Quantum Field Theory

Set 20

Exercise 1

Let's apply charge conjugation on the complex scalar field:

$$\begin{aligned} C\phi(x)C &= \int d\Omega_{\vec{k}} \left[e^{-ikx} Ca(\vec{k})C + e^{ikx} Cb^\dagger(\vec{k})C\right] = \eta_C \phi^\dagger(x) \\ &= \eta_C \int d\Omega_{\vec{k}} \left[e^{-ikx} b(\vec{k}) + e^{ikx} a^\dagger(\vec{k})\right]. \end{aligned}$$

Matching the coefficients of the complex exponentials, one gets

$$Ca(\vec{k})C = \eta_C b(\vec{k}),$$
$$Cb(\vec{k})C = \eta_C^* a(\vec{k}).$$

Exercise 2

Let's repeat the steps of Exercise 2 of Set 19, for charge conjugation. We recall the definition of charge conjugation on the creation and annihilation operators, deduced in Exercise 1,

$$Ca^\dagger(\vec{k})C = \eta_C^* b^\dagger(\vec{k}), \qquad Cb^\dagger(\vec{k})C = \eta_C a^\dagger(\vec{k}),$$

and write explicitly

$$\begin{aligned} C|\Phi_l\rangle &= \int d\Omega_{\vec{p}}\ f_l(\vec{p}, -\vec{p})\, Ca^\dagger(\vec{p})C\, Cb^\dagger(-\vec{p})C|0\rangle \\ &= \eta_C \eta_C^* \int d\Omega_{\vec{p}}\ f_l(\vec{p}, -\vec{p})\, b^\dagger(\vec{p})\, a^\dagger(-\vec{p})|0\rangle \\ &= \int d\Omega_{\vec{p}}\ f_l(-\vec{p}, \vec{p})\, a^\dagger(\vec{p})\, b^\dagger(-\vec{p})\, |0\rangle = (-1)^l |\Phi_l\rangle, \end{aligned}$$

where we have repositioned $a^\dagger$ and $b^\dagger$ in the initial order using the commutation relation $[a^\dagger, b^\dagger] = 0$.
Note that the combined action of the two transformations leaves a state of scalar particle-antiparticle invariant:

$$CP|\Phi_l\rangle = (-1)^{l+l} |\Phi_l\rangle = |\Phi_l\rangle.$$

Let's now move to fermions, recalling that

$$Cb^\dagger(r, \vec{k})C = -\eta_C^* \tilde{d}^\dagger(r, \vec{k}) \quad \Longrightarrow \quad C\tilde{d}^\dagger(r, \vec{k})C = -\eta_C b^\dagger(r, \vec{k}).$$

One has

$$\begin{aligned} C|\Psi_{l,S}\rangle &= \sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p}, -\vec{p})\, \chi_S(r, t)\, Cb^\dagger(t, \vec{p})CC\tilde{d}^\dagger(r, -\vec{p})C|0\rangle \\ &= \sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p}, -\vec{p})\, \chi_S(r, t)\, \tilde{d}^\dagger(t, \vec{p})\, b^\dagger(r, -\vec{p})|0\rangle \\ &= -\sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p}, -\vec{p})\, \chi_S(r, t)\, b^\dagger(r, -\vec{p})\, \tilde{d}^\dagger(t, \vec{p})|0\rangle = (-1)^{l+S} |\Psi_{l,S}\rangle, \end{aligned}$$

where the minus sign in the third line comes from changing the order of the operators, since they antiocommute $\{\tilde{d}^\dagger, b^\dagger\} = 0$.
In the end, the combined action of the two transformations on a state made of fermionic particle-antiparticle is given by:

$$CP|\Psi_{l,S}\rangle = (-1)^{l+S}(-1)^{l+1} |\Psi_{l,S}\rangle = (-1)^{S+1} |\Psi_{l,S}\rangle.$$

Exercise 3

The time reversal transformation property for the scalar field is

$$T\phi(\vec{x},t)T = \eta_T\phi(\vec{x},-t),$$

which leads to

$$Ta(\vec{k})T = \eta_T a(-\vec{k}),$$
$$Tb^\dagger(\vec{k})T = \eta_T b^\dagger(-\vec{k}),$$

for the annihilation and creation operators.
The action of time reversal on the derivative of the field is a bit more tricky. Since

$$[\partial_\mu\phi](\vec{x},t) = \int d\Omega_{\vec{k}}(-ik_\mu)\left[a(\vec{k})e^{-ikx} - b^\dagger(\vec{k})e^{ikx}\right],$$
$$[\partial_\mu\phi](\vec{x},-t) = \int d\Omega_{\vec{k}}(-ik_\mu)\left[a(\vec{k})e^{i\eta^{\mu\mu}k_\mu x^\mu} - b^\dagger(\vec{k})e^{-i\eta^{\mu\mu}k_\mu x^\mu}\right],$$

recalling that T is an anti-linear operator, one has

$$\begin{aligned}
T[\partial_\mu\phi](\vec{x},t)T &= \int d\Omega_{\vec{k}}(ik_\mu)\left[Ta(\vec{k})Te^{ikx} - Tb^\dagger(\vec{k})Te^{-ikx}\right]\\
&= \eta_T\int d\Omega_{\vec{k}}(ik_\mu)\left[a(-\vec{k})e^{ikx} - b^\dagger(-\vec{k})e^{-ikx}\right]\\
&= \eta_T\int d\Omega_{\vec{k}}(i\eta^{\mu\mu}k_\mu)\left[a(\vec{k})e^{i\eta^{\mu\mu}k_\mu x^\mu} - b^\dagger(\vec{k})e^{-i\eta^{\mu\mu}k_\mu x^\mu}\right]\\
&= -\eta_T\eta^{\mu\mu}[\partial_\mu\phi](\vec{x},-t),
\end{aligned}$$

where in the third step we have exchanged as usual $\vec{k}\to-\vec{k}$. Thus one can compute the action of time reversal on the current:

$$\begin{aligned}
TJ_\mu(\vec{x},t)T &= T\{i\phi^\dagger(\vec{x},t)[\partial_\mu\phi](\vec{x},t) - i[\partial_\mu\phi^\dagger](\vec{x},t)\phi(\vec{x},t)\}T\\
&= -iT\phi^\dagger(\vec{x},t)TT[\partial_\mu\phi](\vec{x},t)T + iT[\partial_\mu\phi^\dagger](\vec{x},t)TT\phi(\vec{x},t)T\\
&= -i\eta_T\phi^\dagger(\vec{x},-t)(-\eta_T\eta^{\mu\mu})[\partial_\mu\phi](\vec{x},-t) + i(-\eta_T\eta^{\mu\mu})[\partial_\mu\phi^\dagger](\vec{x},-t)\eta_T\phi(\vec{x},-t),\\
&= \eta^{\mu\mu}J_\mu(\vec{x},-t),
\end{aligned}$$

Where in the last step it has been used the property $\eta_T^2 = 1$.
This result, combined with the transformation properties of the field A^μ (Exercise 3), causes the term $A^\mu J_\mu$, appearing in the scalar QED Lagrangian, to transform as $TA^\mu J_\mu T = -\eta_T A^\mu J_\mu$ so that, by choosing $\eta_T = -1$, it is possible to make this coupling time reversal-invariant.

Exercise 4

Given the transformation properties of a Dirac fermion ψ under parity and charge conjugation ($P = P^\dagger$, $C = C^\dagger$), namely

$$P\psi(t,\vec{x})P = \eta_P\gamma^0\psi(t,-\vec{x}),$$
$$C\psi(t,\vec{x})C = -i\eta_C\gamma^2\psi^*(t,\vec{x}),$$

we want to compute the transformation properties of all the bilinears of the form $\bar{\psi}\Gamma\psi$, where Γ is some 4×4 matrix. In order to do this, it is sufficient to compute the transformation properties for

$$\Gamma = \{1_4, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \gamma^{\mu\nu}\},$$

since we have proved that any 4×4 matrix can be decomposed into a linear combination of these quantities.
Before proceeding further, it is useful to work out some properties of the gamma matrices: in particular we want

to find a close form for $(\gamma^\mu)^\dagger$ and $(\gamma^\mu)^T$. Let's start from the latter. Recalling the expression for the gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

we can immediately see that the only hermitian matrix is γ^0 while the other ones are anti-hermitian. This suggests the following formula:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

Indeed, making use of the Clifford algebra of the gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we get

$$\gamma^0\gamma^\mu\gamma^0 = \begin{cases} (\gamma^0)^3 = \gamma^0, & \mu = 0, \\ \gamma^0\gamma^i\gamma^0 = -\gamma^i(\gamma^0)^2 = -\gamma^i, & \mu = i = 1,2,3, \end{cases}$$

which is $= (\gamma^\mu)^\dagger$.
Similarly one can guess a formula for $(\gamma^\mu)^T$: γ^0, γ^2 are symmetric while γ^1, γ^3 are antisymmetric. Hence

$$\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0 = \begin{cases} \gamma^0\gamma^2\gamma^0\gamma^2\gamma^0 = -(\gamma^2)^2\gamma^0 = \gamma^0 & \mu = 0, \\ \gamma^0(\gamma^2)^3\gamma^0 = -\gamma^0\gamma^2\gamma^0 = \gamma^2 & \mu = 2, \\ \gamma^0\gamma^2\gamma^i\gamma^2\gamma^0 = \gamma^0\gamma^2\gamma^2\gamma^0\gamma^i = -\gamma^i & \mu = i = 1,3, \end{cases}$$

which is $= (\gamma^\mu)^T$.
Let's manipulate a bit the transformation under C:

$$C\psi C = -i\eta_C\gamma^2\psi^* = -i\eta_C\gamma^2(\psi^\dagger)^T = -i\eta_C(\psi^\dagger\gamma^2)^T = -i\eta_C(\bar{\psi}\gamma^0\gamma^2)^T.$$

Similarly:

$$C\bar{\psi}C = C\psi^\dagger C\gamma^0 = -i\eta_C^*\psi^T\gamma^2\gamma^0 = -i\eta_C^*(\gamma^0\gamma^2\psi)^T.$$

Now we are able to compute the transformation properties of the bilinears:

$$P\bar{\psi}(t,\vec{x})\Gamma\psi(t,\vec{x})P = \eta_P^2\bar{\psi}(t,-\vec{x})\gamma^0\Gamma\gamma^0\psi(t,-\vec{x}) \equiv \bar{\psi}\gamma^0\Gamma\gamma^0\psi.$$

From now on we won't write the argument of the field and it will be understood that the parity transformation changes the sign of the spatial coordinates. Charge conjugation instead acts as follows:

$$C\bar{\psi}\Gamma\psi C = -|\eta_C|^2(\gamma^0\gamma^2\psi)^T\Gamma(\bar{\psi}\gamma^0\gamma^2)^T.$$

A bilinear $\bar{\psi}\Gamma\psi$ is a number, since all the spinorial indices are contracted. This means that it is equal to its transpose, however we must pay attention to ordering, since fermions anticommute:

$$\begin{aligned} C\bar{\psi}\Gamma\psi C &= -(\gamma^0\gamma^2)_{im}\psi_m\Gamma_{ij}\bar{\psi}_n(\gamma^0\gamma^2)_{nj} = (\gamma^0\gamma^2)_{im}\bar{\psi}_n\psi_m\Gamma_{ij}(\gamma^0\gamma^2)_{nj} \\ &= \bar{\psi}_n(\gamma^0\gamma^2)_{nj}\Gamma_{ij}(\gamma^0\gamma^2)_{im}\psi_m = \bar{\psi}(\gamma^0\gamma^2)\Gamma^T(\gamma^0\gamma^2)\psi = -\bar{\psi}\gamma^2\gamma^0\Gamma^T\gamma^0\gamma^2\psi \end{aligned}$$

Everything is now reduced to understanding what $\gamma^0\Gamma\gamma^0$ and $-\gamma^2\gamma^0\Gamma^T\gamma^0\gamma^2$ are. Let's see it case by case:

- Let's start from the simplest case: $\Gamma = 1_4$. Then:

  $$\gamma^0\gamma^0 = 1\,, \qquad -\gamma^2\gamma^0\gamma^0\gamma^2 = 1.$$

  This means that:

  $$P\bar{\psi}\psi P = \bar{\psi}\psi, \qquad C\bar{\psi}\psi C = \bar{\psi}\psi.$$

  Due to its transformation under parity, this object is called a *scalar*.

- Let us consider now $\Gamma = \gamma^5$. Hence

  $$\gamma^0\gamma^5\gamma^0 = -\gamma^5\,, \qquad -\gamma^2\gamma^0\gamma^5\gamma^0\gamma^2 = \gamma^5.$$

  This means that:

  $$P\bar{\psi}\gamma^5\psi P = -\bar{\psi}\gamma^5\psi, \qquad C\bar{\psi}\gamma^5\psi C = \bar{\psi}\gamma^5\psi.$$

  Due to its transformation under parity, this object is called a *pseudo-scalar*.

- The next one is $\Gamma = \gamma^\mu$:

$$\begin{aligned}
&\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger = \eta^{\mu\mu}\gamma^\mu, \\
&-\gamma^2\gamma^0(\gamma^\mu)^T\gamma^0\gamma^2 = -\gamma^2\gamma^0\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0\gamma^0\gamma^2 = -\gamma^\mu.
\end{aligned}$$

  This means that

$$P\bar{\psi}\gamma^\mu\psi P = \eta^{\mu\mu}\bar{\psi}\gamma^\mu\psi, \qquad C\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi$$

  Due to its transformation under parity, this object is called a *vector*.

- The following term is $\Gamma = \gamma^\mu\gamma^5$:

$$\begin{aligned}
&\gamma^0\gamma^\mu\gamma^5\gamma^0 = -\eta^{\mu\mu}\gamma^\mu\gamma^5, \\
&-\gamma^2\gamma^0(\gamma^\mu\gamma^5)^T\gamma^0\gamma^2 = -\gamma^2\gamma^0\gamma^5(\gamma^\mu)^T\gamma^0\gamma^2 \\
&\qquad\qquad = -\gamma^5\gamma^2\gamma^0(\gamma^\mu)^T\gamma^0\gamma^2 = -\gamma^5\gamma^\mu = \gamma^\mu\gamma^5.
\end{aligned}$$

  This means that

$$P\bar{\psi}\gamma^\mu\gamma^5\psi P = -\eta^{\mu\mu}\bar{\psi}\gamma^\mu\gamma^5\psi, \qquad C\bar{\psi}\gamma^\mu\gamma^5\psi C = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

  Due to its transformation under parity, this object is called a *pseudo-vector*.

- The last term is $\Gamma = \gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$:

$$\begin{aligned}
&\gamma^0\gamma^{\mu\nu}\gamma^0 = \eta^{\mu\mu}\eta^{\nu\nu}\gamma^{\mu\nu}, \\
&-\gamma^2\gamma^0(\gamma^{\mu\nu})^T\gamma^0\gamma^2 = -\frac{1}{2}\gamma^2\gamma^0[\gamma^{\nu\,T},\, \gamma^{\mu\,T}]\gamma^0\gamma^2.
\end{aligned}$$

  Notice that

$$[\gamma^{\nu\,T},\, \gamma^{\mu\,T}] = \gamma^0\gamma^2\gamma^\nu\gamma^2\gamma^0\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0 - (\mu \leftrightarrow \nu) = -\gamma^0\gamma^2[\gamma^\nu,\, \gamma^\mu]\gamma^2\gamma^0 = \gamma^0\gamma^2[\gamma^\mu,\, \gamma^\nu]\gamma^2\gamma^0,$$

  so that

$$-\gamma^2\gamma^0(\gamma^{\mu\nu})^T\gamma^0\gamma^2 = -\gamma^{\mu\nu}.$$

  This means that

$$P\bar{\psi}\gamma^{\mu\nu}\psi P = \eta^{\mu\mu}\eta^{\nu\nu}\bar{\psi}\gamma^{\mu\nu}\psi, \qquad C\bar{\psi}\gamma^{\mu\nu}\psi C = -\bar{\psi}\gamma^{\mu\nu}\psi.$$

  Due to its transformation under parity, this object is called a tensor.

Let us summarize. We will include also the transformation properties of ∂_μ: this is because we will make use of the CPT invariance to infer the transformation of the above bilinears under time reversal T. However the CPT theorem applies only to Lorentz invariant operator, therefore, when needed, we must contract with the derivative. All transformation properties are summarized in the table. One can verify that all the Lorentz invariant operators that can be constructed satisfy $CPT = 1$.

Table 1: Summary of bilinear transformations.

	∂_μ	$\bar{\psi}\psi$	$\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\gamma^{\mu\nu}\psi$
P	$\eta^{\mu\mu}$	1	-1	$\eta^{\mu\mu}$	$-\eta^{\mu\mu}$	$\eta^{\mu\mu}\eta^{\nu\nu}$
C	1	1	1	-1	1	-1
T	$-\eta^{\mu\mu}$	1	-1	$\eta^{\mu\mu}$	$\eta^{\mu\mu}$	$-\eta^{\mu\mu}\eta^{\nu\nu}$

Note that the compact notation regarding the transformation properties of the derivative in the table actually means that $P\partial_\mu SP = \eta^{\mu\mu}\partial_\mu S$ (and similarly for C and T), where S is a scalar made up with fields: indeed P, C, and T act non trivially only on fields.