Quantum Field Theory

Set 15: solutions

Exercise 1: Maxwell's equations and transverse components

We will now show that Maxwell's equations describe indeed only two dynamical degrees of freedom. We recall that the Bianchi identity $\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0$ translates into the two homogenous Maxwell equations once the field strength is written in terms of the electric and magnetic field:

$$\vec{\nabla} \wedge \vec{E} + \dot{\vec{B}} = 0 \,, \qquad \vec{\nabla} \cdot \vec{B} = 0.$$

The above equations are trivially solved once we express $\vec{E}$ and $\vec{B}$ in terms of the four potential A_μ. On the other hand, we can let $\vec{E} = \vec{E}_L + \vec{E}_\perp$, and similarly for $\vec{B}$, with:

$$E^i_\perp \equiv \left(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2}\right) E^j \,, \qquad E^i_L \equiv E^i - E^i_\perp \,, \qquad \partial^i E^i_\perp = 0.$$

Then the above equations become:

$$\vec{\nabla} \wedge \vec{E}_\perp = -\dot{\vec{B}}_\perp \,, \qquad \vec{\nabla} \cdot \vec{B}_\perp = 0,$$
$$0 = \vec{\nabla} \wedge \vec{E}_L = -\dot{\vec{B}}_L \,, \qquad \vec{\nabla} \cdot \vec{B}_L = 0.$$

The longitudinal component of the magnetic field is constant in time, and has divergence, curl and thus laplacian equal to 0. Hence it is a constant. Since the field has to vanish at infinity, it can only be identically 0, thus:

$$\vec{B}_L = 0 \,, \qquad \vec{\nabla} \wedge \vec{E}_\perp = -\dot{\vec{B}}_\perp.$$

The evolution of the magnetic field is completely determined by the one of the electric field and so only 3 degrees of freedom are present in $F^{\mu\nu}$.
The other two Maxwell equations are $\partial_\mu F^{\mu\nu} = J^\nu$, where we have allowed for a current coupled to the electromagnetic field. In terms of $\vec{E}, \vec{B}$ they read:

$$\vec{\nabla} \cdot \vec{E} = \rho \,, \qquad \vec{\nabla} \wedge \vec{B} - \dot{\vec{E}} = \vec{J}.$$

Again decomposing in longitudinal and transverse components we get:

$$\vec{\nabla} \cdot \vec{E}_L = \rho,$$

which fixes the longitudinal part of the electric field completely. Hence the only dynamical component is $\vec{E}_\perp$, which contains only 2 degrees of freedom.

Let us now see how this can be obtained using the four potential A_μ. The inhomogeneous Maxwell equations split in the following way (recall $\partial_\mu = (\partial_0, \vec{\nabla})$, $A_\mu = (A_0, -\vec{A})$):

$$\nu = 0 : \qquad -\nabla^2 A_0 - \vec{\nabla} \cdot \dot{\vec{A}} = \vec{\nabla} \cdot \vec{E} = J^0,$$
$$\nu = i : \qquad \ddot{\vec{A}} + \vec{\nabla}\dot{A}_0 - \nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \vec{\nabla} \wedge \vec{B} - \dot{\vec{E}} = \vec{J}.$$

The solution of the first equation is formally:

$$A_0 = -\nabla^{-2}(J^0 + \vec{\nabla} \cdot \dot{\vec{A}}).$$

Plugging this into the second equation, we get:

$$\ddot{A}^i - \partial_i \nabla^{-2}(\dot{J}^0 + \vec{\nabla}\cdot\ddot{\vec{A}}) - \nabla^2 A^i + \partial_i(\partial_j A^j) = J^i$$
$$\implies \left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right)\ddot{A}^j - \nabla^2\left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right)A^j = J^i + \partial_i\nabla^{-2}\dot{J}^0.$$

We can recognize a wave equation for the projected component $A^i_\perp = (\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2})A^j$. The combination in parenthesis is indeed a projector since, if squared, it is equal to itself. In momentum space we can see that it projects A^i on the direction orthogonal to the momentum p:

$$A^i_\perp = \left(\delta^{ij} - \frac{p^i p^j}{p^2}\right)A^j \implies p^i A^i_\perp = 0.$$

This is equivalent to imposing the Coulomb condition (in that gauge $\vec{\nabla}\cdot\vec{A}_L = \vec{\nabla}\wedge\vec{A}_L = 0$ thus $A_L = 0$). Indeed the Coulomb gauge identifies the physical degrees of freedom, even if it is not a Lorentz invariant constraint. Moreover the *longitudinal* part $A^i_L = A^i - A^i_\perp$ does not appear anywhere: it is completely decoupled.

Exercise 2: Energy momentum tensor

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial_\rho A^\rho)^2.$$

The equations of motion are:

$$\partial_\mu F^{\mu\nu} + \lambda\partial^\nu(\partial_\rho A^\rho) = \Box A^\nu - (1-\lambda)\partial^\nu(\partial_\rho A^\rho) = 0.$$

The energy momentum tensor can be derived using the usual procedure:

$$x'^\mu = x^\mu - a^\mu,$$
$$A'_\rho(x') = A_\rho(x) \simeq A_\rho(x') + a^\nu \underbrace{\partial_\nu A_\rho(x')}_{\Delta_{\rho\nu}}.$$

Thus we get:

$$J^\mu_i = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta_{a\,i} - \epsilon^\mu_i\mathcal{L} \implies T^\mu_{\ \nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\Delta_{\rho\,\nu} - \delta^\mu_\nu\mathcal{L}.$$

And hence:

$$T^\mu_{\ \nu} = -F^{\mu\rho}\partial_\nu A_\rho + \frac{1}{4}\delta^\mu_\nu F_{\alpha\beta}F^{\alpha\beta} - \lambda(\partial_\alpha A^\alpha)\partial_\nu A^\mu + \frac{\lambda}{2}\delta^\mu_\nu(\partial_\rho A^\rho)^2.$$

We can now check explicitly, that the divergence of the energy momentum tensor vanishes as predicted by Noether's theorem:

$$\partial_\mu T^\mu_{\ \nu} = -\partial_\mu F^{\mu\rho}\partial_\nu A_\rho - F^{\mu\rho}\partial_\mu\partial_\nu A_\rho + \frac{1}{2}F^{\alpha\beta}\partial_\nu F_{\alpha\beta}$$
$$- \lambda(\partial_\alpha A^\alpha)\partial_\nu(\partial_\mu A^\mu) - \lambda\partial_\mu(\partial_\alpha A^\alpha)\partial_\nu A^\mu + \lambda(\partial_\alpha A^\alpha)\partial_\nu(\partial_\mu A^\mu).$$

Using the equation of motion for the first term, we can eliminate all the terms proportional to λ. What remains is only:

$$-F^{\mu\rho}\partial_\mu\partial_\nu A_\rho + \frac{1}{2}F^{\alpha\beta}\partial_\nu F_{\alpha\beta} = -\frac{1}{2}F^{\mu\rho}(\partial_\mu\partial_\nu A_\rho - \partial_\rho\partial_\nu A_\mu) + \frac{1}{2}F^{\alpha\beta}\partial_\nu F_{\alpha\beta},$$

where we have used the antisymmetry of the field strength. Expanding this gives:

$$-\frac{1}{2}F^{\alpha\beta}\left(\partial_\alpha\partial_\nu A_\beta - \partial_\beta\partial_\nu A_\alpha - \partial_\nu F_{\alpha\beta}\right) = -\frac{1}{2}F^{\alpha\beta}\left(\partial_\alpha\partial_\nu A_\beta - \partial_\beta\partial_\nu A_\alpha - \partial_\nu\partial_\alpha A_\beta + \partial_\nu\partial_\beta A_\alpha\right) = 0.$$

In the limit $\lambda \to 0$, the Lagrangian is gauge invariant. Therefore, it could be expected that also the energy momentum tensor is gauge invariant but this is in fact not true, since the Noether formula contains a derivative with respect to the field A_μ which is not gauge invariant (only the tensor $F^{\mu\nu}$ is). Instead under a gauge transformation:

$$T^\mu_{\ \nu} \longrightarrow T^\mu_{\ \nu} + F^{\mu\rho}\partial_\nu\partial_\rho\Lambda.$$

On the contrary however, the charges, which are related to physical quantities, must be gauge invariant and indeed they are:

$$P_\nu = \int d^3x\ T^0_{\ \nu} \longrightarrow \int d^3x\ T^0_{\ \nu} + \int d^3x\ F^{0i}\partial_\nu\partial_i\Lambda = \int d^3x\ T^0_{\ \nu} - \int d^3x\ \partial_i F^{0i}\partial_\nu\Lambda = \int d^3x\ T^0_{\ \nu},$$

where we have integrated by parts and used the equation of motion $\partial_i F^{i0} = 0$. To summarize, although the Noether procedure gives us a non gauge invariant energy momentum tensor, the charges are invariant.
One can always modify the definition of the energy momentum tensor by adding a piece $K^\mu_{\ \nu}$ which is divergenceless and such that $K^0_{\ \nu}$ is a total space derivative. In the present case we can define:

$$\tilde{T}^\mu_{\ \nu} = T^\mu_{\ \nu} + F^{\mu\rho}\partial_\rho A_\nu = F^{\mu\rho}F_{\rho\nu} + \frac{1}{4}\delta^\nu_\mu F_{\alpha\beta}F^{\alpha\beta}.$$

Indeed

$$\partial_\mu(F^{\mu\rho}\partial_\rho A_\nu) = 0\ , \qquad F^{0\rho}\partial_\rho A_\nu = F^{0i}\partial_i A_\nu = \partial_i(F^{0i}A_\nu) + \text{eq. of motion}.$$

Since $\tilde{T}^\mu_{\ \nu}$ depends only on the field strength, it is gauge invariant. This could be argued also by noticing that the gauge variation of $F^{\mu\rho}\partial_\rho A_\nu$ exactly compensates the one of $T^\mu_{\ \nu}$. Notice finally that $\tilde{T}^\mu_{\ \nu}$ is symmetric and traceless:

$$\tilde{T}^\mu_{\ \mu} = F^{\mu\rho}F_{\rho\mu} + \frac{1}{4}4F_{\alpha\beta}F^{\alpha\beta} = 0.$$

When $\lambda \neq 0$, differentiating the equations of motion with respect to x^ν gives:

$$\partial_\nu\partial_\mu F^{\mu\nu} + \lambda\Box(\partial_\rho A^\rho) \equiv \lambda\Box(\partial_\rho A^\rho) = 0,$$

where the first piece is equal to 0 due to the antisymmetry of the $F^{\mu\nu}$. Since $\Box(\partial_\rho A^\rho) = 0$, if $\partial_\rho A^\rho = 0$ and $\partial_t(\partial_\rho A^\rho) = 0$ at *a particular time*, it is true that $\partial_\rho A^\rho = 0$ identically. So if both these conditions are satisfied and $\partial_\rho A^\rho = 0$, then quantities computed with the equations of motion (like the Noether currents and charges) do not depend on λ. But, in general, if $\partial_\rho A^\rho \neq 0$, a λ-dependence remains.

Exercise 3: Coulomb gauge

Let us consider the Lagrangian for a massless vector field

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\ .$$

We can obtain the Euler-Lagrange equations performing a variation of the action with respect to the field and imposing this to be vanishing:

$$\begin{aligned}\delta_A S &= \int d^4x\delta_A\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) = -\frac{1}{4}\int d^4x\, 2\, F^{\mu\nu}\delta_A\left(F_{\mu\nu}\right)\\ &= -\frac{1}{2}\int d^4x F^{\mu\nu}\, 2\,\partial_\mu\delta A_\nu = \int d^4x\left(\partial_\mu F^{\mu\nu}\right)\delta A_\nu = 0,\end{aligned}$$

where in the last equality we have integrated by parts. Finally the equations of motion read

$$\partial_\mu F^{\mu\nu} = 0\ .$$

Let us compute the conjugate momenta of the fields A_μ. In principle we expect four momenta Π^μ:

$$\begin{aligned}\Pi^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_t A_\mu)} = -F^{\alpha\beta}\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_t A_\mu)} = -F^{0\mu},\\ \Pi^0 &= 0, \qquad\qquad \Pi^i = -F^{0i} = \partial^i A^0 - \partial^0 A^i = \partial_t A_i - \partial_i A_0.\end{aligned}$$

The conjugate momentum of the field A_0 is identically vanishing. This suggests that the zero component of the vector potential is not a dynamical variable, even if it appears explicitly in the Lagrangian. This can be seen also looking at the zero component of the equations of motion: it translates in a constraint for the conjugate momenta which can be used to solve the field A_0 with respect to the others:

$$0 = \partial_i F^{i0} = \partial_i\Pi^i = -\vec{\nabla}\cdot\dot{\vec{A}} - \nabla^2 A_0.$$

The above equation has to be thought as a Poisson equation for the field A_0 with source given by the divergence of $\dot{\vec{A}}$. Once one is able to solve the theory for the spatial components of the potential the time component is automatically determined. Once we try to quantize this theory we face with the problem of defining consistently the canonical commutation relations; indeed the usual definition

$$[\Pi_\mu(\vec{x},t),\, A_\nu(\vec{y},t)] = i\eta_{\mu\nu}\delta^3(\vec{x}-\vec{y})$$

is in contrast with the vanishing of Π^0 because the right hand side would not be zero. This is related to the fact that we are describing the theory with a redundant formalism: we are using too many degrees of freedom with respect to what we need in practice. The redundancy is encoded in the big symmetry of the Lagrangian, which is invariant under the so called gauge transformations:

$$A_\mu \longrightarrow A_\mu - \partial_\mu\Lambda,$$

where Λ is an arbitrary function (to be formal, $\Lambda(x)$ must be integrable on $\mathbb{R}^4$). One can use this freedom in the definition of the field A_μ to eliminate the unphysical degrees of freedom present in the theory. Thus, to summarize, A_0 is never a dynamical degree of freedom since can be expressed in terms of A_i using the constraint $\partial_i\Pi^i = 0$, while an additional degree of freedom can be eliminated using the gauge freedom. Therefore one is left with only two degrees of freedom, which are what one expects for a massless spin one particle (we recall that a massless representation of the Poincaré group contains only one helicity plus its image under parity for a total of two degrees of freedom). Let us now see an explicit example where the gauge freedom can be used to impose an additional constraint on A_μ. In general $\vec{\nabla}\cdot\vec{A} \neq 0$. If however we consider the transformed field $A'_\mu = A_\mu - \partial_\mu\Lambda$, with a suitable choice of Λ we can impose :

$$0 = \vec{\nabla}\cdot\vec{A'} = \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda = 0 \quad\Longrightarrow\quad \Lambda = -\nabla^{-2}\vec{\nabla}\cdot\vec{A}.$$

Therefore we can always choose a configuration where, dropping the prime, $\vec{\nabla}\cdot\vec{A} = 0$. In the above expression, with an abuse of notation we have expressed Λ as the solution of the Poisson equation with a source. To be more concrete one can define the Green function $G(x)$ in three dimensions as the solution of the Poisson equation with a pointlike source:

$$\nabla^2 G(\vec{x}) = \delta^3(\vec{x}) \quad\Longrightarrow\quad G(\vec{x}) = \frac{1}{4\pi|\vec{x}|},$$

then the solution for Λ is the convolution

$$\Lambda(\vec{x}) = -\int d^3y\, G(\vec{x}-\vec{y})\vec{\nabla}\cdot\vec{A}(\vec{y}).$$

Notice that in general Λ can depend also on time. The choice of A_μ that satisfies the constraint $\vec{\nabla}\cdot\vec{A} = 0$ is called *Coulomb gauge*. In this case the other constraint reads:

$$0 = \vec{\nabla}\cdot\vec{\Pi} = -\nabla^2 A_0,$$

which only admits the solution $A_0 = 0$. Therefore the time component is exactly zero while the space one are *transverse* (this because in momentum space the Coulomb gauge becomes $\vec{p}\cdot\vec{A} = 0$, so the potential is orthogonal to the direction of $\vec{p}$). A choice of the commutation relations which is consistent with the two constraints $\vec{\nabla}\cdot\vec{\Pi} = \vec{\nabla}\cdot\vec{A} = 0$ can be the following:

$$[\Pi_i(\vec{x},t),\, A_j(\vec{y},t)] = i\delta_{ij}\delta^3(\vec{x}-\vec{y}) + i\partial_i^{(x)}\partial_j^{(y)}\frac{1}{4\pi|\vec{x}-\vec{y}|},$$

and we can easily check the consistency taking the derivative $\partial_i^{(x)}$ of both sides of the relation; from the l.h.s. we must get zero. From the r.h.s. we have

$$i\partial_j^{(x)}\delta^3(\vec{x}-\vec{y}) + i\partial_j^{(y)}\underbrace{\nabla^2\frac{1}{4\pi|\vec{x}-\vec{y}|}}_{\delta^3(\vec{x}-\vec{y})} = i\partial_j^{(x)}\delta^3(\vec{x}-\vec{y}) - i\partial_j^{(x)}\delta^3(\vec{x}-\vec{y}) = 0.$$

Hence the above definition is coherent.