Quantum Field Theory

Set 17: solutions

Exercise 1

A general state $|\Psi\rangle$ of n photons can be written as a linear combination of pure states of n photons $a^\dagger_{\mu_1}(\vec{k}_1)...a^\dagger_{\mu_n}(\vec{k}_n)|0\rangle$ weighted by some wave function $f^{\mu_1\cdots\mu_n}(k_1,...,k_n)$, which expresses the probability of having a particular combination of polarizations and momenta:

$$|\Psi\rangle = \int d\Omega_{\vec{k}_1}...d\Omega_{\vec{k}_n}\, f^{\mu_1\cdots\mu_n}(\vec{k}_1,...,\vec{k}_n)\, a^\dagger_{\mu_1}(\vec{k}_1)...a^\dagger_{\mu_n}(\vec{k}_n)|0\rangle.$$

For the simpler case of only two photons we get:

$$|\Psi\rangle = \int d\Omega_{\vec{k}}\, d\Omega_{\vec{p}}\, f^{\mu\nu}(\vec{k},\vec{p})\, a^\dagger_\mu(\vec{k})a^\dagger_\nu(\vec{p})|0\rangle.$$

The wave function f has to satisfy some constraints. The most obvious one comes from the bosonic nature of the operators $a^\dagger_\mu(\vec{k})$. Since they are associated with a bosonic field they commute among themselves and satisfy:

$$[a^\dagger_\mu(\vec{k}), a^\dagger_\nu(\vec{p})] = [a_\mu(\vec{k}), a_\nu(\vec{p})] = 0 \qquad\qquad [a^\dagger_\mu(\vec{k}), a_\nu(\vec{p})] = \eta_{\mu\nu}2k_0(2\pi)^3\delta^3(\vec{k}-\vec{p}).$$

Note that the sign in the commutation relations is correct: the transverse components have a minus, which is consistent with the commutation relations for the scalar field. Hence

$$|\Psi\rangle = \int d\Omega_{\vec{k}}\, d\Omega_{\vec{p}}\, f^{\mu\nu}(\vec{k},\vec{p})\, a^\dagger_\mu(\vec{k})a^\dagger_\nu(\vec{p})|0\rangle = \int d\Omega_{\vec{k}}\, d\Omega_{\vec{p}}\, f^{\mu\nu}(\vec{k},\vec{p})a^\dagger_\nu(\vec{p})a^\dagger_\mu(\vec{k})|0\rangle = \int d\Omega_{\vec{k}}\, d\Omega_{\vec{p}}\, f^{\nu\mu}(\vec{p},\vec{k})\, a^\dagger_\mu(\vec{k})a^\dagger_\nu(\vec{p})|0\rangle,$$

where in the last equality we have renamed indices and momenta. We thus find that the wave function has to be symmetric under the exchange of a pair of indices and the correspondent momenta:

$$f^{\mu\nu}(\vec{k},\vec{p}) = f^{\nu\mu}(\vec{p},\vec{k}).$$

In addition, if $|\Psi\rangle$ has to describe a physical state, the following condition must hold:

$$L|\Psi\rangle = 0, \qquad \text{with} \;\; L = q^\rho a_\rho(q).$$

Imposing this condition we find:

$$L|\Psi\rangle = \int d\Omega_{\vec{k}}\, d\Omega_{\vec{p}}\, f^{\mu\nu}(\vec{k},\vec{p})\, q^\rho a_\rho(\vec{q})\, a^\dagger_\mu(\vec{k})a^\dagger_\nu(\vec{p})|0\rangle = 0$$

Using the commutation relations we can manipulate this equality:

$$\begin{aligned} a_\rho(\vec{q})a^\dagger_\mu(\vec{k})a^\dagger_\nu(\vec{p})|0\rangle &= [a_\rho(\vec{q}),\, a^\dagger_\mu(\vec{k})]a^\dagger_\nu(\vec{p})|0\rangle + a^\dagger_\mu(\vec{k})a_\rho(\vec{q})a^\dagger_\nu(\vec{p})|0\rangle \\ &= [a_\rho(\vec{q}),\, a^\dagger_\mu(\vec{k})]a^\dagger_\nu(\vec{p})|0\rangle + a^\dagger_\mu(\vec{k})[a_\rho(\vec{q}),\, a^\dagger_\nu(\vec{p})]|0\rangle, \end{aligned}$$

since $a_\rho(\vec{q})|0\rangle = 0$. Substituting the commutators with their explicit expressions we get:

$$a_\rho(\vec{q})a^\dagger_\mu(\vec{k})a^\dagger_\nu(\vec{p})|0\rangle = -\eta_{\rho\mu}2k_0(2\pi)^3\delta^3(\vec{k}-\vec{q})\; a^\dagger_\nu(\vec{p})|0\rangle - \eta_{\rho\nu}2p_0(2\pi)^3\delta^3(\vec{p}-\vec{q})\; a^\dagger_\mu(\vec{k})|0\rangle.$$

Due to the presence of the delta-function we can easily integrate over d^3k and d^3p. The result reads:

$$\begin{aligned} &0 = L|\Psi\rangle = -\int d\Omega_{\vec{p}}\, f^{\rho\nu}(\vec{q},\vec{p})\, q_\rho a^\dagger_\nu(\vec{p})|0\rangle - \int d\Omega_{\vec{k}}\, f^{\mu\rho}(\vec{k},\vec{q})\, q_\rho a^\dagger_\mu(\vec{k})|0\rangle = -2\int d\Omega_{\vec{p}}\, f^{\rho\nu}(\vec{q},\vec{p})\, q_\rho a^\dagger_\nu(\vec{p})|0\rangle \\ &\implies \quad f^{\rho\nu}(\vec{q},\vec{p})\, q_\rho = 0. \end{aligned}$$

Therefore f has to be transverse in any index associated to the corresponding momenta.
Finally we can compute the condition to have a positive norm state (recall that a physical state has a positive norm and is also annihilated by L):

$$\langle\Psi|\Psi\rangle = \int d\Omega_{\vec{k}}\, d\Omega_{\vec{p}} d\Omega_{\vec{k}'}\, d\Omega_{\vec{p}'}\, f^{*\rho\sigma}(\vec{k}',\vec{p}') f^{\mu\nu}(\vec{k},\vec{p})\, \langle 0|a_\rho(\vec{k}')a_\sigma(\vec{p}')\, a^\dagger_\mu(\vec{k})a^\dagger_\nu(\vec{p})|0\rangle.$$

$$\begin{aligned}\langle 0|a_\rho(\vec{k}')a_\sigma(\vec{p}')\, a^\dagger_\mu(\vec{k})a^\dagger_\nu(\vec{p})|0\rangle &= -\eta_{\sigma\mu}2k_0(2\pi)^3\delta^3(\vec{k}-\vec{p}')\ \langle 0|a_\rho(\vec{k}')a^\dagger_\nu(\vec{p})|0\rangle - \eta_{\sigma\nu}2p_0(2\pi)^3\delta^3(\vec{p}-\vec{p}')\ \langle 0|a_\rho(\vec{k}')a^\dagger_\mu(\vec{k})|0\rangle \\ &= \eta_{\rho\nu}\eta_{\sigma\mu}4p_0k_0(2\pi)^6\delta^3(\vec{k}-\vec{p}')\ \delta^3(\vec{k}'-\vec{p}) + \eta_{\rho\mu}\eta_{\sigma\nu}4k_0p_0(2\pi)^6\delta^3(\vec{p}-\vec{p}')\delta^3(\vec{k}-\vec{k}').\end{aligned}$$

Then, integrating over d^3k and d^3p, we get

$$\langle\Psi|\Psi\rangle = \int d\Omega_{\vec{k}'}\, d\Omega_{\vec{p}'}\, \left[f^{*\nu\mu}(\vec{k}',\vec{p}')f_{\mu\nu}(\vec{p}',\vec{k}') + f^{*\mu\nu}(\vec{k}',\vec{p}')f_{\mu\nu}(\vec{k}',\vec{p}')\right] = 2\int d\Omega_{\vec{k}}\, d\Omega_{\vec{p}}\, \left[f^{*\mu\nu}(\vec{k},\vec{p})f_{\mu\nu}(\vec{k},\vec{p})\right],$$

and the positivity condition is $|f|^2 \geq 0$, where the norm is defined implicitly in the above expression.

Exercise 2

The polarization of a photon of momentum k_μ is defined by the constraint:

$$\varepsilon_\mu k^\mu = 0.$$

Let's define a four-vector $\bar{k}^\mu$ with components $\bar{k}^0 = k^0$ and $\bar{k}^i = -k^i$. Note that k^μ and $\bar{k}^\mu$ form a complete basis of the longitudinal-time subspace. In terms of k^μ, $\bar{k}^\mu$ and ε^μ, the transverse polarization vector is written as:

$$\varepsilon^\perp_\mu = \left(g_{\mu\nu} - \frac{k_\mu\bar{k}_\nu + \bar{k}_\mu k_\nu}{k\cdot\bar{k}}\right)\varepsilon^\nu = \varepsilon_\mu - \left(\frac{\bar{k}\cdot\varepsilon}{\bar{k}\cdot k}\right)k_\mu.$$

Note that $\varepsilon^\perp_\mu$ satisfies $\varepsilon^\perp_\mu k^\mu = \varepsilon^\perp_\mu \bar{k}^\mu = 0$, and that these conditions are of course Lorentz-invariant because written in terms of dot products. Moreover, writing more explicitly $k^\mu = k^0(1,\vec{n})$ and $\bar{k}^\mu = k^0(1,-\vec{n})$ it is easy to prove that the condition $\varepsilon_\mu k^\mu = 0$ implies $\varepsilon_0 = \vec{\varepsilon}\cdot\vec{n}$ and in turn that:

$$\begin{aligned}\varepsilon^\perp_0 &= 0, \\ \frac{\bar{k}\cdot\varepsilon}{\bar{k}\cdot k} &= \frac{\varepsilon^0}{k^0}.\end{aligned}$$

Finally, $\varepsilon^\perp_\mu k^\mu = \varepsilon^\perp_\mu \bar{k}^\mu = 0$ implies $\varepsilon^\perp_i k^i = 0$.
Now consider a generic Lorentz transformation acting on $\varepsilon^\perp_\mu$ and transforming it into $\varepsilon'^\perp_\mu = \varepsilon'_\mu - \frac{\varepsilon^0}{k^0}k'_\mu$. We get:

$$\begin{aligned}\varepsilon'^\perp_0 &= \varepsilon'_0 - \frac{\varepsilon^0}{k^0}k'_0 &&\neq 0, \\ \varepsilon'^\perp_i k'^i &= -\varepsilon'^\perp_0 k'^0 &&\neq 0.\end{aligned}$$

There is only one case in which this is not true (i.e. in which $\varepsilon'^\perp_0 = 0$, and consequently $\varepsilon'^\perp_i k'^i = 0$), namely the case of a longitudinal boost: such a boost leaves the transverse components of any fourvector untouched and mixes the time and longitudinal components, which for $\varepsilon'^\perp_\mu$ are both 0.
For generic transformation one can define:

$$\tilde{\varepsilon}^\perp_\mu \equiv \varepsilon'^\perp_\mu + \left(\frac{\varepsilon^0}{k^0} - \frac{\varepsilon'^0}{k'^0}\right)k'_\mu,$$

finding:

$$\begin{aligned}\tilde{\varepsilon}^\perp_0 &= \varepsilon'_0 - \frac{\varepsilon'^0}{k'^0}k'_0 &&= 0, \\ \tilde{\varepsilon}^\perp_i k'^i &= -\tilde{\varepsilon}^\perp_0 k'^0 &&= 0.\end{aligned}$$

Note that in the special case of longitudinal boost one has $\tilde{\varepsilon}^{\perp}_{\mu} = \varepsilon'^{\perp}_{\mu}$, as can be seen from the definition of $\varepsilon^{\perp}_{\mu}$ replacing all fourvectors by their primed counterparts. In general, $\frac{\varepsilon'^0}{k'^0}$ is not the Lorentz transform of $\frac{\varepsilon^0}{k^0}$, since $\vec{k}' \neq -\vec{k}'$.
One important point to notice is the following. Since the condition of orthogonality ($\tilde{\varepsilon}^{\perp}_{i} k'^{i} = 0$) and of null time component ($\tilde{\varepsilon}^{\perp}_{0} = 0$) are not Lorentz-invariant, if an observer defines a fourvector which just contains the two physical transverse photon polarizations, another generic observer will see that fourvector as containing three photon polarizations, meaning that the projection on the physical subspace is an observer-dependent statement. So when we define the vector $\varepsilon^{\perp}_{\mu}$, we are defining an object that transforms as a fourvector, but in a weak sense: it is true that $\Lambda : \varepsilon^{\perp}_{\mu} \to \Lambda_{\mu}{}^{\nu} \varepsilon^{\perp}_{\nu} \equiv \varepsilon'^{\perp}_{\mu}$, but $\varepsilon'^{\perp}_{\mu}$ does *not* share the basic, defining property of $\varepsilon^{\perp}_{\mu}$, namely the '$\perp$'. If instead we want a Lorentz-transformed vector which shares the same defining properties as $\varepsilon^{\perp}_{\mu}$ we have to implement a *nonlinear* transformation $\Lambda_{NL} : \varepsilon^{\perp}_{\mu} \to \Lambda_{\mu}{}^{\nu} \varepsilon^{\perp}_{\nu} - \frac{\varepsilon'^0}{k'^0} \Lambda_{\mu}{}^{\nu} k_{\nu} \equiv \tilde{\varepsilon}^{\perp}_{\mu}$.
At the end, as far as physical applications are concerned, these remarks, even though conceptually important, are quite harmless since we'll see that gauge invariance implies $M^{\mu} k_{\mu} = 0$, with M^{μ} a physical scattering amplitude, so that $M'^{\mu} \varepsilon'^{\perp}_{\mu} = M'^{\mu} \tilde{\varepsilon}^{\perp}_{\mu}$ for all observators, but it is important to keep in mind the distinction between $\varepsilon'^{\perp}_{\mu}$ and $\tilde{\varepsilon}^{\perp}_{\mu}$ in cases in which even the longitudinal part enters the game.