Quantum Field Theory

Set 3: solutions

After the correction of the three exercises there is a little review on totally antisymmetric tensors, i.e. ϵ, and on the contraction of tensors. Some properties that are used in the correction are listed.

Exercise 1

The Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

give

$$\partial_\mu \left(\frac{\partial}{\partial(\partial_\mu \phi)} \frac{1}{2} \partial_\rho \phi \partial_\sigma \phi \eta^{\rho\sigma} \right) = \partial_\mu (\eta^{\rho\sigma} \delta^\mu_\rho \partial_\sigma \phi) \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi = \frac{\partial V(\phi)}{\partial \phi}.$$

For the massive $\lambda\phi^4$ theory one gets:

$$\partial_\mu \partial^\mu \phi = -m^2 \phi - \frac{\lambda}{6} \phi^3.$$

Let us now consider the term

$$\alpha(\partial_\mu \phi \, \partial^\mu \phi)^2 \equiv \alpha(\partial_\mu \phi \, \partial^\mu \phi)(\partial_\nu \phi \, \partial^\nu \phi).$$

In order to compute the dimension in energy of the constant α let us compute first the dimension of the field ϕ. We recall that the dimension of the action is power of $\hbar$ (depending on the spacetime dimension), therefore in natural units the action is dimensionless. From this one can extract the dimension of the Lagrangian density $\mathcal{L}$:

$$S = \int dt \, d^{D-1}x \mathcal{L}(\vec{x}, t)$$

$$[S] = E^0 \,, \qquad [dx] = [dt] = E^{-1} \qquad \Rightarrow \qquad [\mathcal{L}] = E^D$$

Therefore in 3+1 dimensions all the terms appearing in the Lagrangian density must have dimension E^4. Let us consider the kinetic term: knowing the dimension $[\partial_\mu] = [\frac{\partial}{\partial x^\mu}] = E$, we can extract the dimension of ϕ:

$$[\partial_\mu \phi \, \partial^\mu \phi] = [\partial_\mu]^2 \times [\phi]^2 = E^2 \times [\phi]^2 \qquad \Rightarrow \qquad [\phi] = E$$

The field ϕ has dimension of a mass. One can repeat in arbitrary dimension and get $[\phi] = E^{\frac{D-2}{2}}$. Let us come back to the parameter α:

$$[\mathcal{L}] = E^4 \qquad \Rightarrow \qquad [\alpha] \times [\partial_\mu]^4 \times [\phi]^4 = E^4 \qquad \Rightarrow \qquad [\alpha] = E^{-4}$$

Finally let us compute the Euler-Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu (\partial^\mu \phi + 4\alpha(\partial_\nu \phi \, \partial^\nu \phi) \partial^\mu \phi) = -\frac{\partial V}{\partial \phi}.$$

Exercise 2

The Lagrangian of the system is

$$L = \int d^3x \mathcal{L}(x,t) \qquad \mathcal{L}(x,t) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.$$

The conjugate momentum is

$$\pi(x,t) = \frac{\delta \, L}{\delta(\partial_0 \phi)} = \frac{\partial \, \mathcal{L}(x,t)}{\partial(\partial_0 \phi(x,t))} = \partial_0 \phi(x,t).$$

The Hamiltonian reads

$$H = \int d^3x \mathcal{H}(x,t) = \int d^3x \left[\pi\partial_0\phi - \mathcal{L}\right] = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i\phi)(\partial_i\phi) + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right].$$

Given two functionals of ϕ, π

$$F[\phi,\pi](t) = \int f(\phi(x,t),\pi(x,t))d^3x,$$

One defines the equal time Poisson brackets between the two as

$$\{F(t),G(t)\} = \int \left(\frac{\delta F}{\delta\pi(z,t)}\frac{\delta G}{\delta\phi(z,t)} - \frac{\delta F}{\delta\phi(z,t)}\frac{\delta G}{\delta\pi(z,t)}\right) d^3z,$$

where as usual $\frac{\delta F}{\delta\phi(z,t)} = \frac{\partial f(z,t)}{\partial\phi(z,t)}$. In particular

$$\begin{aligned}
\{\pi(x,t),\phi(y,t)\}_t = \int \Bigg(&\frac{\delta\left(\int d^3x_1\pi(x_1,t)\delta^3(x-x_1)\right)}{\delta\pi(z,t)}\frac{\delta\left(\int d^3x_2\phi(x_2,t)\delta^3(y-x_2)\right)}{\delta\phi(z,t)} \\
&- \frac{\delta\left(\int d^3x_1\pi(x_1,t)\delta^3(x-x_1)\right)}{\delta\phi(z,t)}\frac{\delta\left(\int d^3x_1\phi(x_2,t)\delta^3(y-x_2)\right)}{\delta\pi(z,t)}\Bigg) d^3z \\
= \int &\left(\delta^3(x-z)\delta^3(y-z)\right) d^3z = \delta^3(x-y).
\end{aligned}$$

The equations of motion become:

$$\begin{aligned}
\dot{\phi} &= \{H,\phi\}, \\
\dot{\pi} &= \{H,\pi\},
\end{aligned}$$

and therefore

$$\begin{aligned}
\dot{\phi}(y,t) = \{H,\phi(y,t)\} &= \int d^3x \left\{\frac{1}{2}\pi^2(x,t),\phi(y,t)\right\} = \int d^3x\, \pi(x,t)\{\pi(x,t),\phi(y,t)\} = \pi(y,t), \\
\dot{\pi}(y,t) = \{H,\pi(y,t)\} &= \int d^3x \left\{\frac{1}{2}(\partial_i\phi(x,t))^2 + \frac{1}{2}m^2\phi^2(x,t) + \frac{\lambda}{4!}\phi^4(x,t),\pi(y,t)\right\} = \\
&= \int d^3x \left(\partial_i\phi(x,t)\{\partial_i\phi(x,t),\pi(y,t)\} + m^2\phi(x,t)\{\phi(x,t),\pi(y,t)\} + \frac{\lambda}{3!}\phi^3(x,t)\{\phi(x,t),\pi(y,t)\}\right) \\
&= -\int d^3x \left(\partial_i\phi(x,t)\frac{\partial}{\partial x^i}\delta^3(x-y) + (m^2\phi(x,t) + \frac{\lambda}{3!}\phi^3(x,t))\delta^3(x-y)\right) \\
&= \partial_i\partial_i\phi(y,t) - m^2\phi(y,t) - \frac{\lambda}{3!}\phi^3(y,t).
\end{aligned}$$

Substituting the former in the latter on can show the equivalence with the Lagrangian formalism:

$$\partial_t^2\phi(y,t) - \partial_i\partial_i\phi(y,t) = \Box\phi(y,t) = -m^2\phi(y,t) - \frac{\lambda}{3!}\phi^3(y,t).$$

Exercise 3

The Maxwell equations read

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \rho, \qquad \vec{\nabla}\wedge\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0, \\
\vec{\nabla}\cdot\vec{B} &= 0, \qquad \vec{\nabla}\wedge\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{\vec{J}}{c}.
\end{aligned}$$

One can also rewrite the latter expression in components; recalling that:

$$\begin{aligned}
&\vec{E} = (E^1,E^2,E^3), \qquad \vec{B} = (B^1,B^2,B^3), \qquad \vec{J} = (J^1,J^2,J^3), \qquad \rho = J^0, \\
&\vec{\nabla} = (\partial_1,\partial_2,\partial_3), \qquad \frac{\partial}{\partial t} = \partial_0,
\end{aligned} \tag{1}$$

one obtains

$$\partial_i E^i = J^0, \qquad \epsilon_{ijk}\partial_j E^k + \frac{1}{c}\partial_0 B^i = 0,$$
$$\partial_i B^i = 0, \qquad \epsilon_{ijk}\partial_j B^k - \frac{1}{c}\partial_0 E^i = \frac{J^i}{c}.$$

In defining four components quantities one must pay attention to the position of spatial indices; since these are lowered and raised with a metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, with Minkosky signature the spacial indices acquire a minus sign in the transition. For example:

$$V^\mu = (V^0, V^i) = (V_0, -V_i), \qquad \partial_\mu = (\partial_0, \partial_i) = (\partial^0, -\partial^i).$$

The field strength F can be expressed in terms of the vector potential $A_\mu = (A_0, A_i)$ as follows:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

One should notice the antisymmetric nature of the tensor F for the exchange of the indices $\mu \leftrightarrow \nu$. In order to verify that this expression reflects the definition of F in terms of the physical fields E, B on can compute

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_0 A^i + \partial_i A_0 = (\partial_0 \vec{A} + \nabla A_0)^i = -E^i,$$
$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\partial_1 A^2 + \partial_2 A^1 = -(\vec{\nabla} \wedge \vec{A})^3 = -B^3,$$

where we have lowered indices of derivatives to get the standard form as defined before.
In order to compute the equations of motion for the field A_μ one has to apply the Euler-Lagrange equation. This time the field with respect to which we differentiate carries an additional space-time index:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu}.$$

The previous equation contains a free index ν, which selects the component of the field A_ν with respect to which one derives (therefore one has 4 "independent" equations), and a summed index μ. The equations read

$$\begin{aligned}
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) &= \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[-\frac{1}{4} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right) \left(\partial^\rho A^\sigma - \partial^\sigma A^\rho \right) \right] = \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[-\frac{1}{2} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right) \partial^\rho A^\sigma \right] \\
&= \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[-\frac{1}{2} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right) \partial_\alpha A_\beta \eta^{\rho\alpha} \eta^{\sigma\beta} \right] \\
&= \partial_\mu \left[-\frac{1}{2} \left(\delta^\mu_\rho \, \delta^\nu_\sigma - \delta^\mu_\sigma \, \delta^\nu_\rho \right) \partial_\alpha A_\beta \eta^{\rho\alpha} \eta^{\sigma\beta} - \frac{1}{2} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right) \delta^\mu_\alpha \, \delta^\nu_\beta \, \eta^{\rho\alpha} \eta^{\sigma\beta} \right] = -\partial_\mu F^{\mu\nu}, \\
\frac{\partial \mathcal{L}}{\partial A_\nu} &= \frac{\partial}{\partial A_\nu} \left(-J^\rho A_\rho \right) = -J^\rho \delta^\nu_\rho = -J^\nu,
\end{aligned} \tag{2}$$

where we have derived using the relation

$$\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\rho A_\sigma)} = \delta^\rho_\mu \delta^\sigma_\nu,$$

that is to say that one gets non-vanishing contribution only if the indices of derivative and of the vector A match. Otherwise the derivative gives zero since it's like deriving a variable with respect to an independent one. Finally the equations of motion are given by

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

At this point it's straightforward to verify that one has obtained exactly the Maxwell equations: the component 0 reads

$$\partial_\mu F^{\mu 0} = \partial_i F^{i0} = J^0 = -\partial_i F^{0i} = \partial_i E^i = \rho,$$

while the component i is

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} = -\partial_0 E^i - \partial_j \epsilon_{jik} B^k = J^i,$$

where we have used the property $F^{mn} = F_{mn} = -\epsilon_{mnp} B^p$. However we immediately see that the Euler-Lagrange equations reproduce only the inhomogeneous Maxwell equations, that is to say the ones with a source on their l.h.s.. The homogeneous equations derive from the so called *Bianchi identity*:

$$\epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 2\epsilon_{\mu\nu\rho\sigma} \partial^\nu \partial^\rho A^\sigma = 0,$$

since the tensor $\partial^\nu \partial^\rho$ is contracted with the total antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$. Expanding the identity one gets:

$$\epsilon_{0\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \epsilon_{0ijk}\partial^i F^{jk} = -\epsilon_{ijk}\partial^i \epsilon^{jkl} B^l = -2\partial_i B^i = 0$$

$$\begin{aligned}\epsilon_{i\nu\rho\sigma}\partial^\nu F^{\rho\sigma} &= \epsilon_{i0jk}\partial^0 F^{jk} + 2\epsilon_{ij0k}\partial^j F^{0k} \\ &= \epsilon_{ijk}\partial^0 \epsilon^{jkl} B^l + 2\epsilon_{ijk}\partial^j(-E^k) = 2\partial_0 B^i + 2\epsilon_{ijk}\partial_j E^k = 0.\end{aligned}$$

The Bianchi identities are relation encoded in the structure of the field strength and find their natural explication in the formalism of differential forms.
One can solve the latter equation for a simple external source. The simplest current one can think about is the one generated by a static pointlike charge. In general the current generated by a pointlike particle has the form

$$J^\mu = \left(e\delta^3(x - x(t)),\, e\vec{v}(t)\delta^3(x - x(t))\right),$$

however, since the particle doesn't move, its velocity $\vec{v}$ is null and the current has only the 0-component. Therefore $J^\mu = \left(e\delta^3(x),\, \vec{0}\right)$. For such a configuration one should expect to find the Coulomb potential generated by a charge e. Since the current is time independent we can look for a static solution. We start considering the equation for the scalar potential $A_0(\vec{x})$:

$$\partial_i E^i = \partial_i(-\partial_i A_0 - \partial_0 A^i) = -\nabla^2 A_0(x) = e\delta^3(x).$$

The solution of this Laplace equation can be easily obtained in Fourier transform; define

$$A_0(\vec{x}) = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\tilde{A}(\vec{p})e^{i\vec{p}\cdot\vec{x}}, \qquad \delta^3(\vec{x}) = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}}.$$

therefore the equation becomes

$$\int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\left(\tilde{A}(\vec{p})\partial_i\partial_i + e\right)e^{i\vec{p}\cdot\vec{x}} = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\left(-\tilde{A}(\vec{p})|\vec{p}|^2 + e\right)e^{i\vec{p}\cdot\vec{x}} = 0.$$

The latter expression states that the function $\tilde{A}(\vec{p})\left(-|\vec{p}|^2 + e\right)$, thought as an element of the Hilbert space where the Fourier transform is defined, has vanishing scalar product with all the functions $e^{i\vec{p}\cdot\vec{x}}$ which is a complete basis in that space. Therefore it must be

$$\tilde{A}(\vec{p}) = \frac{e}{|\vec{p}|^2}.$$

The solution in coordinate space is simply obtained by Fourier transforming the one in momentum space:

$$\begin{aligned}A_0(\vec{x}) &= e\int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{x}}}{|\vec{p}|^2} = e\int_0^{\infty} dp\frac{p^2}{(2\pi)^3}\int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\phi\frac{e^{ipx\cos\theta}}{p^2} \\ &= \frac{e}{4\pi^2}\int_0^{\infty} dp\left(\frac{e^{-ipx} - e^{ipx}}{-ipx}\right) = \frac{e}{2\pi^2}\int_0^{\infty} dp\left(\frac{\sin(px)}{px}\right) \\ &= \frac{e}{2\pi^2 x}\underbrace{\int_0^{\infty} dy\frac{\sin y}{y}}_{\pi/2} = \frac{e}{4\pi}\frac{1}{|\vec{x}|},\end{aligned}$$

which is the usual Coulomb scalar potential generated by a static charge and gives an electric field

$$\vec{E} = -\vec{\nabla}A_0(\vec{x}) = \frac{e}{4\pi^2}\frac{\vec{x}}{|\vec{x}|^3}.$$

The vector potential satisfies the homogeneous equation $\nabla^2\vec{A}(\vec{x}) - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}(\vec{x})) = 0$. Thus, we can choose $\vec{A} = 0$.

Antisymmetric tensors

In d dimension one can define a total antisymmetric tensor $\epsilon_{a_1 \dots a_d}$ with the following properties:

- ϵ is antisymmetric for the exchange of any two near indices:

$$\epsilon_{a_1..a_i\, a_{i+1}..a_d} = -\epsilon_{a_1..a_{i+1}\, a_i..a_d}$$

- $\epsilon_{a_1 \dots a_d}$ is different from zero only if all the indices have different value. In particular

$$\epsilon_{1\,2\dots d} = 1 \qquad \epsilon_{a_1 \dots a_d} = (-1)^n$$

  where n is the number of permutations of two near indices to reach the configuration $12...d$ from $a_1 a_2 ... a_d$.

For example in three dimension one has ε_{ijk} which has entries:

$$\varepsilon_{123} = 1 \qquad \varepsilon_{132} = -1 \qquad \varepsilon_{213} = -1 \qquad \varepsilon_{231} = 1 \qquad ....$$

In four dimension one has $\epsilon_{\mu\nu\rho\sigma}$. It's interesting to note that, due to the antisymmetric property defined above, if one of the indices is temporal (that is to say equal to 0) all the other must be spacial (i, j, k) in order to have a non vanishing contribution. For example, in the 0-component of the Bianchi identity $\epsilon_{0\nu\rho\sigma}\partial^\nu F^{\rho\sigma}$, the summed indices can be only spacial, otherwise ϵ is zero. Then

$$\epsilon_{0\nu\rho\sigma} \to \epsilon_{0ijk} \to \varepsilon_{ijk}.$$

Contraction between ϵ tensors can be expressed in terms of delta functions. The most frequent are the following:

$$\begin{aligned}
&\varepsilon_{ijk}\varepsilon_{imn} = \delta_{im}\delta_{kn} - \delta_{in}\delta_{km}, \\
&\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}, \\
&\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\alpha\beta} = 2\left(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu\right), \\
&\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\nu\alpha\beta} = (3!)\delta^\rho_\mu,
\end{aligned} \tag{3}$$

Frequently one deals with symmetric or antisymmetric tensor, contracted among themselves or with some other generic tensor. A useful way to manipulate these expression is the following. First of all one can observe that every tensor (with two indices) can be written as symmetric part plus the antisymmetric one:

$$T_{ab} = T^S_{ab} + T^A_{ab},$$

where

$$T^S_{ab} = T^S_{ba} = \frac{1}{2}\left(T_{ab} + T_{ba}\right) \qquad T^A_{ab} = -T^A_{ba} = \frac{1}{2}\left(T_{ab} - T_{ba}\right).$$

Denoting by A_{mn} and S_{mn} an antisymmetric and a symmetric tensor respectively, one has:

$$\mathrm{Tr}\left[A \cdot S\right] \equiv A_{ab}S_{ba} = -A_{ba}S_{ab} = -A_{ab}S_{ba} = 0,$$

where we have first used the (anti)symmetry property of the tensor and then renamed the indices to show that this contraction is equal to minus itself. More in general for a tensor T_{ab} one has:

$$\mathrm{Tr}\left[A \cdot T\right] = A_{ab}T_{ba} = A_{ab}T^A_{ba} \qquad \mathrm{Tr}\left[S \cdot T\right] = S_{ab}T_{ba} = S_{ab}T^S_{ba}.$$

Only the part with the same symmetry property survives in the contraction.