Quantum Field Theory

Set 21: solutions

Exercise 1

Recalling the transformation properties of a Weyl fermion under time-reversal,

$$U_T^\dagger\,\chi_L^\alpha\,U_T = \eta_T\,\epsilon^{\alpha\beta}\chi_L^\beta(-t,\vec{x}),$$

let us apply U_T to the Weyl Lagrangian:

$$U_T^\dagger\,\mathcal{L}_W\,U_T = U_T^\dagger(i\chi_L^\dagger\,\bar{\sigma}^\mu\partial_\mu\,\chi_L)\,U_T.$$

We recall that U_T is anti-unitary, that means:

$$U_T^\dagger\,(aO_1 + bO_2)\,U_T = a^* U_T^\dagger\,O_1\,U_T + b^* U_T^\dagger\,O_2\,U_T,$$

where a, b are constant and O_1, O_2 are operators. Hence we have

$$\begin{aligned} U_T^\dagger\,\mathcal{L}_W(t,\vec{x})\,U_T &= -i(U_T^\dagger\chi_L^\dagger(t,\vec{x})U_T)\,(U_T^\dagger\bar{\sigma}^\mu U_T)\partial_\mu\,(U_T^\dagger\,\chi_L(t,\vec{x})\,U_T) = -i|\eta_T|^2(\epsilon\,\chi_L(-t,\vec{x}))^\dagger(\bar{\sigma}^\mu)^*\partial_\mu\,(\epsilon\,\chi_L(-t,\vec{x})) \\ &= -i\chi_L^\dagger\epsilon^T\,(\bar{\sigma}^\mu)^*\epsilon\partial_\mu\,\chi_L, \end{aligned}$$

where the argument $(-t,\vec{x})$ is understood. Now we use that fact that

$$\epsilon^T(\bar{\sigma}^\mu)^*\epsilon = (\mathbb{1}_2, \epsilon(\sigma^i)^*\epsilon) = (\mathbb{1}_2, \sigma^i) = \sigma^\mu\,,$$

to get finally:

$$U_T^\dagger\,\mathcal{L}_W\,U_T(t,\vec{x}) = -i\chi_L^\dagger(-t,\vec{x})\,\sigma^\mu\partial_\mu\,\chi_L(-t,\vec{x}) = i\chi_L^\dagger(-t,\vec{x})\,\bar{\sigma}^\mu\widetilde{\partial}_\mu\,\chi_L(-t,\vec{x}) = \mathcal{L}_W(-t,\vec{x}),$$

where we have denoted $\widetilde{\partial}_\mu \equiv (-\partial_t, \partial_i)$. Note that the transformed Lagrangian is unchanged in form but is evaluated at $-t$.

Exercise 2

Let us consider a one dimensional quantum system. We want to study the scattering from a potential of the general form illustrated in Figure 1: V is significantly different from zero in a region $x \in [-L\,, L]$, while it rapidly approaches 0 for $|x| > L$. Therefore in the regions far away from the potential we can take $V \equiv 0$ and consider

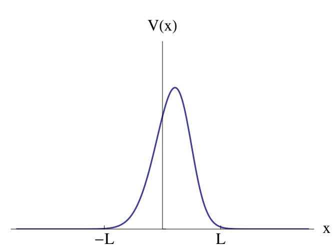

Figure 1: Generic barrier

the theory as free: the solutions with positive energy E_k are complex exponentials:

$$e^{ikx}\,, \qquad e^{-ikx}\,, \qquad E_k = \frac{k^2}{2m}\,,$$

where m is the mass of the particle scattered by the potential. This system represents a simple setup to understand the meaning of *in* and *out* states. Let us first recall their definition in general and then apply it to the this specific case. Given the Hamiltonian H_0, which usually represents the free theory, we call $|\phi_\alpha\rangle$ its eigenvectors:

$$H_0|\phi_\alpha\rangle = E_\alpha|\phi_\alpha\rangle.$$

A general state of the theory will be a wave packet made of a superposition of the states $|\phi_\alpha\rangle$:

$$|\phi\rangle = \int d\alpha\, g(\alpha)|\phi_\alpha\rangle. \tag{1}$$

Let us assume that at a certain time an interaction H_I is switched on. The eigenstates of the complete Hamiltonian $H = H_0 + H_I$ are now modified. Let us assume that they can be labeled with the same index α:

$$(H_0 + H_I)|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle.$$

We can now recall the definition of in- and out- states:

- the *in*-states $|\psi^+\rangle$ tend to $|\phi\rangle$ for $t \to -\infty$;
- the *out*-states $|\psi^-\rangle$ tend to $|\phi\rangle$ for $t \to \infty$.

A more formal definition is given using the Moeller operators. However, as far as this exercise is concerned, the above definition is enough. In the present case the role of H_0 is given by the free Hamiltonian $H_0 = \frac{p^2}{2m}$ while the interaction is $H_I = V(x)$. In a scattering process the states of interest are those representing a particle incoming at time $-\infty$ from the left (or the right) and a particle outgoing at time $+\infty$ to the left (or the right). These states are described by wave packets of the form (1).
We claim that the in-state associated to a incoming particle from the left is given by the following solution:

$$|\psi^+\rangle \equiv \int dk\, g(k)|\psi_k^+\rangle\ , \qquad\qquad \langle x|\psi_k^+\rangle = \begin{cases} e^{ikx} + Re^{-ikx}, & \text{for } x \ll -L, \\ Te^{ikx}, & \text{for } x \gg L, \\ \text{not specified} & \text{otherwise,} \end{cases}$$

where R and T are two coefficients. The above state is defined at time $t = 0$. In order to see if the definition of in-state applies we need to evolve it back in time up to $t \to -\infty$. Since the dynamics of $|\psi^+\rangle$ is determined by the full theory we must evolve it using the complete Hamiltonian H. Hence:

$$|\psi^+(t)\rangle = e^{-iHt}|\psi^+\rangle = \int dk\, g(k)e^{-iHt}|\psi_k^+\rangle = \int dk\, g(k)e^{-i\frac{k^2}{2m}t}|\psi_k^+\rangle.$$

Up to now we haven't specified the form of the wave packet $g(k)$. Let us take for simplicity a gaussian distribution centered around a momentum p:

$$g(k) = e^{-\frac{(p-k)^2}{2\sigma^2}}$$

If the gaussian is very narrow the main contribution in the integral on dk will come from the neighborhood of the momentum p and all the rest will be suppressed exponentially. So we can write $k = p + \epsilon$:

$$\begin{aligned} |\psi^+(t)\rangle &= \int dk\, e^{-\frac{(p-k)^2}{2\sigma^2}}\, e^{-i\frac{k^2}{2m}t}|\psi_k^+\rangle \simeq \int dk\, e^{-\frac{(p-k)^2}{2\sigma^2}}\, e^{-i\frac{p^2}{2m}t - i\frac{p\epsilon}{m}t}|\psi_k^+\rangle\ . \\ &\simeq e^{-i\frac{p^2}{2m}t} \int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t}|\psi_{p+\epsilon}^+\rangle. \end{aligned}$$

where $v \equiv p/m$, and we neglected the term $i\frac{\epsilon^2}{2m}t$ in the exponential.

Let us now consider the above state in the regions $|x| \gg L$, where we make use of the explicit form for $\langle x|\psi_k^+\rangle$. Hence:

$$\langle x|\psi^+(t)\rangle = e^{-i\frac{p^2}{2m}t} \times \begin{cases} \int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t}\left(e^{i(p+\epsilon)x} + Re^{-i(p+\epsilon)x}\right) = \frac{\sqrt{2\pi}}{\sigma}\left(e^{ipx}e^{-\frac{(x-vt)^2}{2\sigma^2}} + Re^{-ipx}e^{-\frac{(x+vt)^2}{2\sigma^2}}\right), & \text{for } x \ll -L \\ T\int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t}e^{i(p+\epsilon)x} = T\frac{\sqrt{2\pi}}{\sigma}e^{ipx}e^{-\frac{(x-vt)^2}{2\sigma^2}}, & \text{for } x \gg L. \end{cases}$$

Finally let us consider the limit $t \to -\infty$: in this limit some terms vanish because they are exponentially suppressed:

$$\langle x|\psi^+(t)\rangle \longrightarrow \begin{cases} \frac{\sqrt{2\pi}}{\sigma} e^{ipx} e^{-\frac{(x-vt)^2}{2\sigma^2}} & \text{for } x \ll -L,\ t \to -\infty, \\ 0 & \text{for } x \gg L\,, t \to -\infty. \end{cases}$$

The above solution describes as announced an incoming wave packet moving from left to right. Similarly one could find the out-states:

$$|\psi^-\rangle \equiv \int dk\, g(k)|\psi_k^-\rangle\,, \qquad \langle x|\psi_k^-\rangle = \begin{cases} T' e^{+ikx} & \text{for } x \ll -L \\ e^{+ikx} + R' e^{-ikx} & \text{for } x \gg L \\ \text{not specified} & \text{otherwise} \end{cases}$$

Evolving in time as before we get:

$$\langle x|\psi^-(t)\rangle = e^{-i\frac{p^2}{2m}t} \times \begin{cases} T' \int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t} e^{i(p+\epsilon)x}, & \text{for } x \ll -L, \\ \int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t} \left(e^{i(p+\epsilon)x} + R' e^{-i(p+\epsilon)x}\right), & \text{for } x \gg L. \end{cases}$$

Again in the limit $t \to \infty$ some integrals vanish and we are left with:

$$\langle x|\psi^-(t)\rangle \longrightarrow \begin{cases} 0 & \text{for } x \ll -L, \\ \frac{\sqrt{2\pi}}{\sigma} e^{ipx} e^{-\frac{(x-vt)^2}{2\sigma^2}} & \text{for } x \gg L. \end{cases}$$

The above solution represents an outgoing wave packet moving from left to right. One could also find a in-state incoming at time $-\infty$ from the right and out-state escaping towards the left at time $+\infty$. Finally we could compute the matrix element between the in- and out-state, which corresponds to the S-matrix element between the incoming wave packet and the outgoing one:

$$\langle \psi^-|\psi^+\rangle.$$

Exercise 3

Suppose we have some state $|\phi_k\rangle$ which is an eigenstate of a free Hamiltonian H_0. For simplicity let us consider $H_0 = \frac{p^2}{2m}$. Let us assume that at certain finite time t and a finite distance L the states start interacting with a potential V. The system is now described by the full Hamiltonian $H = H_0 + V$. We also assume that the interaction with the potential is localized in space, so that the system, far away from the interaction point and after enough time, can still be described in terms of eigenstates of H_0.
We want to extract an expression for the asymptotic states for this framework. Recalling the Lippmann-Schwinger equation we have:

$$|\Psi_k^\pm\rangle = |\phi_k\rangle + \frac{1}{E_k - H_0 \pm i\epsilon} H_I |\Psi_k^\pm\rangle,$$

where in this case $H_I = V$ and we are labeling the states with the index k: $E_k = \frac{k^2}{2m}$. We can describe the theory in coordinate space taking the bracket with an eigenstate of position operator $\langle x|$ or in momentum space considering the bracket with an eigenstate of momentum $\langle p|$. We choose the former description. Then:

$$\Psi_k^\pm(x) = \langle x|\Psi_k^\pm\rangle = \langle x|\phi_k\rangle + \langle x|\frac{1}{E_k - H_0 \pm i\epsilon} V|\Psi_k^\pm\rangle.$$

Let us insert a complete set of states $\int d^3x' |x'\rangle\langle x'| = 1$ before the operator V:

$$\Psi_k^\pm(x) = \langle x|\phi_k\rangle + \int d^3x'\, \underbrace{\langle x|\frac{1}{E_k - H_0 \pm i\epsilon}|x'\rangle}_{\mathcal{G}_{k\pm}(x-x')} \langle x'|V|\Psi_k^\pm\rangle.$$

Let us first compute the Green function $\mathcal{G}_\pm(x - x')$:

$$\mathcal{G}_{k\pm}(x - x') = \int d^3p d^3p' \langle x|p\rangle\langle p|\frac{1}{E_k - H_0 \pm i\epsilon}|p'\rangle\langle p'|x'\rangle,$$

where again we inserted two set of complete eigenstates of momentum. Recalling standard results of QM we have:

$$\langle x|p\rangle = \frac{e^{i\vec{x}\cdot\vec{p}}}{\sqrt{(2\pi)^3}},$$

and therefore:

$$\begin{aligned}\mathcal{G}_{k\pm}(x-x') &= \int \frac{d^3p}{(2\pi)^3} d^3p' \frac{1}{E_k - \frac{p'^2}{2m} \pm i\epsilon} \langle p|p'\rangle e^{i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{x}')} = \int \frac{d^3p}{(2\pi)^3} d^3p' \frac{1}{E_k - \frac{p'^2}{2m} \pm i\epsilon} \delta^3(\vec{p}-\vec{p}') e^{i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{x}')} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_k - \frac{p^2}{2m} \pm i\epsilon} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')},\end{aligned}$$

where we have used the fact that the free Hamiltonian applied to an eigenstate of momentum simply gives $H_0|p\rangle = \frac{p^2}{2m}|p\rangle$. Hence:

$$\begin{aligned}\mathcal{G}_{k\pm}(x-x') &= \int \frac{d^3p}{(2\pi)^3} \frac{2m}{k^2 - p^2 \pm i\epsilon} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = \int_0^\infty \frac{p^2\,dp}{(2\pi)^2} \frac{2m}{k^2 - p^2 \pm i\epsilon} \int_{-1}^1 d(\cos\theta) e^{ip|\vec{x}-\vec{x}'|\cos\theta} \\ &= \int_0^\infty \frac{p\,dp}{(2\pi)^2} \frac{2m}{i|\vec{x}-\vec{x}'|} \frac{e^{ip|\vec{x}-\vec{x}'|} - e^{-ip|\vec{x}-\vec{x}'|}}{k^2 - p^2 \pm i\epsilon} \\ &= \frac{im}{4\pi^2|\vec{x}-\vec{x}'|} \int_{-\infty}^\infty p\,dp \frac{e^{ip|\vec{x}-\vec{x}'|} - e^{-ip|\vec{x}-\vec{x}'|}}{p^2 - k^2 \mp i\epsilon},\end{aligned}$$

where in the last step we have used the symmetry of the integrand under $p \to -p$ to extend the integral from $-\infty$ to $+\infty$ and we have therefore divided by 2. The above integral is composed by two pieces and contains two poles at $p^2 = k^2 \pm i\epsilon$ or $p \simeq \pm(k \pm i\epsilon)$ (note that the two $\pm$'s are unrelated and that the ϵ appearing here is *not* the same as before). According to how we close the contour and the $\pm$ prescription we can enclose or not a pole.

Let us separate the discussions and start with $\mathcal{G}_{k+}(x-x')$ where the poles are at $p = \pm(k+i\epsilon)$: for the first term in the integral we close in the upper plane and therefore we encircle only the pole at $p = k + i\epsilon$; for the second piece we close the contour in the lower half-plane and we enclose only the pole at $p = -k - i\epsilon$. Hence:

$$\begin{aligned}\mathcal{G}_{k+}(x-x') &= \frac{im}{4\pi^2|\vec{x}-\vec{x}'|}(2\pi i) \left[\left(\frac{p\,e^{ip|\vec{x}-\vec{x}'|}}{p+k} \right)\Bigg|_{p=k} + \left(\frac{p\,e^{-ip|\vec{x}-\vec{x}'|}}{p-k} \right)\Bigg|_{p=-k} \right] \\ &= -\frac{2m}{4\pi|\vec{x}-\vec{x}'|} e^{ik|\vec{x}-\vec{x}'|}.\end{aligned}$$

Let us consider now $\mathcal{G}_{k-}(x-x')$ where the poles are at $p = \pm(k - i\epsilon)$: for the first term in the integral we close again in the upper half-plane and therefore we encircle only the pole at $p = -k + i\epsilon$; for the second piece we close the contour in the lower half-plane and we enclose only the pole at $p = k - i\epsilon$. Hence:

$$\begin{aligned}\mathcal{G}_{k-}(x-x') &= \frac{im}{4\pi^2|\vec{x}-\vec{x}'|}(2\pi i) \left[\left(\frac{p\,e^{ip|\vec{x}-\vec{x}'|}}{p-k} \right)\Bigg|_{p=-k} + \left(\frac{p\,e^{-ip|\vec{x}-\vec{x}'|}}{p+k} \right)\Bigg|_{p=k} \right] \\ &= -\frac{2m}{4\pi|\vec{x}-\vec{x}'|} e^{-ik|\vec{x}-\vec{x}'|}.\end{aligned}$$

At this point we can come back to our Lippmann-Schwinger equation and plug in the expression for the Green function:

$$\Psi_k^\pm(x) = \langle x|\phi_k\rangle - \frac{2m}{4\pi} \int d^3x' \frac{e^{\pm ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \langle x'|V|\Psi_k^\pm\rangle.$$

Let us now perform the first approximation: we consider the form of $\Psi_k^\pm(x)$ for $x \gg L$, where L is the characteristic scale of the potential V (and $x' \le L$, otherwise the matrix element $\langle x'|V|\Psi_k^\pm\rangle$ is 0). Then we can write:

$$|\vec{x}-\vec{x}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta} = r\left(1 - 2\frac{r'}{r}\cos\theta + \frac{r'^2}{r^2}\right)^{1/2} \simeq r\left(1 - \frac{r'}{r}\cos\theta\right) = r - r'\cos\theta,$$

where $r \equiv |\vec{x}|$ and $r' \equiv |\vec{x}'|$ and $\vec{x} \cdot \vec{x}' = rr' \cos\theta$. Finally

$$\frac{e^{\pm ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \simeq \frac{e^{\pm ikr}}{r} e^{\mp ik\hat{x}\cdot\vec{x}'}$$

Plugging this in the expression for $\Psi_k^{\pm}(x)$ and inserting again a complete set of states we have:

$$\Psi_k^{\pm}(x) = \langle x|\phi_k\rangle - \frac{2m}{4\pi}\frac{e^{\pm ikr}}{r}\int d^3x' d^3x'' \langle x'|V|x''\rangle\langle x''|\Psi_k^{\pm}\rangle e^{\mp ik\hat{x}\cdot\vec{x}'}.$$

If the potential is such that $\langle x'|V|x''\rangle = V(x')\langle x'|x''\rangle = V(x')\delta^3(\vec{x}' - \vec{x}'')$, we have

$$\Psi_k^{\pm}(x) = e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{4\pi}\frac{e^{\pm ikr}}{r}\underbrace{\int d^3x' V(x')\langle x'|\Psi_k^{\pm}\rangle e^{\mp ik\hat{x}\cdot\vec{x}'}}_{f(k,\hat{x})}.$$

The above function completely describes the effect of the potential at large distances from it.