Quantum Field Theory

Set 25: solutions

Exercise 1

Let us start from the matrix element squared

$$|\mathcal{M}|^2 = \lambda^4 \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2 .$$

Making use of the 2-body phase space in the case of final state particles with equal masses, we have

$$d\sigma = \frac{1}{4\sqrt{(p_a \cdot p_b)^2 - m^4}} |\mathcal{M}|^2 \beta \frac{d\cos\theta d\varphi}{64\pi^2},$$

where $\beta \equiv \sqrt{1-\frac{4m^2}{s}}$ is the velocity of the incoming particles in the center of mass frame. We have put 64 instead of 32 as a factor accounting for the identity of the two final state particles: thus we will integrate over all the phase space. Since we consider four identical masses we can further simplify the previous expression, because the flux factor is proportional to $1/\beta$:

$$d\sigma = |\mathcal{M}|^2 \frac{d\cos\theta d\varphi}{128\pi^2 s}.$$

Hence

$$\frac{d\sigma}{d\cos\theta} = \frac{\lambda^4}{64\pi s}\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2,$$

where we have integrated in $d\varphi$ because nothing depends on the azimuthal angle. Indeed one can show that, parametrizing momenta as

$$\begin{aligned}
p_a &= \frac{\sqrt{s}}{2}(1,0,0,\beta),\\
p_b &= \frac{\sqrt{s}}{2}(1,0,0,-\beta),\\
p_c &= \frac{\sqrt{s}}{2}(1,0,\beta\sin\theta,\beta\cos\theta),\\
p_d &= \frac{\sqrt{s}}{2}(1,0,-\beta\sin\theta,-\beta\cos\theta),
\end{aligned}$$

the independent Mandelstam invariants are simply

$$\begin{aligned}
t \equiv (p_a - p_c)^2 = -\frac{s\beta^2}{2}(1-\cos\theta) = -2\left(\frac{s}{4} - m^2\right)(1-\cos\theta),\\
u \equiv (p_a - p_d)^2 = -\frac{s\beta^2}{2}(1+\cos\theta) = -2\left(\frac{s}{4} - m^2\right)(1+\cos\theta).
\end{aligned}$$

Therefore the l.h.s of the differential cross section $\frac{d\sigma}{d\cos\theta}$ is only a function of θ.
Before performing the integration let us consider the high energy limit $s \gg m^2$. In this limit one would expect the total cross section not to depend on masses, and therefore, by dimensional analysis,

$$\sigma \simeq \frac{\lambda^4}{s^3}.$$

This behavior is wrong or, to say better, is not the dominant one in this limit. To see this, one can indeed expand the differential cross section in powers of m^2/s. Note that the mass acts as a regulator of the integral in $d\cos\theta$ (in the strict massless case there are non integrable singularities at $\cos\theta = \pm 1$ coming from the $t-$ and $u-$channels).

Therefore the expansion in powers of m^2/s at leading order takes place neglecting the mass for the $s-$channel (never singular) and in the *definitions* of t and u, but retaining it in the denominators of the $t-$ and $u-$channels. This yields

$$\sigma \simeq \frac{\lambda^4}{64\pi s}\int_{-1}^{1} d\cos\theta \left[\frac{1}{s} - \frac{2}{s(1-\cos\theta)+2m^2} - \frac{2}{s(1+\cos\theta)+2m^2}\right]^2$$

To perform the above integration we can notice that in the massless limit the square of the first piece is finite, the interference terms of the $s-$channel with the others has a single pole, and the interference between the $t-$ and $u-$channels has two single poles: all these terms will diverge at most as $\log(m)$ in the small mass limit. Conversely, the squares of the $t-$ and $u-$channels have double poles, thus they will result in a m^{-2} dependence. By changing variable $y \to -y$ in one of the two contributions, it is immediate to notice that they are equal, so that the main contribution to the cross section will be

$$\sigma \simeq \frac{\lambda^4}{64\pi s} 2\int_{-1}^{1} d\cos\theta \left[\frac{2}{s(1+\cos\theta)+2m^2}\right]^2 = \frac{\lambda^4}{64\pi s}\left[\frac{4}{m^2 s}\right].$$

The leading behavior has a different power with respect to what we guessed by dimensional analysis.
One can understand this behavior in the following way: if we put the mass to zero the cross section diverges. This is because a massless particle can mediate long range (actually infinite range) interaction and therefore two particles interact even if they are far apart.
Another way to see this is to consider the dominant contribution to the integral which arises from small values of the denominator:

$$\begin{aligned}(1-\cos\theta) &\simeq \frac{2m^2}{s} \implies \theta = \theta_{\rm max} \simeq \frac{2m}{\sqrt{s}},\\ (1+\cos\theta) &\simeq \frac{2m^2}{s} \implies \theta \simeq \pi - \theta_{\rm max}.\end{aligned}$$

The leading contribution comes from scattering at small angles between the beamline and the outgoing particles (the case of $\theta \simeq \pi$ is small angle between $\vec{p}_a$ and $\vec{p}_d$), which corresponds to large-distance interactions. Indeed the transverse momentum of a particle scattered at angular distance $\theta_{\rm max}$ will be $(p_\perp)_{\rm max} \simeq \frac{\sqrt{s}}{2}\sin\theta_{\rm max} \simeq m$, which corresponds to an impact parameter (i.e. the shortest mutual distance that the two interacting particles reach) of $b_{\rm min} \simeq \frac{1}{(p_\perp)_{\rm max}} \simeq \frac{1}{m}$.
In the limit of vanishing mass the region that gives the largest contribution to the cross section is that of large distances.

In the opposite limit, the non-relativistic one, the differential cross section is perfectly finite over all the phase space, so it is possible to expand the integrand in Taylor series without any particular treatment (in the ultra relativistic limit we had to retain the mass in the propagators). At zeroth order in $s-4m^2$ one has $t=u=0$, thus

$$\frac{d\sigma}{d\cos\theta} \simeq \frac{\lambda^4}{64\pi 4m^2}\left[\frac{1}{4m^2-m^2} + \frac{1}{-m^2} + \frac{1}{-m^2}\right]^2 = \frac{25\lambda^4}{2304\pi m^6},$$

and

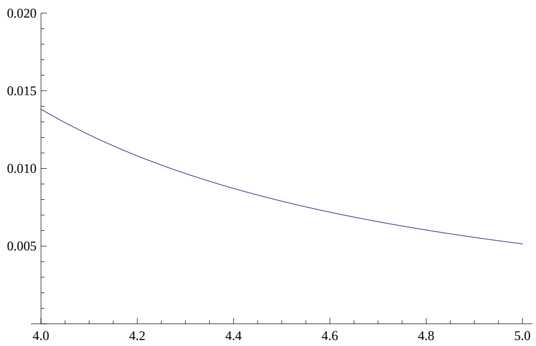
$$\sigma \simeq \frac{25\lambda^4}{1152\pi m^6}.$$

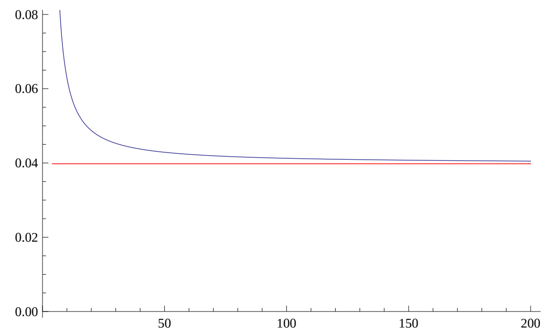
For completeness, we give the result for the total cross section without approximations (from which one can deduce that the limits presented above are indeed correct), namely

$$\sigma = \frac{\lambda^4}{64\pi s}\left[\frac{4}{m^2\left(s-3m^2\right)} + \frac{2}{\left(m^2-s\right)^2} + \frac{4\log\left(\frac{m^2}{s-3m^2}\right)}{4m^4-5m^2 s+s^2} + \frac{4s\log\left(\frac{s}{m^2}-3\right)}{-8m^6+14m^4 s-7m^2 s^2+s^3}\right],$$

and we show its behavior as a function of s (in GeV^2) for $\lambda = m = 1$ GeV. The plot on the left displays 2σ in the limit of small masses, which correctly tends to $\frac{25}{576\pi}\text{GeV}^{-2}$ for this choice of parameters. The plot on the right shows the observable $2s^2 \times \sigma$ in the high energy region. The asymptotic flatness of the plot underlines the s^{-2}

behavior of the cross section, and the value of the plotted quantity at $s \to \infty$ is $\frac{1}{8\pi}\,\mathrm{GeV}^{-2}$.





Exercise 2

Recalling the anticommutation relation of the Dirac matrices $\{\gamma^\mu\,,\gamma^\nu\} = 2\eta^{\mu\nu}$, and using the cyclic property of the trace, we have:

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}[\gamma^\nu\gamma^\mu] = \frac{1}{2}\mathrm{Tr}[\{\gamma^\mu,\,\gamma^\nu\}] = \eta^{\mu\nu}\mathrm{Tr}[1] = 4\eta^{\mu\nu},\\
\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 2\eta^{\mu\nu}\mathrm{Tr}[\gamma^\rho\gamma^\sigma] - \mathrm{Tr}[\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma] = 8\eta^{\mu\nu}\eta^{\rho\sigma} - \mathrm{Tr}[\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma]\\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + \mathrm{Tr}[\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma]\\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + 8\eta^{\mu\sigma}\eta^{\rho\nu} - \mathrm{Tr}[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu].
\end{aligned}$$

Using the cyclicity of the trace we have:

$$\mathrm{Tr}[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu] = \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] \quad\Longrightarrow\quad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}).$$

Recalling the anticommutation relation $\{\gamma^\mu\,,\gamma^5\} = 0$ we can consider the trace of an odd number of γ matrices and insert $\gamma^5\gamma^5 = 1$:

$$\mathrm{Tr}[\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = \mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}].$$

Using the cyclicity we have

$$\mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = \mathrm{Tr}[\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}\gamma^5],$$

while if we were to pass γ_5 through all the other matrices we would get a minus sign for each anticommutation:

$$\mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = (-1)^{2n+1}\mathrm{Tr}[\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}\gamma^5],$$

so that

$$\mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = -\mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = 0.$$

From this results it follows straightforwardly that

$$\mathrm{Tr}[\gamma^5\cdot(\text{odd number of }\gamma\text{'s})] = \mathrm{Tr}[(\text{odd number of }\gamma\text{'s})] = 0,$$

since γ^5 can be written as the product of 4 Dirac matrices and hence the total number is still odd.

We now show that $\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0$. Let's insert the factor $1 = \eta^{\rho\rho}\gamma^\rho\gamma^\rho$, (not summed over ρ), where $\rho \neq \mu$, $\rho \neq \nu$:

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = \eta^{\rho\rho}\mathrm{Tr}[\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu] = -\eta^{\rho\rho}\mathrm{Tr}[\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho] = -\eta^{\rho\rho}\mathrm{Tr}[\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu] = 0,$$

where we have anticommuted γ^ρ and used then cyclicity.

Finally let us consider

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma].$$

First of all we notice that whenever two Lorentz indices are equal, this expression vanishes since it becomes proportional to $\mathrm{Tr}[\gamma^5\gamma\gamma]$. Moreover, because all indices are different, one gets a minus sign after every anticommutation of two Dirac matrices. Thus this object is completely antisymmetric in its Lorentz indices and cannot but be proportional to the Levi-Civita tensor:

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = k\epsilon^{\mu\nu\rho\sigma}.$$

To work out the coefficient it is convenient to consider the particular case $\{\mu,\nu,\rho,\sigma\} = \{3,2,1,0\}$ and use the definition $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, to get

$$\mathrm{Tr}[i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0] = k\epsilon^{3210} \quad\Longrightarrow\quad -4i = k.$$

Thus

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4i\epsilon^{\mu\nu\rho\sigma}.$$