generators for the group $U(N)$ is given by a complete basis of the complex hermitian $N \times N$ matrices.

- $$SU(N) \equiv \left\{U \in GL(N,\mathbb{C}) \middle| UU^\dagger = U^\dagger U = 1_N,\ \det(U) = 1\right\}.$$

  The latter group is similar to the previous one but with an additional constraint: if in $U(N)$ the determinant of a matrix satisfies $|\det(U)| = 1$, here we choose only $\det(U) = 1$. This corresponds to considering only the subgroup of $U(N)$ connected to the identity. The additional requirement can be translated to the algebra using the relation

  $$\det(e^A) = e^{\mathrm{Tr}[A]}.$$

  Therefore the algebra is now composed by complex hermitian *traceless* $N \times N$ matrices. The tracelessness constraint consists in only one relation between the components of an hermitian matrix T since one already knows that all diagonal elements are real. The dimension of the algebra is therefore:

  $$\dim(su(N)) = N^2 - 1.$$

- $$SO(N) \equiv \left\{ R \in GL(N,\mathbb{R}) \middle| RR^T = R^T R = 1_N,\ \det(R) = 1 \right\}.$$

  This is the group of orthogonal real $N \times N$ matrices. Still using the exponential map

  $$R_\alpha = e^{\alpha^a T^a} \simeq 1_N + \alpha^a T^a + O(\alpha^2).$$

  This time it's better to define the generator without the i in the exponent: in this way, since R is real also the T^a are real instead of purely imaginary. The orthogonality implies:

  $$1_N = R_\alpha R_\alpha^T \simeq (1_N + \alpha^a T^a)\left(1_N + \alpha^a (T^a)^T\right) \simeq 1_N + \alpha^a T^a + \alpha^a (T^a)^T,$$

  that is to say the algebra is formed by antisymmetric real matrices. The tracelessness is automatically satisfied since antisymmetric matrices have all zero components in the diagonal. The number of components of such a matrix are $N(N-1)/2$, which corresponds to the dimension of the algebra $so(N)$.

- $$O(N) \equiv \left\{ R \in GL(N,\mathbb{R}) \middle| RR^T = R^T R = 1_N \right\}.$$

  The structure of the algebra is the same as the previous one since the removed constraint has no implication at the algebra level. However the group is not the same: one can think about $O(N)$ as $SO(N)$ with additional parities that invert an odd number of coordinates. For example $O(3)$ can be thought as the rotation group $SO(3)$ together with the following matrices

  $$P_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

  The latter are discrete symmetries: composing a generic element of $SO(N)$ with one of these, one can generate the whole $O(N)$. Note that in this case the exponential map doesn't cover all the group since it's formed by several disconnected pieces: the one containing the identity is the subgroup $SO(N)$ and one can reach the others acting with the parities.

- $$SL(N,\mathbb{C}) \equiv \left\{ V \in GL(N,\mathbb{C}) \middle| \det(V) = 1 \right\}.$$

  This is the group of complex $N \times N$ matrices with unitary determinant. Using the exponential map one obtains the constraint for the algebra:

  $$\det(V) = 1 = e^{i\alpha^a \mathrm{Tr}[T^a]} \ \Rightarrow\ Tr[T^a] = 0.$$

  Since the tracelessness this time is a complex statement, it contains two independent constraints and the dimension of the algebra is

  $$\dim(sl(N,\mathbb{C})) = 2N^2 - 2 = 2(N^2 - 1).$$

Exercise 4

Given an algebra

$$\left[T^a, T^b\right] = i f^{abc} T^c\ ,$$

one can consider the following identity

$$\begin{aligned}
&\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] = \\
&T^a\left(T^b T^c - T^c T^b\right) - \left(T^b T^c - T^c T^b\right) T^a + T^b\left(T^c T^a - T^a T^c\right) - \left(T^c T^a - T^a T^c\right) T^b \\
&+ T^c\left(T^a T^b - T^b T^a\right) - \left(T^a T^b - T^b T^a\right) T^c = 0.
\end{aligned}$$

Substituting in the first line the result of each commutator one gets

$$\begin{aligned}
&\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] \\
&= \sum_d i f^{bcd}\left[T^a, T^d\right] + i f^{cad}\left[T^b, T^d\right] + i f^{abd}\left[T^c, T^d\right] \\
&= \sum_{d,f} -f^{bcd} f^{adf} T^f - f^{cad} f^{bdf} T^f - f^{abd} f^{cdf} T^f.
\end{aligned}$$

The latter is a vanishing linear combination of generators that are a basis of the algebra, therefore the whole coefficient has to be zero:

$$\sum_d \left(f^{adf}f^{bcd} + f^{bdf}f^{cad} + f^{cdf}f^{abd}\right) = 0.$$

This identity can also be used to show that the quantities f^{abc}, called *structure constants*, provide themselves a representation of the group. Let's define a set of matrices $\{A^a\}$ as

$$(A^a)_b{}^c \equiv -if^{abc}.$$

Then the Jacobi identity can be rewritten as

$$\begin{aligned}
&f^{adf}f^{bcd} - f^{bdf}f^{acd} + f^{cdf}f^{abd} = 0,\\
&(A^b)_c{}^d(A^a)_d{}^f - (A^a)_c{}^d(A^b)_d{}^f = -if^{abd}(A^d)_c{}^f,\\
&\left[A^b, A^a\right] = if^{bad}A^d.
\end{aligned}$$

Thus the matrices satisfy the algebra and therefore provide a representation of the group. The vector space on which these matrices act is the algebra itself. This is called *adjoint representation.*