Quantum Field Theory

Set 18: solutions

Exercise 1

Let us consider the Lagrangian of a massive vector field:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu,$$

from which we can compute the conjugate momentum of the field A_μ:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu}.$$

Π^0 is vanishing, which is a consequence of the fact that the field A_0 is not a dynamical variable. The only non trivial quantities are then:

$$\Pi^i = -\partial^0 A^i + \partial^i A^0.$$

The equations of motion following from the above Lagrangian can be divided into a set of three dynamical equations:

$$0 = \partial_\mu F^{\mu j} + M^2 A^j = \partial_0 F^{0j} + \partial_i F^{ij} + M^2 A^j = \ddot{A}^j - \partial^j \dot{A}_0 + \partial_i F^{ij} + M^2 A^j,$$

and constraint:

$$0 = \partial_\mu F^{\mu 0} + M^2 A^0 \implies A_0 = -\frac{1}{M^2} \partial_i \Pi^i,$$

which lets us express the non-dynamical variable as a function of the momenta. The Hamiltonian reads:

$$H = \int d^3x \left(\Pi^\mu \dot{A}_\mu - \mathcal{L} \right) = \int d^3x \left(\Pi^i \dot{A}_i - \mathcal{L} \right) = \int d^3x \left(-\Pi^i \Pi_i + \Pi^i \partial_i A_0 - \mathcal{L} \right),$$

where in the last step we have used the definition of Π^i in terms of $\dot{A}^i$. Expanding and eliminating A_0, we get

$$\begin{aligned} H &= \int d^3x \left(-\Pi^i \Pi_i - \frac{1}{M^2} \Pi^i \partial_i (\partial_j \Pi^j) - \mathcal{L} \right) = \int d^3x \left(-\Pi^i \Pi_i - \frac{1}{M^2} \Pi^i \partial_i (\partial_j \Pi^j) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} M^2 A_\mu A^\mu \right) \\ &= \int d^3x \left(\Pi^i \Pi^i + \frac{1}{M^2} (\partial_i \Pi^i)(\partial_j \Pi^j) + \frac{1}{2} F_{0j} F^{0j} + \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} M^2 A_0^2 - \frac{1}{2} M^2 A_i A^i \right) \\ &= \int d^3x \, \frac{1}{2} \left(\Pi^i \Pi^i + \frac{1}{M^2} (\partial_i \Pi^i)^2 + \frac{1}{2} F^{ij} F^{ij} + M^2 A^i A^i \right). \end{aligned}$$

In order to quantize this theory we need to impose a set of canonical commutation relations. Here a subtlety arises since, because of the constraint relating A_0 and the Π^i, we cannot impose a vanishing commutation relation between all the A_μ. A correct set of commutation relation is instead:

$$\begin{aligned} &\left[A_i(\vec{x}, t),\, \Pi^j(\vec{y}, t) \right] = i \delta_i^j \delta^3(\vec{x} - \vec{y}), \\ &\left[A_i(\vec{x}, t),\, A_j(\vec{y}, t) \right] = \left[\Pi^i(\vec{x}, t),\, \Pi^j(\vec{y}, t) \right] = \left[A_0(\vec{x}, t),\, \Pi^j(\vec{y}, t) \right] = 0, \\ &\left[A_i(\vec{x}, t),\, A_0(\vec{y}, t) \right] = \left[A_i(\vec{x}, t),\, -\frac{1}{M^2} \partial_m \Pi^m(\vec{y}, t) \right] = \frac{i}{M^2} \partial_i^{(x)} \delta^3(\vec{x} - \vec{y}). \end{aligned}$$

We can check the consistency of the above commutation relations considering the commutator of the field and the momenta with the Hamiltonian. Notice that the commutation relations are defined at equal time but since H is independent of time we can compute it at any time:

$$\begin{aligned} \left[H,\, \Pi^j(\vec{x}, t) \right] \equiv -i \dot{\Pi}^j(\vec{x}, t) &= \int d^3y \left(\frac{1}{2} F^{mn} \left[F_{mn}(\vec{y}, t),\, \Pi^j(\vec{x}, t) \right] - M^2 A^m \left[A_m(\vec{y}, t),\, \Pi^j(\vec{x}, t) \right] \right) \\ &= \int d^3y \left(F^{mn} \left[\partial_m^{(y)} A_n(\vec{y}, t),\, \Pi^j(\vec{x}, t) \right] - M^2 A^m \left[A_m(\vec{y}, t),\, \Pi^j(\vec{x}, t) \right] \right) \\ &= i \int d^3y \left(F^{mj} \partial_m^{(y)} \delta^3(\vec{x} - \vec{y}) - M^2 A^j \delta^3(\vec{x} - \vec{y}) \right) = -i \partial_m \, F^{mj}(\vec{x}, t) - i M^2 A^j(\vec{x}, t), \end{aligned}$$

$$[H,\, A_j(\vec{x},t)] \equiv -i\dot{A}_j(\vec{x},t) = \int d^3y \left(-\Pi_i \left[\Pi^i(\vec{y},t),\, A_j(\vec{x},t)\right] + \frac{1}{M^2}(\partial_m^{(y)}\Pi^m) \left[(\partial_n^{(y)}\Pi^n),\, A_j(\vec{x},t)\right] \right)$$
$$= \int d^3y \left(i\Pi_j(\vec{y},t)\delta^3(\vec{x}-\vec{y}) + i\frac{1}{M^2}(\partial_m\Pi^m)\partial_j^{(x)}\delta^3(\vec{x}-\vec{y}) \right) = i\Pi_j(\vec{x},t) + \frac{i}{M^2}\partial_j(\partial_m\Pi^m)(\vec{x},t).$$

Finally taking the time derivative of the second equation and using the first we get:

$$\ddot{A}^j = -\dot{\Pi}^j - \frac{1}{M^2}\partial_0\partial^j(\partial_m\Pi^m) = -\dot{\Pi}^j + \partial_0\partial^j A_0 = -\partial_m\, F^{mj} - M^2A^j + \partial_0\partial^j A_0$$
$$\Longrightarrow\; \partial_0\partial^0 A^j - \partial_0\partial^j A^0 + \partial_m\, F^{mj} + M^2A^j = 0.$$

Exercise 2

We want to compute the Noether charge associated to rotations for a massive vector field. Lorentz transformations act as usual:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \simeq x^\mu + w^\mu{}_\nu x^\nu = x^\mu - \frac{1}{2}\epsilon^\mu_{\alpha\beta}w^{\alpha\beta} \;\Longrightarrow\; \epsilon^\mu_{\alpha\beta} = -\left(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha\right),$$
$$A'_\rho(x') = \Lambda_\rho{}^\nu A_\nu(x) = \Lambda_\rho{}^\nu A_\nu(\Lambda^{-1}x') \simeq \left(\delta_\rho^\nu + w_\rho{}^\nu\right) A_\nu(x'^\mu - w^\mu{}_\sigma x'^\sigma) \simeq A_\rho(x') + w_\rho{}^\nu A_\nu(x') - w^{\mu\nu}x'_\nu\partial'_\mu A_\rho(x').$$

Dropping the primes on x' we have:

$$A'_\rho(x) - A_\rho(x) = \frac{1}{2}w^{\alpha\beta}\Delta_{\rho,\alpha\beta} \;\Longrightarrow\; \Delta_{\rho,\alpha\beta} = (\eta_{\rho\alpha}A_\beta - \eta_{\rho\beta}A_\alpha) + (x_\alpha\partial_\beta - x_\beta\partial_\alpha)\, A_\rho.$$

Notice that we have defined $\epsilon^\mu_{\alpha\beta}$ and $\Delta_{\rho,\alpha\beta}$ without the factor $1/2$. It is clear that all the definitions are equivalent, as long as they are all consistent, however this is the choice that provides the correct normalization of the generators of rotations: $[J_i, J_j] = i\epsilon_{ijk}J_k$. The Noether current is then:

$$J^\mu_{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\Delta_{\rho,\alpha\beta} - \epsilon^\mu_{\alpha\beta}\mathcal{L} = -F^{\mu\rho}\Delta_{\rho,\alpha\beta} + \left(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha\right)\mathcal{L}$$
$$= -F^{\mu\rho}\left(\eta_{\rho\alpha}A_\beta + x_\alpha\partial_\beta A_\rho\right) + F^{\mu\rho}\left(\eta_{\rho\beta}A_\alpha + x_\beta\partial_\alpha A_\rho\right) + \left(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha\right)\mathcal{L}.$$

We can now focus on the current associated to rotations: $(\alpha\beta) = (ij)$. In addition we take $\mu = 0$ in order to obtain the charge:

$$Q_{ij} \equiv \int d^3x\, J^0_{ij} = -\left(\int d^3x\left(F^0{}_iA_j + F^{0m}x_i\partial_jA_m\right) - (i\leftrightarrow j)\right) = \int d^3x\left(-F_{0i}A_j + F_{0m}x_i\partial_jA_m\right) - (i\leftrightarrow j),$$

where we have lowered all the indices. We finally define the three generators of rotations as:

$$J_k = \frac{1}{2}\epsilon_{ijk}Q_{ij} = \epsilon_{ijk}\int d^3x\left(-F_{0i}A_j + F_{0m}x_i\partial_jA_m\right).$$

From now on we will keep all indices down. Recalling the definition of the conjugate momenta ($\Pi^i = -F^{0i}$) we can also rewrite:

$$J_k = \epsilon_{ijk}\int d^3x\left(\Pi_iA_j - \Pi_m\, x_i\partial_jA_m\right).$$

One can check that this definition is correctly normalized. We now proceed and expand the above expressions in terms of raising and lowering operators. Recall the definitions:

$$\Pi_i(x) = \int\frac{d^3k}{(2\pi)^3}\; e^{ikx}\Pi_i(\vec{k},t) \qquad A_i(x) = \int\frac{d^3k}{(2\pi)^3}\; e^{ikx}A_i(\vec{k},t),$$
$$\Pi_i(\vec{k}) = \frac{i}{2}\left[\left(a_\perp(\vec{k}) - a_\perp^\dagger(-\vec{k})\right)_i + \frac{M}{\omega_k}\left(a_L(\vec{k}) - a_L^\dagger(-\vec{k})\right)_i\right],$$
$$A_j(\vec{k}) = \frac{1}{2\omega_k}\left[\left(a_\perp(\vec{k}) + a_\perp^\dagger(-\vec{k})\right)_j + \frac{\omega_k}{M}\left(a_L(\vec{k}) + a_L^\dagger(-\vec{k})\right)_j\right],$$

where $\omega_k = \sqrt{k^2 + M^2}$ and we have decomposed the operator in transverse and longitudinal pieces:

$$a_{\perp\, i}(\vec{k}) = P^{\perp}_{ij} a_j(\vec{k}), \qquad a_{L\, i}(\vec{k}) = P^{L}_{ij} a_j(\vec{k}),$$
$$P^{\perp}_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}, \qquad P^L = 1 - P^{\perp}.$$

In the appendix we compute the expression of the angular momentum in terms of the operators $a(k)$ explicitly. The result has the simple form:

$$J_k = i\epsilon_{ijk} \int d\Omega_{\vec{k}} \left\{ a_i(\vec{k}) a^{\dagger}_j(\vec{k}) - a_m(\vec{k}) \left(k_i \frac{\partial}{\partial k^j} \right) a^{\dagger}_m(\vec{k}) \right\}.$$

The above expression is composed of two pieces, corresponding to the intrinsic spin carried by a state and the orbital angular momentum. One can look for eigenstates of the 3rd component J_3. In the reference frame in which the particle is at rest this must correspond to the spin of the state. Let us consider the states:

$$|3\rangle = a^{\dagger}_3(0)|0\rangle, \qquad |\pm\rangle = \left(\frac{a^{\dagger}_1(0) \pm i a^{\dagger}_2(0)}{\sqrt{2}} \right) |0\rangle.$$

Applying J_3 yields:

$$J_3|\pm\rangle = i \int d\Omega_{\vec{k}} \left\{ a_1(k) a^{\dagger}_2(k) - a_2(k) a^{\dagger}_1(k) \right\} \left(\frac{a^{\dagger}_1(0) \pm i a^{\dagger}_2(0)}{\sqrt{2}} \right) |0\rangle = \pm \left(\frac{a^{\dagger}_1(0) \pm i a^{\dagger}_2(0)}{\sqrt{2}} \right) |0\rangle = \pm|\pm\rangle,$$
$$J_3|0\rangle = 0.$$

Angular momentum in terms of creation operators

Let us start from the first piece:

$$\begin{aligned}
&\epsilon_{ijk} \int d^3x\, \Pi_i A_j \\
&= \epsilon_{ijk} \int d^3x \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3}\, \Pi_i(\vec{k}_1) A_j(\vec{k}_2) e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \\
&= \epsilon_{ijk} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3}\, \Pi_i(\vec{k}_1) A_j(\vec{k}_2) (2\pi)^3 \delta^3(\vec{k}_1+\vec{k}_2) = \epsilon_{ijk} \int \frac{d^3k}{(2\pi)^3}\, \Pi_i(-\vec{k}) A_j(\vec{k}) \\
&= \epsilon_{ijk} \int \frac{d^3k}{(2\pi)^3} \frac{i}{4w} \left[\left(a_{\perp}(-\vec{k}) - a^{\dagger}_{\perp}(\vec{k}) \right)_i \left(a_{\perp}(\vec{k}) + a^{\dagger}_{\perp}(-\vec{k}) \right)_j + \frac{M}{w} \left(a_L(-\vec{k}) - a^{\dagger}_L(\vec{k}) \right)_i \left(a_{\perp}(\vec{k}) + a^{\dagger}_{\perp}(-\vec{k}) \right)_j \right. \\
&\qquad \left. + \frac{w}{M} \left(a_{\perp}(-\vec{k}) - a^{\dagger}_{\perp}(\vec{k}) \right)_i \left(a_L(\vec{k}) + a^{\dagger}_L(-\vec{k}) \right)_j + \left(a_L(-\vec{k}) - a^{\dagger}_L(\vec{k}) \right)_i \left(a_L(\vec{k}) + a^{\dagger}_L(-\vec{k}) \right)_j \right].
\end{aligned}$$

Keeping on expanding we get:

$$\begin{aligned}
&\epsilon_{ijk} \int d^3x\, \Pi_i A_j \\
&= \epsilon_{ijk} \int \frac{d^3k}{(2\pi)^3} \frac{i}{4w} \left[\left(a_{\perp i}(-\vec{k}) a^{\dagger}_{\perp j}(-\vec{k}) - a^{\dagger}_{\perp i}(\vec{k}) a_{\perp j}(\vec{k}) \right) + \frac{M}{w} \left(a_{Li}(-\vec{k}) a^{\dagger}_{\perp j}(-\vec{k}) - a^{\dagger}_{Li}(\vec{k}) a_{\perp j}(\vec{k}) \right) \right. \\
&\qquad \left. + \frac{w}{M} \left(a_{\perp i}(-\vec{k}) a^{\dagger}_{Lj}(-\vec{k}) - a^{\dagger}_{\perp i}(\vec{k}) a_{Lj}(\vec{k}) \right) + \left(a_{Li}(-\vec{k}) a^{\dagger}_{Lj}(-\vec{k}) - a^{\dagger}_{Li}(\vec{k}) a_{Lj}(\vec{k}) \right) \right]
\end{aligned}$$

The additional terms we haven't written are in some case odd in k, such that $\epsilon_{ijk} a_{\perp i}(\vec{k}) a_{\perp j}(-\vec{k})$ or they will cancel out with other terms coming from other pieces. We don't need to keep track of them because all these terms consist of two a or two $a^{\dagger}$ and we know a priori that they must cancel. The final expression for the Noether charge must contain only terms linear in a and $a^{\dagger}$ (it cannot mix states with different particle content) because, since it is a conserved quantity, it commutes with the Hamiltonian, so it is diagonal in the basis of multiparticle

states $|\vec{k}_1, ..., \vec{k}_n\rangle$ where H is diagonal.
We can finally rewrite the above expression as:

$$\epsilon_{ijk} \int \frac{d^3k}{(2\pi)^3} \; \frac{i}{4w} \left[2a_{\perp i}(\vec{k})a^\dagger_{\perp j}(\vec{k}) + 2a_{Li}(\vec{k})a^\dagger_{Lj}(\vec{k}) + \left(\frac{M}{w} + \frac{w}{M}\right) \left(a_{Li}(\vec{k})a^\dagger_{\perp j}(\vec{k}) + a_{\perp i}(\vec{k})a^\dagger_{Lj}(\vec{k})\right)\right]. \quad (1)$$

The second piece is:

$$\begin{aligned}
& -\epsilon_{ijk} \int d^3x \; \Pi_m \, x_i \partial_j A_m \\
&= -\epsilon_{ijk} \int d^3x \int \frac{d^3k_1}{(2\pi)^3} \, \frac{d^3k_2}{(2\pi)^3} \Pi_m(\vec{k}_1) A_m(\vec{k}_2) e^{i\vec{k}_1\cdot\vec{x}} \left(x_i \partial_j e^{i\vec{k}_2\cdot\vec{x}}\right) \\
&= -\epsilon_{ijk} \int d^3x \int \frac{d^3k_1}{(2\pi)^3} \, \frac{d^3k_2}{(2\pi)^3} \Pi_m(\vec{k}_1) A_m(\vec{k}_2) e^{i\vec{k}_1\cdot\vec{x}} \left(k_{2j} \frac{\partial}{\partial k_2^i} e^{i\vec{k}_2\cdot\vec{x}}\right)
\end{aligned}$$

where we have used the usual trick:

$$\frac{\partial}{\partial x^j} e^{ik_m x_m} = -ik_j e^{ik_m x_m}, \qquad\qquad i\frac{\partial}{\partial k^i} e^{ik_m x_m} = x_i e^{ik_m x_m}.$$

Thus, integrating over d^3x we get:

$$-\epsilon_{ijk} \int d^3x \; \Pi_m \, x_i \partial_j A_m = -\int \frac{d^3k_1}{(2\pi)^3} \, d^3k_2 \, \Pi_m(\vec{k}_1) A_m(\vec{k}_2) \epsilon_{ijk} k_{2j} \frac{\partial}{\partial k_2^i} \delta^3(\vec{k}_1 + \vec{k}_2).$$

We can integrate by parts and get:

$$\begin{aligned}
-\epsilon_{ijk} \int d^3x \; \Pi_m \, x_i \partial_j A_m &= \int \frac{d^3k_1}{(2\pi)^3} \, d^3k_2 \, \Pi_m(\vec{k}_1) \delta^3(\vec{k}_1 + \vec{k}_2) \left(\epsilon_{ijk} k_{2j} \frac{\partial}{\partial k_2^i}\right) A_m(\vec{k}_2) \\
= -\int \frac{d^3k}{(2\pi)^3} \; \Pi_m(\vec{k}) \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) A_m(-\vec{k}).
\end{aligned}$$

Notice that the minus in the last equality arises by changing the order of the indices i, j. This time Π and A are contracted in a scalar product (but there is a differential operator in the middle). Rewriting the expressions in terms of ladder operators will give four contributions which we consider one by one. The first one is:

$$\int \frac{d^3k}{(2\pi)^3} \frac{-i}{4w} \left(a_\perp(-\vec{k}) - a^\dagger_\perp(\vec{k})\right)_m \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) \left(a_\perp(\vec{k}) + a^\dagger_\perp(-\vec{k})\right)_m = \int \frac{d^3k}{(2\pi)^3} \; \frac{-i}{2w} a_{\perp m}(\vec{k}) \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) a^\dagger_{\perp m}(\vec{k}).$$

Recalling that $a_{\perp m}(\vec{k}) = P^\perp_{mr} a_r(\vec{k})$ we have:

$$= \int \frac{d^3k}{(2\pi)^3} \; \frac{-i}{2w} a_{\perp m}(\vec{k}) \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) a^\dagger_m(\vec{k}) + \int \frac{d^3k}{(2\pi)^3} \; \frac{-i}{2w} a_{\perp m}(\vec{k}) a_r(\vec{k})^\dagger \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) P^\perp_{mr}.$$

We need the following expression:

$$\left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) P^\perp_{mr} = \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) \left(\delta_{mr} - \frac{k_m k_r}{k^2}\right) = \epsilon_{imk} P^L_{ir} + \epsilon_{irk} P^L_{im},$$

where we must take care of the relation $\frac{\partial k_i}{\partial k^j} = -\delta_{ij}$. Hence, substituting this gives:

$$\int \frac{d^3k}{(2\pi)^3} \frac{-i}{2w} a_{\perp m}(\vec{k}) \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) a^\dagger_m(\vec{k}) + \int \frac{d^3k}{(2\pi)^3} \frac{-i}{2w} \epsilon_{imk} a_{\perp m}(\vec{k}) a^\dagger_{Li}(\vec{k}). \quad (2)$$

The second contribution is:

$$\int \frac{d^3k}{(2\pi)^3} \; \frac{-i}{4w} \frac{M}{w} \left(a_L(-\vec{k}) - a^\dagger_L(\vec{k})\right)_m \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) \left(a_\perp(\vec{k}) + a^\dagger_\perp(-\vec{k})\right)_m.$$

This time the piece where the derivative acts on a_m is absent since we would get a $P^\perp$ acting on a_L which gives zero. Hence we only have:

$$\begin{aligned}
&= \int \frac{d^3k}{(2\pi)^3} \; \frac{-i}{4w} \frac{M}{w} \left(a_L(-\vec{k}) - a^\dagger_L(\vec{k})\right)_m \left(a(\vec{k}) + a^\dagger(-\vec{k})\right)_r \left(\epsilon_{ijk} k_i \frac{\partial}{\partial k^j}\right) P^\perp_{mr} \\
&= \int \frac{d^3k}{(2\pi)^3} \; \frac{-i}{4w} \frac{M}{w} \left\{\epsilon_{imk} \left(a_L(-\vec{k}) - a^\dagger_L(\vec{k})\right)_m \left(a_L(\vec{k}) + a^\dagger_L(-\vec{k})\right)_i + \epsilon_{irk} \left(a_L(-\vec{k}) - a^\dagger_L(\vec{k})\right)_i \left(a(\vec{k}) + a^\dagger(-\vec{k})\right)_r\right\} \\
&= \int \frac{d^3k}{(2\pi)^3} \; \frac{-i}{4w} \frac{M}{w} \left\{\epsilon_{imk} \left(a_{Li}(\vec{k}) a^\dagger_{\perp m}(\vec{k}) + a_{\perp i}(\vec{k}) a^\dagger_{Lm}(\vec{k})\right)\right\}. \qquad (3)
\end{aligned}$$

The third contribution is:

$$\int \frac{d^3k}{(2\pi)^3}\ \frac{-i}{4w}\frac{w}{M}\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_m\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_m .$$

Also here we find the piece where the derivative acts on a_m to be absent since we would get a P^L acting on $a_\perp$ which is also zero. Hence the only contribution is

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3}\ \frac{-i}{4w}\frac{w}{M}\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_m\left(a(\vec{k})+a^\dagger(-\vec{k})\right)_r\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)P^L_{mr}\\
&=\int \frac{d^3k}{(2\pi)^3}\ \frac{i}{4w}\frac{w}{M}\left\{\epsilon_{imk}\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_m\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_i\right\}\\
&=\int \frac{d^3k}{(2\pi)^3}\ \frac{-i}{4w}\frac{w}{M}\left\{\epsilon_{imk}\left(a_{\perp i}(\vec{k})a^\dagger_{Lm}(\vec{k})+a_{Li}(\vec{k})a^\dagger_{\perp m}(\vec{k})\right)\right\}. \qquad (4)
\end{aligned}$$

Finally, the last term:

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3}\ \frac{-i}{4w}\left(a_L(-\vec{k})-a_L^\dagger(\vec{k})\right)_m\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_m=\int \frac{d^3k}{(2\pi)^3}\frac{-i}{2w}a_{Lm}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a^\dagger_{Lm}(\vec{k})\\
&=\int \frac{d^3k}{(2\pi)^3}\ \frac{-i}{2w}a_{Lm}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a^\dagger_m(\vec{k})+\int \frac{d^3k}{(2\pi)^3}\ \frac{i}{2w}\left\{\epsilon_{imk}a_{Lm}(\vec{k})a^\dagger_{Li}(\vec{k})+\epsilon_{imk}a_{Li}(\vec{k})a^\dagger_m(\vec{k})\right\}\\
&=\int \frac{d^3k}{(2\pi)^3}\ \frac{-i}{2w}a_{Lm}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a^\dagger_m(\vec{k})+\int \frac{d^3k}{(2\pi)^3}\ \frac{-i}{2w}\left\{\epsilon_{imk}a_{Lm}(\vec{k})a^\dagger_{\perp i}(\vec{k})\right\}. \qquad (5)
\end{aligned}$$

Collecting formulas (1),(2),(3),(4),(5), we find:

$$\begin{aligned}
J_k=\epsilon_{ijk}\int \frac{d^3k}{(2\pi)^3}\ \frac{i}{2w}&\left\{a_{\perp i}(\vec{k})a^\dagger_{\perp j}(\vec{k})+a_{Li}(\vec{k})a^\dagger_{Lj}(\vec{k})+a_{Li}(\vec{k})a^\dagger_{\perp j}(\vec{k})+a_{\perp i}(\vec{k})a^\dagger_{Lj}(\vec{k})\right.\\
&\left.-\ (a_{Lm}(\vec{k})+a_{\perp m}(\vec{k}))\left(k_i\frac{\partial}{\partial k^j}\right)a^\dagger_m(\vec{k})\right\}\\
=i\epsilon_{ijk}\int d\Omega_{\vec{k}}\ &\left\{a_i(\vec{k})a^\dagger_j(\vec{k})-a_m(\vec{k})\left(k_i\frac{\partial}{\partial k^j}\right)a^\dagger_m(\vec{k})\right\}.
\end{aligned}$$