# Quantum Field Theory

## Set 24: solutions

### Exercise 1

Let us consider the Lagrangian of two interacting scalar fields:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 - \frac{\lambda}{2}\phi^2\Phi.$$

We want to study the decay $\Phi \to \phi\phi$. In what follows $p_a$ will be the four-momentum of $\Phi$ while $p_b$ and $p_c$ are the four-momenta of the final particles. The differential decay width in the $\Phi$ center of mass reads

$$d\Gamma = \frac{1}{2M}|\mathcal{M}_{fi}|^2\sqrt{1-\frac{4m^2}{M^2}}\frac{d\cos\theta d\varphi}{32\pi^2},$$

where we have used the form of the 2 body phase space for equal final particles. We only need to compute the matrix element between the final state $|f\rangle$ and the initial state $|i\rangle$ which is defined by

$$S_{fi} \equiv \langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4\delta^4(p_a - p_b - p_c)\mathcal{M}_{fi}.$$

Let's first obtain the expression for the $S$-matrix element from the operator equation

$$S = T\exp\left(-i\int_{-\infty}^{\infty} dt\ H_I(t)\right) \equiv \mathbb{1} - i\int_{-\infty}^{\infty} dt\ H_I(t) + (-i)^2\int_{-\infty}^{\infty} dt\ H_I(t)\int_{-\infty}^{t} dt'\ H_I(t') + ...,$$

where $T$ is the time-ordering symbol (and it is defined by the previous Taylor expansion), and $H_I$ is the interaction Hamiltonian written in the interaction picture, $H_I(t) = e^{iH_0t}H_{int}(0)e^{-iH_0t}$. In the Born approximation, one only retains the first two terms in the Taylor expansion, so that

$$\begin{aligned}
\langle f|S|i\rangle &\simeq \langle f|\left(\mathbb{1} - i\int_{-\infty}^{\infty} dt\ H_I(t)\right)|i\rangle = \delta_{fi} - i\int_{-\infty}^{\infty} dt\ \langle f|e^{iH_0t}H_{int}(0)e^{-iH_0t}|i\rangle \\
&= \delta_{fi} - i\int_{-\infty}^{\infty} dt\ \langle f|e^{iE_ft}H_{int}(0)e^{-iE_it}|i\rangle = \delta_{fi} - i\langle f|H_{int}(0)|i\rangle\int_{-\infty}^{\infty} dt\ e^{i(E_f-E_i)t} \\
&= \delta_{fi} - 2\pi i\delta(E_f - E_i)\langle f|H_{int}(0)|i\rangle.
\end{aligned}$$

Thus we can identify the amplitude:

$$(2\pi)^3\delta^3(\vec{p}_f - \vec{p}_i)\mathcal{M}_{fi} = -\langle f|H_{int}(0)|i\rangle.$$

Let us now use the decomposition for real fields in order to compute the matrix element of the interaction hamiltonian.

$$\begin{aligned}
\Phi(\vec{x},0) &= \int d\Omega_{\vec{p}}\left(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right), \\
\phi(\vec{x},0) &= \int d\Omega_{\vec{p}}\left(b(\vec{p})e^{i\vec{p}\cdot\vec{x}} + b^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right).
\end{aligned}$$

Therefore in the rest frame of $\Phi$ ($\vec{p}_a = \vec{0}$) we have

$$\langle\vec{p_a}|H_{int}(0)|\vec{p}_b,\vec{p}_c\rangle = \langle 0|a(\vec{0})H_{int}(0)b^\dagger(\vec{p}_b)b^\dagger(\vec{p}_c)|0\rangle = \frac{\lambda}{2}\int d^3x\ \langle 0|a(\vec{0})\ \Phi(\vec{x},0)\phi^2(\vec{x},0)\ b^\dagger(\vec{p}_b)b^\dagger(\vec{p}_c)|0\rangle =$$

$$\frac{\lambda}{2}\int d^3x\ d\Omega_{\vec{q}}d\Omega_{\vec{k}}d\Omega_{\vec{t}}\ \langle 0|a(\vec{0})a^\dagger(\vec{q})b(\vec{k})b(\vec{t})b^\dagger(\vec{p}_b)b^\dagger(\vec{p}_c)|0\rangle e^{-i(\vec{q}-\vec{k}-\vec{t})\cdot\vec{x}},$$

where we have taken only the proper terms in order to have the same number of $a$, $a^\dagger$ and of $b$, $b^\dagger$ (only a state with the same number of daggered and undaggered operators has non zero overlap with $|0\rangle$). Making use of the equal time commutation relations

$$[a(\vec{p}), a^\dagger(\vec{q})] = [b(\vec{p}), b^\dagger(\vec{q})] = (2\pi)^3 2p_0 \delta^3(\vec{p}-\vec{q}),$$

we get

$$\langle\vec{0}|H_{int}(0)|\vec{p}_b,\vec{p}_c\rangle = \frac{\lambda}{2}\int d^3x\; d\Omega_{\vec{q}} d\Omega_{\vec{k}} d\Omega_{\vec{t}}\; e^{-i(\vec{q}-\vec{k}-\vec{t})\cdot\vec{x}}(2\pi)^9 2q_0 2k_0 2t_0 \delta^3(\vec{q})\left[\delta^3(\vec{k}-\vec{p}_b)\delta^3(\vec{t}-\vec{p}_c)+\delta^3(\vec{k}-\vec{p}_c)\delta^3(\vec{t}-\vec{p}_b)\right].$$

Performing all the integrations on momenta and recalling that $\int d^3x\; e^{-i(\vec{q}-\vec{k}-\vec{t})\cdot\vec{x}} = 2\pi^3\delta^3(\vec{q}-\vec{k}-\vec{t})$ we finally get

$$\langle\vec{0}|H_{int}(0)|\vec{p}_b,\vec{p}_c\rangle = \lambda(2\pi)^3\delta^3(\vec{p}_c+\vec{p}_b) = \langle\vec{p}_b,\vec{p}_c|H_{int}(0)|\vec{0}\rangle,$$

where the last equality holds since $\lambda$ is real. Therefore

$$(2\pi)^3\delta^3(\vec{p}_b+\vec{p}_c)\mathcal{M}_{fi} = -\lambda(2\pi)^3\delta^3(\vec{p}_b+\vec{p}_c) \implies \mathcal{M}_{fi} = -\lambda.$$

Finally the decay width reads

$$\frac{d\Gamma}{d\cos\theta} = \frac{\lambda^2}{32\pi M}\sqrt{1-\frac{4m^2}{M^2}},$$

and in the end

$$\Gamma = \frac{\lambda^2}{32\pi M}\sqrt{1-\frac{4m^2}{M^2}}.$$

In the last formula we have integrated over half of the phase space since the two final-state particles are identical. Notice that the decay width goes to zero when $m \to M/2$ which is correct because for bigger values of the final state mass the decay is kinematically forbidden. The lifetime $\tau$ is simply $1/\Gamma$.

## Exercise 2

$$[\phi(x),\phi(y)] = \int d\Omega_p d\Omega_k \left[a_p e^{-ipx} + a_p^\dagger e^{ipx}, a_k e^{-ikx} + a_k^\dagger e^{ikx}\right] = \int d\Omega_p \left(e^{-ip(x-y)} - e^{ip(x-y)}\right) = D(x-y) - D(y-x)$$

The function $D(x)$ is Lorentz invariant:

$$D(\Lambda x) = \int d\Omega_p e^{-ip(\Lambda x)} = \int d\Omega_p e^{-i(\Lambda^{-1}p)x} = \int d\Omega_{p'} e^{-ip'x} = D(x)$$

where $p' = \Lambda^{-1}p$ and $d\Omega' = d\Omega$ is a Lorentz-invariant measure. Therefore, if $(x-y)^2 < 0$, the vector $x-y$ can be brought to a pure spacial form: $(x-y) \to z$, $z^0 = 0$. Then the original expression becomes:

$$[\phi(x),\phi(y)] = \int d\Omega_p \left(e^{-i\vec{p}\cdot\vec{z}} - e^{i\vec{p}\cdot\vec{z}}\right) = 0$$

Let's end up with a discussion on the meaning of the condition $[\phi(x),\phi(y)] = [\phi(x),\pi(y)] = 0, (x-y)^2 < 0$. Consider a classical equation for the evolution of field $\phi$ (for instance, Klein-Gordon plus interactions). How do express the notion of causality in this theory? $\phi(x)$ is determined for every $x$ by the equations of motions upon fixing the initial conditions at Cauchy surface, for instance:

$$\phi(t=0,\vec{x}) = \phi_0(x)$$
$$\dot{\phi}(t=0,\vec{x}) = \pi_0(x)$$

Now, if one changes the these initial condition at a point $x$, this should not affect the value of the field at some $y$ such that $(x^2-y^2) < 0$, because no perturbation should propagate faster than the speed of light.

In other words:

$$\frac{\delta\phi(y)}{\delta\phi_0(x)} = \frac{\delta\phi(y)}{\delta\pi_0(x)} = 0 \quad \text{if}\,(x-y)^2 < 0 \tag{1}$$

or equivalently, in the language of the Poisson brackets:

$$\{\phi(y),\pi_0(x)\} = \{\phi(y),\phi_0(x)\} = 0 \quad \text{if}\,(x-y)^2 < 0 \tag{2}$$

After the canonical quantization $\{\cdot\,,\cdot\} \to i\hbar[\cdot\,,\cdot]$ this becomes the condition on the quantum commutators.

## Exercise 3

This exercise could be solved along the lines of the previous, but since the interaction Hamiltonian has four identical fields, and the scattering is $2 \to 2$, then the blind application of the above formulas rapidly becomes unfeasible. Then we can use Wick's theorem to evaluate the matrix element efficiently.
Recall that Wick's theorem states

$$T\{\phi(x_1)\cdots\phi(x_n)\} \equiv \; :[\phi(x_1)\cdots\phi(x_n) + \text{contractions}]: \quad ,$$

where ": :" is the normal-ordering symbol, meaning that all the creation operators appear on the left, while "contractions" is a sketchy way to indicate all the possible contractions of two fields, namely

$$\wick{\c{\phi}(x_1)\c{\phi}(x_2)} = D(x_1 - x_2),$$

$D$ representing the propagator. To be concrete, let's work out the $T$-ordered product of four identical scalar fields (understood in the interaction picture), according to Wick's theorem:

$$\begin{aligned}
T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} \quad \equiv \quad & :[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\
& +\wick{\c{\phi}(x_1)\c{\phi}(x_2)}\phi(x_3)\phi(x_4) + \wick{\c{\phi}(x_1)\phi(x_2)\c{\phi}(x_3)}\phi(x_4) + \wick{\c{\phi}(x_1)\phi(x_2)\phi(x_3)\c{\phi}(x_4)} \\
& +\phi(x_1)\wick{\c{\phi}(x_2)\c{\phi}(x_3)}\phi(x_4) + \phi(x_1)\wick{\c{\phi}(x_2)\phi(x_3)\c{\phi}(x_4)} + \phi(x_1)\phi(x_2)\wick{\c{\phi}(x_3)\c{\phi}(x_4)} \\
& +\wick{\c1{\phi}(x_1)\c1{\phi}(x_2)\c2{\phi}(x_3)\c2{\phi}(x_4)} + \wick{\c1{\phi}(x_1)\c2{\phi}(x_2)\c1{\phi}(x_3)\c2{\phi}(x_4)} + \wick{\c1{\phi}(x_1)\c2{\phi}(x_2)\c2{\phi}(x_3)\c1{\phi}(x_4)}]: \\
\\
= \quad & :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): \\
& +D(x_1-x_2):\phi(x_3)\phi(x_4): + D(x_1-x_3):\phi(x_2)\phi(x_4): + D(x_1-x_4):\phi(x_2)\phi(x_3): \\
& +D(x_2-x_3):\phi(x_1)\phi(x_4): + D(x_2-x_4):\phi(x_1)\phi(x_3): + D(x_3-x_4):\phi(x_1)\phi(x_2): \\
& +D(x_1-x_2)D(x_3-x_4) + D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3). \qquad (3)
\end{aligned}$$

To compute the scattering amplitude $\mathcal{M}_{fi}$ we need to evaluate

$$\langle \vec{p}_c, \vec{p}_d | T\left(-i\int_{-\infty}^{\infty} dt\; H_I(t)\right) |\vec{p}_a, \vec{p}_b\rangle = -\frac{i\lambda}{4!}\langle \vec{p}_c, \vec{p}_d | T \int d^4x\; \phi^4(x) |\vec{p}_a, \vec{p}_b\rangle,$$

Note that $T(\phi(x)^4) = \phi(x)^4$, since the four fields are evaluated at the same time $x^0$. Nonetheless, it turns out convenient to write the ordinary product as a time-ordered product, because the application of Wick's theorem on $T(\phi(x)^4)$ gives huge simplifications. We can thus use the explicit form found in equation (3).

The last line in (3) only contains C-numbers (pairs of propagators), so the contribution coming from there is non vanishing only if initial and final state momenta are identical. This corresponds to the trivial part of the $S$-matrix, so we discard it because, computing scattering amplitudes, we always look for contribution coming from nontrivial kinematics.
If we now consider the terms with one propagator, the structure we find is

$$A \equiv \langle 0| a(\vec{p}_c) a(\vec{p}_d) a^\dagger(\vec{k}) a(\vec{q}) a^\dagger(\vec{p}_a) a^\dagger(\vec{p}_b)|0\rangle,$$

where the terms with 2 $a$'s and 0 $a^\dagger$'s or viceversa have not been written because vanishing. Moving the creation operator to the left and the annihilation operator to the right one gets

$$\begin{aligned}
A \propto\; & \delta^3(\vec{p}_c - \vec{p}_b)\delta^3(\vec{p}_d - \vec{k})\delta^3(\vec{p}_a - \vec{q}) + \delta^3(\vec{p}_c - \vec{p}_a)\delta^3(\vec{q} - \vec{p}_b)\delta^3(\vec{p}_d - \vec{k}) \\
& +\delta^3(\vec{p}_c - \vec{p}_k)\delta^3(\vec{q} - \vec{p}_a)\delta^3(\vec{p}_d - \vec{p}_b) + \delta^3(\vec{q} - \vec{p}_b)\delta^3(\vec{p}_c - \vec{k})\delta^3(\vec{p}_d - \vec{p}_a),
\end{aligned}$$

in which always appears a Dirac delta involving two external momenta. So again, these pieces contribute to the trivial part of the scattering amplitude (the nontrivial one being the one in which the incoming and outgoing momenta are not related to each other).
The only contribution comes from the term with four uncontracted fields.
Let's start by defining $\phi^+(x)$ ($\phi^-(x)$) as the component of the field associated with the creation (annihilation)

operator, and noticing that

$$\phi^-(x)|\vec{p}\rangle = \int d\Omega_{\vec{k}}\ e^{-ikx} a(\vec{k}) a^\dagger(\vec{p})|0\rangle = e^{-ipx}|0\rangle,$$
$$\langle\vec{p}|\phi^+(x) = \int d\Omega_{\vec{k}}\ e^{ikx}\langle 0|a(\vec{p})a^\dagger(\vec{k}) = \langle 0|e^{ipx},$$

where we have used the equal-time commutation relations.
Now, in terms of these components at defined frequency, one gets

$$\langle\vec{p}_c,\vec{p}_d| : \phi(x)^4 : |\vec{p}_a,\vec{p}_b\rangle = 6\ \langle\vec{p}_c,\vec{p}_d|\phi^+(x)^2\phi^-(x)^2|\vec{p}_a,\vec{p}_b\rangle = 6\cdot 2\cdot 2\ e^{i(p_c+p_d-p_a-p_b)x},$$

where we have retained only terms with 2 creators and 2 annihilators. The two factors of 2 come from commuting $\phi^-(x)^2$ at the right of $a^\dagger(\vec{p}_a)a^\dagger(\vec{p}_b)$ and $\phi^+(x)^2$ at the left of $a(\vec{p}_c)a(\vec{p}_d)$. In fact one has

$$\int d\Omega_{\vec{k}} d\Omega_{\vec{q}} a(\vec{q})a(\vec{k})a^\dagger(\vec{p}_a)a^\dagger(\vec{p}_b) = \int d\Omega_{\vec{k}} d\Omega_{\vec{q}} \left[(2\pi)^3 2k_0\delta^3(\vec{p}_a-\vec{k})a(\vec{q})a^\dagger(\vec{p}_b) + a(\vec{q})a^\dagger(\vec{p}_a)a(\vec{k})a^\dagger(\vec{p}_b)\right]$$
$$\int d\Omega_{\vec{k}} d\Omega_{\vec{q}} \left[(2\pi)^3 2k_0(2\pi)^3 2q_0\delta^3(\vec{p}_a-\vec{k})\delta^3(\vec{p}_b-\vec{q}) + (2\pi)^3 2k_0(2\pi)^3 2q_0\delta^3(\vec{p}_a-\vec{q})\delta^3(\vec{p}_b-\vec{k})\right] = 2.$$

From what we have obtained here, we can deduce a set of rules to evaluate matrix elements of time-ordered products. First of all, to obtain scattering amplitudes, one can discard all terms that lead to a Dirac delta between two external momenta, since these will contribute to the trivial part of the $S$-matrix. Second, once everything is normal-ordered, the amplitude will receive a contribution from every commutation of a $\phi^-$ ($\phi^+$) trough a creation (annihilation) operator.
We can better reformulate these two rules by defining

$$\begin{aligned} \overbrace{\phi(x)|\vec{p}\rangle} \equiv \phi^-(x)|\vec{p}\rangle &= e^{-ipx}|0\rangle, \\ \langle\overbrace{\vec{p}|\phi}(x) \equiv \langle\vec{p}|\phi^+(x) &= \langle 0|e^{ipx}, \end{aligned}$$

as the contractions of fields with external states, and by stating that

- a scattering amplitude gets a vanishing contribution when *not* all external states are contracted with fields (trivial part of the $S$-matrix),
- a scattering amplitude (for scalars) gets a contribution equal to $e^{-ipx}$ ($e^{ipx}$) for each contraction of an initial (final) state with momentum $\vec{p}$ with a field $\phi(x)$.

The latter item is the Feynman rule for external legs in coordinate space. Next time a complete set of rules (including also internal propagators) will be provided in momentum space.

At the end of the day, one gets:

$$-\frac{i\lambda}{4!}\langle\vec{p}_c,\vec{p}_d|T\int d^4x\ \phi^4(x)|\vec{p}_a,\vec{p}_b\rangle = -\frac{i\lambda}{4!}\int d^4x\ (4!)\ e^{i(p_c+p_d-p_a-p_b)x} = -i\lambda(2\pi)^4\delta^4(p_c+p_d-p_a-p_b)$$
$$= i\mathcal{M}_{fi}(2\pi)^4\delta^4(p_c+p_d-p_a-p_b),$$

thus again $\mathcal{M}_{fi} = -\lambda$.
From this expression it is now straightforward to deduce the total cross section which, using the standard definitions, is

$$d\sigma = \frac{1}{2s}\lambda^2\frac{d\Omega}{32\pi^2} \quad\Longrightarrow\quad \sigma = \frac{\lambda^2}{32\pi s}.$$

In the last formula we have integrated on half of the phase space since the two final-state particles are identical.