Quantum Field Theory

Set 2: solutions

Exercise 1

Let us consider a Dark Matter (DM) particle moving with energy $\tilde{E}$ in the reference frame in which the nucleus is at rest. When the scattering is elastic the minimum energy that the nucleus can have after the scattering corresponds to its mass: the DM particles keeps all the kinetic energy. In practice the scattering is equivalent to a process in which the two particle don't interact and the DM simply goes straight (but it can happen even if the DM particle and the nucleus do interact). Continuity processes in which the recoil energy of the nucleus is very small compared to its mass are equally allowed.

Let us now consider a model of inelastic DM. This time the scattering is inelastic: the final products are the same initial nucleus but a different DM particle that we call Y. For instance we can think about the DM as a system with two discrete energy levels, one at energy M_X, the second at energy $M_Y = M_X + \delta$, with $\delta \ll M_X, M_N$. The scattering of X with the nucleus induces a level transition and produces the state Y. We want to compute the minimum energy required in order for this process to happen. The key observation is that when the energy is minimum the particles are produced at rest in the center of mass. This is intuitive because if the particles move in the center of mass they have some additional kinetic energy that must be provided by some of the incoming particles. We can therefore impose the four momentum conservation for this configuration. Notice that if two particles move with the same velocity in one reference frame (they are at rest in the center of mass) they move with the same velocity in all the reference frames. Let us call $\tilde{P}_X$ and $\tilde{P}_N$ the four momenta in the laboratory before the scattering, $P_Y\,, P_N$ the four momenta in the center of mass after the scattering and $\tilde{P}_Y$, $\tilde{P}'_N$ the four momenta in the laboratory after the scattering.

$$
\begin{aligned}
&\tilde{P}_{X\mu} = (\tilde{E}_{\rm min}, 0, 0, \tilde{p}_{\rm min}) \qquad \tilde{P}_{N\mu} = (M_N, 0, 0, 0)\\
&P_{Y\mu} = (M_Y, 0, 0, 0) \qquad P'_{N\mu} = (M_N, 0, 0, 0)\\
&\tilde{P}_{Y\mu} = (\tilde{E}_Y, 0, 0, \tilde{p}_Y) \qquad \tilde{P}'_{N\mu} = (\tilde{E}_Y, 0, 0, \tilde{p}_Y)
\end{aligned}
$$

Then we have:

$$
(P_{Y\mu} + P'_{N\mu})^2 = (\tilde{P}_{Y\mu} + \tilde{P}'_{N\mu})^2 = (\tilde{P}_{X\mu} + \tilde{P}_{N\mu})^2
$$

where the second equality is the momentum conservation before and after the scattering while the first equality is due to the fact that the value of the square of a four momentum is independent of the the reference frame in which it is computed. Expanding the first and last parenthesis we get

$$
P_Y^2 + P_N^{'2} + 2P_Y P'_N = \tilde{P}_X^2 + \tilde{P}_N^2 + 2\tilde{P}_X\tilde{P}_N \quad \Rightarrow \quad (M_X + \delta + M_N)^2 = M_X^2 + M_N^2 + 2M_N\tilde{E}_{\rm min}
$$

Expanding at linear order in δ (here we assume $\delta \ll M_X,\, M_N$) we have

$$
\tilde{E}_{\rm min} = M_X + \delta\left(1 + \frac{M_X}{M_N}\right) + O(\delta^2)
$$

Let us assume that a DM particle with energy exactly $\tilde{E}_{\rm min}$ strikes a nucleus. We can easily compute the energy of the nucleus in the laboratory frame using the fact that the nuclus is at rest in the center of mass. Hence we can simply write:

$$
\tilde{E}_N = \gamma(v)(E_N + \beta p_N) = \gamma(v)E_N = \gamma(v)M_N
$$

where v is the velocity that connects the center of mass frame and the laboratory frame. If one defines the total four momentum in the laboratory frame as

$$
\tilde{P}_{\rm tot} = \tilde{P}_X + \tilde{P}_N
$$

then the velocity of the center of mass in the laboratory frame is

$$
\vec{v} = \frac{\tilde{\vec{p}}_{\rm tot}}{\tilde{E}_{\rm tot}} = \frac{\tilde{\vec{p}}_X}{\tilde{E}_X + M_N} = \frac{\sqrt{\tilde{E}^2_{\rm min} - M_X^2}}{\tilde{E}_{\rm min} + M_N} \quad \Rightarrow \quad v^2 = \frac{2\delta}{M_X}\left(1 + \frac{M_X}{M_N}\right)\left(1 + \frac{M_N}{M_X}\right)^{-2} + O(\delta^2)
$$

Since v^2 is of order δ we can expand also the boost factor $\gamma(v) = \sqrt{\frac{1}{1-v^2}} \simeq 1 + \frac{1}{2}v^2$. Finally

$$\tilde{E}_N = M_N + \delta \frac{M_X}{M_X + M_N}.$$

The main difference between elastic and inelastic dark matter is that in the latter case the scattering processes are not allowed for all the energies of the incoming DM particle. Moreover, if the energy of the incoming particle is the minimum, the recoil energy of the nucleus is not infinitesimal but is finite. In order to have a *smaller* recoil energy in the laboratory frame, one needs the DM particle coming with higher energies. Since the velocity distribution of the DM particles is assumed to be a Maxwell-Boltzman distribution, the probability to have an incoming particle with energy higher than $\tilde{E}_{\min}$ is exponentially suppressed and therefore the most probable event (if any) is that with finite nucleus recoil energy.

Let us now compute the value of δ that could explain the absence of observations by CDMS but that allows for some process at DAMA. We know that the scattering takes place if

$$\tilde{E} = M_X \left(1 + \frac{\beta^2}{2}\right) \gtrsim \tilde{E}_{\min} \equiv M_X + \delta \left(1 + \frac{M_X}{M_N}\right) \quad \Longrightarrow \quad \delta \lesssim \frac{\beta^2}{2} \frac{M_X M_N}{M_X + M_N},$$

where we have used the non-relativistic approximation $\tilde{E}_{\min} - M_X \sim \frac{1}{2}\beta^2$. Plugging in the values of $\beta \sim 220$ Km/s and the atomic number of Germanium (73) and Iodine (127), we get that the process can take place if

$$\begin{aligned} \delta &\lesssim 11 \text{ KeV} \qquad \text{at CDMS}, \\ \delta &\lesssim 15 \text{ KeV} \qquad \text{at DAMA}. \end{aligned}$$

If β were exactly 220 km/s It would exist a window, 11 keV $\leq \delta \leq$ 15 keV, such that no process could be observed by CDMS but some process could be observed by DAMA. The fact that the velocity distribution of the DM particles is not a delta function modifies a bit the above discussion but still an interval exists.

Exercise 2

We call $y_i(t)$ the displacement of the i-th mass from the x axis in the $\hat{y}$ direction. The potential of the system is given by the sum of the elastic potentials of each spring. The length of the spring stretching between the i-th and the $i+1$-th mass is given by

$$L_{i,i+1} = \sqrt{(y_i(t) - y_{i+1}(t))^2 + a^2}$$

therefore the total potential is

$$V_{tot} = \sum_i \frac{1}{2} k L_{i,i+1}^2 = \frac{k}{2} \sum_i \left((y_i(t) - y_{i+1}(t))^2 + a^2\right)$$

Notice that a given y_i enters two times in the potential, since each mass is attached to two springs. In order to visualize this one can write some terms explicitly. For instance:

$$V_{tot} = ... + \frac{k}{2}\left((y_1(t) - y_2(t))^2 + a^2\right) + \frac{k}{2}\left((y_2(t) - y_3(t))^2 + a^2\right) + \frac{k}{2}\left((y_3(t) - y_4(t))^2 + a^2\right) + ...$$

and therefore all the times we derive the potential with respect to one coordinate two pieces will appear (indeed there are two strengths acting to each mass). The Lagrangian of the system is

$$L = T - V = \sum_i \frac{1}{2} m \dot{y}_i^2 - \frac{k}{2} \sum_i \left((y_i(t) - y_{i+1}(t))^2 + a^2\right)$$

The equation of motion are obtained as usual from

$$\begin{aligned} &\partial_t \left(\frac{\partial L}{\partial \dot{y}_i}\right) = \frac{\partial L}{\partial y_i} \\ \Rightarrow \quad &\partial_t(m\dot{y}_i(t)) = m\ddot{y}_i(t) = -k(y_i(t) - y_{i+1}(t)) + k(y_{i-1}(t) - y_i(t)) \end{aligned}$$

We can also describe the system using the Hamiltonian formalism. Let us define the conjugate momentum of the i-th variable

$$p_i = \frac{\partial L}{\partial \dot{y}_i} = m\dot{y}_i$$

Then the Hamiltonian reads

$$\begin{aligned} H = \sum p_i \dot{y}_i - L &= \sum_i \left(\frac{\dot{y}_i}{m} \dot{y}_i - \frac{m}{2} \left(\frac{\dot{y}_i}{m} \right)^2 \right) + V \\ &= \sum_i \frac{p_i^2}{2m} + \frac{k}{2} \sum_i \left((y_i(t) - y_{i+1}(t))^2 + a^2 \right) \end{aligned}$$

In the Hamiltonian formalism the equations of motion are of the first order and read

$$\begin{aligned} \dot{y}_i &= \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \\ \dot{p}_i &= -\frac{\partial H}{\partial y_i} = -k(y_i - y_{i+1}) + k(y_{i-1} - y_i). \end{aligned}$$

The equivalence of the two formalisms (Lagrangian and Hamiltonian) is manifest once we differentiate the first equation with respect to time and plug it into the second one: we recover exactly the Lagrangian equation of motion.

The continuum limit $a \to 0$ corresponds to considering the oscillators closer and closer together so that the position of the $i-$th oscillator (which would be $i \cdot a$) can be labelled by a continuos variable x. Instead of having a discrete set of functions of time one has a function of two variables, space and time:

$$y_i(t) \longrightarrow y(x,t)$$

The Lagrangian becomes

$$\lim_{a\to 0} L = \lim_{a \to 0} a \sum_i \frac{1}{2} \left(\frac{m}{a} \dot{y}(x,t)^2 - ka \left(\frac{y(x,t) - y(x+a)}{a} \right)^2 \right)$$

In the $a \to 0$ limit, the term appearing in the parenthesis represents the derivative with respect to the x variable while the sum translates in an integral over x:

$$\begin{aligned} \lim_{a\to 0} \frac{y(x,t) - y(x+a,t)}{a} &= -\frac{\partial y(x,t)}{\partial x} \\ \lim_{a \to 0} a \sum_i &= \int dx \end{aligned}$$

Finally the Lagrangian reads

$$\int dx \underbrace{\frac{1}{2} \left(\mu \left(\frac{\partial y(x,t)}{\partial t} \right)^2 - Y \left(\frac{\partial y(x,t)}{\partial x} \right)^2 \right)}_{\mathcal{L}(x,t)}$$

In the above expression the function $y(x,t)$ is called *field*. In Field Theory the quantity $\mathcal{L}(x,t)$ is called *Lagrangian density* and is a function of space and time. The integral on space of the Lagrangian density is the *Lagrangian* $L(t)$ while the integral on time as well defines the *Action*:

$$S = \int dt L(t) = \int dt\, dx \mathcal{L}(x,t)$$

The equation of motion that determines the dynamic of the field $y(x,t)$ is given by the Euler Lagrange equations:

$$\sum_I \partial_I \left(\frac{\partial \mathcal{L}}{\partial(\partial_I y)} \right) = \frac{\partial \mathcal{L}}{\partial y}$$

where ∂_I stands for derivative with respect to t or x. Therefore:

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t y)} \right) + \partial_x \left(\frac{\partial \mathcal{L}}{(\partial_x y)} \right) = \frac{\partial \mathcal{L}}{\partial y}$$

In order to compute $\partial_I\left(\frac{\partial\mathcal{L}}{\partial(\partial_I y)}\right)$ one has to think of $\mathcal{L}$ as a function of the variable $\partial_I y$ and derive as usual. Hence.

$$\partial_t(\mu\partial_t y(x,t)) - \partial_x(Y\partial_x y(x,t)) = 0 \;\;\Rightarrow\;\; \partial_t^2 y(x,t) = \frac{Y}{\mu}\partial_x^2 y(x,t)$$

One can compare with the direct $a\to 0$ limit in the equation of motion derived previously:

$$\ddot{y}_i(t) \to \partial_t^2 y(x,t) = \lim_{a\to 0} -\frac{ka^2}{m}\frac{\frac{y(x,t)-y(x+a,t)}{a} - \frac{y(x-a,t)-y(x)}{a}}{a} = \lim_{a\to 0}(-)\frac{Y}{\mu}\frac{-\partial_x y(x,t)+\partial_x y(x-a)}{a} = \frac{Y}{\mu}\partial_x^2 y(x,t)$$

The result is the same as that obtained from the Euler Lagrange equation for the field $y(x,t)$. In order to find a solution of the above *wave equation* let us define two set of variables

$$\xi = vt + x\,, \qquad \eta = vt - x\,, \qquad v = \sqrt{\frac{Y}{\mu}}$$

$$x = \frac{\xi-\eta}{2}\,, \quad t = \frac{\eta+\xi}{2v}$$

Then the derivatives with respect to x and t become

$$\frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta} = \frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}$$
$$\frac{\partial}{\partial t} = \frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta} = v\frac{\partial}{\partial\xi} + v\frac{\partial}{\partial\eta}$$

The equation of motion then becomes

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)y(x,t) = 4\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}y(\xi,\eta) = 0$$

The general solution of the above equation has the following form:

$$y(x,t) = f(\xi) + g(\eta) = f(x+vt) + g(vt-x)$$

where f and g are two arbitrary functions. The above solution corresponds to a superposition of two waves, one moving in the right direction and one moving in the left direction.

Exercise 3

The magnetic moment produced by a particle with charge e and velocity v is defined as:

$$\vec{\mu} = \frac{1}{2}e\,\vec{r}\wedge\vec{v},$$

where $\vec{r}$ is the spatial position. If we have a set of many particles we can sum the single contributions since electromagnetism is a linear theory and the superposition principle holds:

$$\vec{\mu} = \frac{1}{2}\sum_{i=1}^{N} e_i\vec{r}_i\wedge\vec{v}_i.$$

In the hypothesis of equal charge-mass ratios one gets

$$\vec{\mu} = \frac{1}{2}\sum_{i=1}^{N}\frac{e_i}{m_i}m_i\vec{r}_i\wedge\vec{v}_i = \frac{e}{2m}\sum_{i=1}^{N}m_i\vec{r}_i\wedge\vec{v}_i = \frac{e}{2m}\vec{L}.$$

The magnetic moment of the system is proportional to its angular momentum. In general one can rewrite

$$\vec{\mu} = g\frac{e}{2m}\vec{L},$$

and we have just seen that classically $g=1$. One can try to make a simple model of a proton considering it as made by a distribution of matter and charge. However if we try to apply the previous formula we get into trouble since the measured constant g is approximately 2 (but not exactly). Therefore the classical argument cannot apply. In the following we will see that a relativistic theory of particles predicts exactly $g=2$. However one needs quantum effects to compute deviations from 2 and compare with experiments. The agreement is amazing.