Quantum Field Theory

Set 22: solutions

Exercise 1

Let's first rewrite the Lippman-Schwinger equation,

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H_0 + i\epsilon} H_I |\psi_\alpha^+\rangle,$$

in such a way as to have all the $|\psi_\alpha^+\rangle$ on the right hand side, namely

$$|\phi_\alpha\rangle = |\psi_\alpha^+\rangle - \frac{1}{E_\alpha - H_0 + i\epsilon} H_I |\psi_\alpha^+\rangle.$$

Now, let's apply on both sides the operator $(E_\alpha - H + i\epsilon)^{-1} H_I$, where H is the complete Hamiltonian and H_I is the interaction potential. We recall that the asymptotic states $|\psi_\alpha^+\rangle$ are eigenstates of H with eigenvalue E_α, while $|\phi_\alpha\rangle$ are eigenstates of the free Hamiltonian $H_0 \equiv H - H_I$ with the same eigenvalue. Then

$$\begin{aligned}
\frac{1}{E_\alpha - H + i\epsilon} H_I |\phi_\alpha\rangle &= \frac{1}{E_\alpha - H + i\epsilon} \left[1 - H_I \frac{1}{E_\alpha - H_0 + i\epsilon}\right] H_I |\psi_\alpha^+\rangle \\
&= \frac{1}{E_\alpha - H + i\epsilon} \left[E_\alpha - H_0 + i\epsilon - H_I\right] \left[\frac{1}{E_\alpha - H_0 + i\epsilon}\right] H_I |\psi_\alpha^+\rangle \\
&= \frac{1}{E_\alpha - H_0 + i\epsilon} H_I |\psi_\alpha^+\rangle,
\end{aligned}$$

where in the last step we have used the definition $H \equiv H_0 + H_I$. The term we have found in the third step is nothing but the one that appears on the right hand side of the Lippman-Schwinger equation, so that at the end we can write

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H + i\epsilon} H_I |\phi_\alpha\rangle,$$

and, consequently, the T-matrix element is

$$T^\dagger_{\beta\alpha} \equiv \langle\phi_\beta|H_I|\psi_\alpha^+\rangle = \langle\phi_\beta|H_I|\phi_\alpha\rangle + \langle\phi_\beta|H_I \frac{1}{E_\alpha - H + i\epsilon} H_I|\phi_\alpha\rangle. \tag{1}$$

As an application of this formula, let's consider the case in which $H_I = V_I = V_1(\vec{x}) + V_2(\vec{x})$, and in particular $V_2(\vec{x}) = V_1(\vec{x} + \vec{A}) \equiv e^{i\vec{P}\cdot\vec{A}} V_1(\vec{x}) e^{-i\vec{P}\cdot\vec{A}}$. Suppose $V_1(\vec{x})$ to be significantly different from 0 only in a small region around a given point $\vec{x}_0$, so that the points $\vec{x}$ belonging to that region satisfy $|\vec{A}| \gg |\vec{x} - \vec{x}_0|$, i.e. we can neglect multiple scattering.
In particular the product of the two operators V_1, V_2 can be seen to vanish, indeed:

$$V_1 V_2 = \int dx V_1 V_2 |x\rangle\langle x| = \int dx V_1(x) V_2(x) |x\rangle\langle x| \simeq 0$$

Another useful property is the following: consider the operator products coming from Eq.(1), which can be formally expanded in a geometric series, such as:

$$\frac{1}{E_\alpha - H + i\epsilon} V_1 = \frac{1}{E_\alpha - H_0 - V_1 - V_2 + i\epsilon} V_1 = \frac{1}{E_\alpha - H_0} \frac{1}{1 - \frac{1}{E - H_0}(V_1 + V_2)} V_1 =$$
$$\frac{1}{E_\alpha - H_0} \sum_{n=0}^\infty \left[\frac{1}{E - H_0}(V_1 + V_2)\right]^n V_1 \simeq \frac{1}{E_\alpha - H_0} \sum_{n=0}^\infty \left[\frac{1}{E - H_0} V_1\right]^n V_1 =$$
$$\frac{1}{E_\alpha - H_0 - V_1 + i\epsilon} V_1.$$

In the above calculation we assumed that:

$$V_2 \frac{1}{E_\alpha - H_0} V_1 \simeq 0 \tag{2}$$

even though $\frac{1}{E_\alpha - H_0}$ and V_1, V_2 don't commute and $\frac{1}{E_\alpha - H_0}$ is a non-local operator. However in position-space representation the above term can be written as:

$$\int dxdy V_2(x) \frac{1}{\frac{\nabla^2}{2m} + E}(x-y) V1(y) \propto \int dxdy V_2(x) \frac{e^{-\sqrt{2mE}|x-y|}}{|x-y|} V_2(y) \tag{3}$$

because $\frac{1}{\frac{\nabla^2}{2m}+E}(x-y)$ is similar to the Green function for a potential mediated by a massive particle (the Yukawa potential), therefore the mixing between V_1 and V_2 is exponentially suppressed.
At the end, what one gets is

$$\begin{aligned}
T^\dagger_{\beta\alpha} &= \langle\phi_\beta|H_I|\phi_\alpha\rangle + \langle\phi_\beta|H_I \frac{1}{E_\alpha - H + i\epsilon} H_I|\phi_\alpha\rangle \\
&= \langle\phi_\beta|(V_1+V_2)|\phi_\alpha\rangle + \langle\phi_\beta|V_1 \frac{1}{E_\alpha - H_0 - V_1 + i\epsilon} V_1|\phi_\alpha\rangle + \langle\phi_\beta|V_2 \frac{1}{E_\alpha - H_0 - V_2 + i\epsilon} V_2|\phi_\alpha\rangle \\
&= (T_1)^\dagger_{\beta\alpha} + (T_2)^\dagger_{\beta\alpha}.
\end{aligned}$$

Now we can express $(T_2)^\dagger_{\beta\alpha}$ in terms of $(T_1)^\dagger_{\beta\alpha}$:

$$\begin{aligned}
&(T_2)^\dagger_{\beta\alpha} = \langle\phi_\beta|e^{i\vec{P}\cdot\vec{A}} V_1 e^{-i\vec{P}\cdot\vec{A}}|\phi_\alpha\rangle + \langle\phi_\beta|e^{i\vec{P}\cdot\vec{A}} V_1 e^{-i\vec{P}\cdot\vec{A}} \sum_{n=0}^{\infty} \left[\frac{1}{E-H_0}\right]^{n+1} e^{i\vec{P}\cdot\vec{A}} V_1^n e^{-i\vec{P}\cdot\vec{A}} e^{i\vec{P}\cdot\vec{A}} V_1 e^{-i\vec{P}\cdot\vec{A}}|\phi_\alpha\rangle \\
&= \left(e^{i\vec{P}\cdot\vec{A}} T_1 e^{-i\vec{P}\cdot\vec{A}}\right)^\dagger_{\beta\alpha},
\end{aligned}$$

where we have used the invariance of the free Hamiltonian under translations.
This reasoning can be generalized to an arbitrary number of hamiltonians, so that, in the approximation of large mutual distances, one simply gets

$$H_I = \sum_{j=1}^{N} e^{i\vec{P}\cdot\vec{A}_j} V_1 e^{-i\vec{P}\cdot\vec{A}_j} \quad \Longrightarrow \quad T = \sum_{j=1}^{N} e^{i\vec{P}\cdot\vec{A}_j} V_1 e^{-i\vec{P}\cdot\vec{A}_j},$$

where $\vec{A}_1 \equiv \vec{0}$.
This way, sending a wave with momentum $\vec{k}_i$ on a bunch of scattering potentials, and calling $\vec{q}$ the difference between the outgoing momentum $\vec{k}_f$ and $\vec{k}_i$, one gets

$$\langle\vec{k}_f|T|\vec{k}_i\rangle \equiv F(\vec{q}) = \sum_{j=1}^{N} \langle\vec{k}_f|e^{i\vec{P}\cdot\vec{A}_j} T_1 e^{-i\vec{P}\cdot\vec{A}_j}|\vec{k}_i\rangle = f(\vec{q}) \sum_{j=1}^{N} e^{i\vec{q}\cdot\vec{A}_j},$$

where $f(\vec{q}) \equiv \langle\vec{k}_f|T_1|\vec{k}_i\rangle$. In the continuum limit (looking from distance $|\vec{L}| \gg |\vec{A}_j|$, $\forall\ j$) one has

$$F(\vec{q}) = f(\vec{q}) \int d^3A\ e^{i\vec{q}\cdot\vec{A}} \rho(\vec{A}),$$

which explains why the scattering amplitude $F(\vec{q})$ in a diffraction experiment is the Fourier transform of the matter distribution $\rho(\vec{A})$ (up to an overall form factor $f(\vec{q})$).

Exercise 2

The cross section for a scattering process $AB \to CD$ is given by

$$d\sigma = \frac{1}{4E_A E_B |\vec{v}_A - \vec{v}_B|} |\mathcal{M}_{AB\to CD}|^2 d\Phi_2,$$

where $\mathcal{M}_{AB\to CD}$ is the matrix element associated to the process and $d\Phi_2$ is the 2-body phase space. In general

$$d\Phi_n = \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^4 \left(P_A + P_B - \sum_i P_i \right).$$

In our case $i = C, D$ only. In this exercise P_i represents a 4 momentum while p_i is the spatial momentum. Thanks to the presence of the δ^4 we can easily perform 4 integrals in a straightforward way, without caring about the particular form of the matrix element, that here we leave unexpressed. We recall that this definition of the cross section holds for a reference frame in which the velocities of the incoming particles are collinear (the matrix element and the phase space are Lorentz invariant, while the flux factor depends on the reference frame). Let us take the velocities in the $\hat{z}$ direction. Before performing the integrations we can write the flux factor in a different way:

$$\begin{aligned}\frac{1}{4\sqrt{(P_A\cdot P_B)^2 - m_A^2 m_B^2}} &= \frac{1}{4\sqrt{(E_A E_B - p_A^z p_B^z)^2 - (E_A^2 - p_A^{z2})(E_B^2 - p_B^{z2})}} \\ &= \frac{1}{4\sqrt{(E_B p_A^z - E_A p_B^z)^2}} = \frac{1}{4E_A E_B |v_A^z - v_B^z|},\end{aligned}$$

where we have used the definition of velocity $v \equiv p/E$.
Let's now move to the expression of $d\sigma$. We want to compute it in the center of mass frame, which is defined by requiring the sum of the spatial momenta of the colliding particles to be zero: $\vec{p}_A + \vec{p}_B = 0$.
Let us integrate over $d^3 p_D$. This can be done easily using $\delta^3(\vec{p}_A + \vec{p}_B - \vec{p}_C - \vec{p}_D)$. This Dirac delta enforces the momentum conservation $\vec{p}_D = \vec{p}_A + \vec{p}_B - \vec{p}_C = -\vec{p}_C$ and in addition express the energy E_D as a function of the momentum $\vec{p}_C$ we still have to integrate over:

$$E_D = \sqrt{m_D^2 + \vec{p}_D^2} = \sqrt{m_D^2 + \vec{p}_C^2}.$$

Hence we get:

$$d\sigma = \frac{1}{4\sqrt{(P_A\cdot P_B)^2 - m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d^3 p_C}{(2\pi)^3 2E_C 2E_D} (2\pi)\delta\left(E_A + E_B - E_C - E_D\right).$$

There is still a delta function left that we can use to integrate over another variable. Let us pass in polar coordinates and call $p_C \equiv |\vec{p}_C|$:

$$d\sigma = \frac{1}{4\sqrt{(P_A\cdot P_B)^2 - m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\varphi\ d\cos\theta\ p_C^2 dp_C}{(2\pi)^3 2E_C 2E_D} (2\pi)\delta\left(E_A + E_B - E_C - E_D\right).$$

In the expression just deduced, the remaining delta function will eliminate one of the three integrations (which we can choose to be the one over dp_C), so that two degrees of freedom remain. If one makes the further assumption of scalar particles or of unpolarized scattering, the process is invariant under azimuthal rotations, so that $|\mathcal{M}_{AB\to CD}|^2$ won't depend on φ, and at the end we will be left with only one effective variable, which can be chosen to coincide with θ: $\cos\theta = \frac{\vec{p}_A\cdot\vec{p}_C}{p_A p_C}$. But in general, without assumptions, two integration variables are left.

Therefore we keep this variable and we integrate over dp_C. This integral is not trivial since the dependence of the delta function on p_C is complicated (p_C enters in E_C and E_D). Let us use the following change of variables:

$$\begin{aligned}&\frac{d(E_C + E_D)}{dp_C} = \frac{d(\sqrt{m_C^2 + p_C^2} + \sqrt{m_D^2 + p_C^2})}{dp_C} = \frac{p_C}{\sqrt{m_C^2 + p_C^2}} + \frac{p_C}{\sqrt{m_D^2 + p_C^2}} = \frac{p_C}{E_C} + \frac{p_C}{E_D} \\ &\Longrightarrow \quad dp_C = \frac{E_C E_D}{p_C(E_C + E_D)} d(E_C + E_D).\end{aligned}$$

Note that this change of variables gives the same result that we would deduce by using the property of the Dirac delta

$$\delta(f(x) - A) = \frac{\delta(x - \bar{x})}{\left|\frac{df}{dx}\right|_{x=\bar{x}}},$$

where $\bar{x} \equiv f^{-1}(A)$.
Substituting we get:

$$d\sigma = \frac{1}{4\sqrt{(P_A\cdot P_B)^2 - m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\varphi\ d\cos\theta\ p_C}{16\pi^2} \frac{d(E_C + E_D)}{E_C + E_D} \delta\left(E_A + E_B - E_C - E_D\right).$$

Note that at this stage the variable p_C is no longer independent (we won't integrate over it), so it must be expressed in terms of the integration variables. In particular one can invert the relation between $E_C + E_D$ and p_C to get

$$p_C \equiv p_C(E_C + E_D) = \sqrt{\frac{(E_C + E_D)^4 + (m_C^2 - m_D^2)^2 - 2(E_C + E_D)^2(m_C^2 + m_D^2)}{4(E_C + E_D)^2}}.$$

Now we can perform the integral in $d(E_C + E_D)$, which is trivial: calling $X \equiv (E_C + E_D)$ one has

$$\frac{dX}{X} p_C(X)\delta\left(E_A + E_B - X\right) = \frac{p_C(E_A + E_B)}{E_A + E_B}.$$

Finally, let's introduce the Mandelstam variable $s = (P_A + P_B)^2 = (P_C + P_D)^2$. This quantity represents the total center of mass energy squared: $E_A + E_B = E_C + E_D = \sqrt{s}$.
In the end:

$$d\sigma = \frac{1}{4\sqrt{\left(\frac{s-m_A^2-m_B^2}{2}\right)^2 - m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\varphi\ d\cos\theta}{16\pi^2} \frac{p_C(\sqrt{s})}{\sqrt{s}}$$

Using the expression for p_C in the center of mass (the same as above with the substitution $E_C + E_D \to \sqrt{s}$),

$$p_C(\sqrt{s}) = \sqrt{\frac{s^2 + (m_C^2 - m_D^2)^2 - 2s(m_C^2 + m_D^2)}{4s}},$$

we can obtain simple expressions for the particular cases:

$$m_C = m_D = m \qquad d\sigma = \frac{1}{2\sqrt{\left(s - m_A^2 - m_B^2\right)^2 - 4m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\Omega}{32\pi^2} \sqrt{1 - \frac{4m^2}{s}},$$

$$m_C = m,\ m_D = 0 \qquad d\sigma = \frac{1}{2\sqrt{\left(s - m_A^2 - m_B^2\right)^2 - 4m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\Omega}{32\pi^2} \left(1 - \frac{m^2}{s}\right),$$

where $d\Omega$ is the solid angle $d\varphi d\cos\theta$.