Quantum Field Theory

Set 14: solutions

Exercise 1

Consider a massless Dirac fermion $\psi = \binom{\psi_L}{\psi_R}$ which has the usual free Lagrangian density:

$$\mathcal{L} = i\bar{\psi}\,\partial\!\!\!/\,\psi \equiv i\psi^\dagger\gamma^0\,\partial\!\!\!/\,\psi.$$

Recalling the definition of the gamma matrices and their algebra,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \sigma^\mu = (1, \sigma^i), \qquad \bar{\sigma}^\mu = (1, -\sigma^i), \qquad \{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu},$$

one can express the Lagrangian in terms of the left and right Weyl spinors:

$$\mathcal{L} \;=\; i\left(\psi_L^\dagger,\, \psi_R^\dagger\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \partial_\mu\psi_L \\ \partial_\mu\psi_R \end{pmatrix} \;=\; i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\,\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\,\psi_R.$$

Notice the presence of different σ's in the two terms. The theory is manifestly invariant under two distinct $U(1)$ transformations

$$U(1)_L: \qquad \begin{cases} \psi_L' = e^{i\alpha}\psi_L \\ \psi_R' = \psi_R, \end{cases}$$

$$U(1)_R: \qquad \begin{cases} \psi_L' = \psi_L \\ \psi_R' = e^{i\beta}\psi_R, \end{cases}$$

which rotate separately left and right spinors while leaving unchanged the coordinates. The Noether's currents associated to the symmetries are given by

$$J_L^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_L)}\Delta_{\psi_L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_L^\dagger)}\Delta_{\psi_L^\dagger} = i\psi_L^\dagger\bar{\sigma}^\mu(i\psi_L) = -\psi_L^\dagger\bar{\sigma}^\mu\psi_L,$$

$$J_R^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_R)}\Delta_{\psi_R} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_R^\dagger)}\Delta_{\psi_R^\dagger} = i\psi_R^\dagger\sigma^\mu(i\psi_R) = -\psi_R^\dagger\sigma^\mu\psi_R.$$

In the derivation we have used the fact that the Lagrangian density is written in an asymmetric way with respect to $\psi_{L(R)}^\dagger$ and $\psi_{L(R)}$: only the former appear in the Lagrangian with the derivative ∂_μ. However one can make the symmetry manifest simply integrating 'half' Lagrangian density by parts: $\frac{i}{2}\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\,\psi_L \longrightarrow -\frac{i}{2}\partial_\mu\psi_L^\dagger\bar{\sigma}^\mu\,\psi_L$; this doesn't not modify the theory nor the present discussion.

Considering the sum of the two Noether's currents gives:

$$J_L^\mu + J_R^\mu = -\psi_L^\dagger\bar{\sigma}^\mu\psi_L - \psi_R^\dagger\sigma^\mu\psi_R = -\bar{\psi}\gamma^\mu\psi \equiv J_V^\mu.$$

J_V^μ can be thought of as the Noether's current associated to a $U(1)$ transformation acting in the same way on left-handed and on right-handed fields (it is called *vectorial* $U(1)$, or $U(1)_V$, since the spatial components of its Noether's current form a true vector which changes sign under parity, giving rise to a parity-invariant Lagrangian $\mathcal{L}$ when contracted with the true vector ∂_i):

$$U(1)_V: \quad \psi' = e^{i\alpha}\psi.$$

$U(1)_V$ is the subgroup of $U(1)_L \times U(1)_R$ obtained by performing left and right transformations with equal parameters $\alpha = \beta$, but can also be seen as a transformation acting on the Dirac field.

On top of considering the sum of the original Noether's currents one can also take their difference, $J^\mu_A \equiv J^\mu_R - J^\mu_L$. We now show that the following transformation is a symmetry of the theory and it gives rise to the current J^μ_A:

$$U(1)_A : \quad \psi' = e^{i\alpha\gamma^5}\psi = \exp\left[i\alpha\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]\psi = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

Still this transformation corresponds to a subgroup of $U(1)_L \times U(1)_R$ where this time the left and right parameters are taken opposite in sign. In order to implement this transformation on the Dirac field we need the matrix γ^5: the action on ψ doesn't consist in a simple phase multiplication but involves the 4×4 matrix $e^{-i\alpha\gamma^5}$ which however is parametrized by a single parameter $\alpha \in [0, 2\pi]$.
The free Lagrangian density is invariant under this *axial* transformations. In fact

$$\bar{\psi}'\gamma^\mu\partial_\mu\psi' = \left(e^{i\alpha\gamma^5}\psi\right)^\dagger \gamma^0\gamma^\mu e^{i\alpha\gamma^5}\partial_\mu\psi = \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0\gamma^\mu e^{i\alpha\gamma^5}\partial_\mu\psi,$$

and recalling that γ^5 anticommutes with *all* the Dirac matrices, $\{\gamma^5, \gamma^\mu\} = 0$, and that $(\gamma^5)^2 = 1$, one has

$$\bar{\psi}'\gamma^\mu\partial_\mu\psi' = \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0\gamma^\mu e^{i\alpha\gamma^5}\partial_\mu\psi = \psi^\dagger\gamma^0\gamma^\mu e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\partial_\mu\psi = \bar{\psi}\gamma^\mu\partial_\mu\psi.$$

In the end the Noether's current reads:

$$J^\mu_A = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Delta_\psi = -\bar{\psi}\gamma^\mu\gamma^5\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi = \psi^\dagger_L\bar{\sigma}^\mu\psi_L - \psi^\dagger_R\sigma^\mu\psi_R = J^\mu_R - J^\mu_L.$$

Therefore the independent symmetries $U(1)_L \times U(1)_R$, acting on Weyl spinors, can be recast into two equivalent symmetries, $U(1)_V \times U(1)_A$, acting on the Dirac field. The reason why the transformation with γ^5 is called axial is that the spatial components of its Noether's current form an axial vector, which does not change sign under parity, as it can be checked by explicit computation.

Let's add a mass term for the Dirac field:

$$\mathcal{L} = i\bar{\psi}\,\partial\!\!\!/\,\psi - m\bar{\psi}\psi = i\psi^\dagger_L\bar{\sigma}^\mu\partial_\mu\,\psi_L + i\psi^\dagger_R\sigma^\mu\partial_\mu\,\psi_R - m(\psi^\dagger_L\psi_R + \psi^\dagger_R\psi_L).$$

One can immediately realize that now the left-handed and right-handed Weyl spinors cannot be rotated independently as in the massless case; however the transformation $U(1)_V$ is still a symmetry of the massive theory. This is not the case for the axial transformation, since

$$\bar{\psi}'\psi' = \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 e^{i\alpha\gamma^5}\psi = \bar{\psi}e^{2i\alpha\gamma^5}\psi.$$

The introduction of a mass term preserves the $U(1)_V$ while it explicitly breaks the invariance under $U(1)_A$.

The invariance under the vectorial group has important physical applications. Extending the analysis to a Dirac Lagrangian with two fermion fields, the *up* and *down* quarks,

$$\tilde{\mathcal{L}} = \sum_{ij} i\bar{\psi}_i(\delta_{ij}\,\partial\!\!\!/ - m_{ij})\psi_j,$$

one can notice that if the mass matrix m_{ij} is proportional to the identity, i.e. if the two quarks have the same mass, then $\tilde{\mathcal{L}}$ is invariant under the group $U(2)_V \equiv U(1) \times SU(2)$. The $U(1)$ factor is associated to barion number conservation, while the $SU(2)$ subgroup is the isospin. Since the mass degeneracy in Nature is only approximate, the vectorial group is not an exact symmetry, implying the slight mass difference among barions in the same isospin multiplet, like neutron and proton.

Exercise 2

Let's first write the Dirac Lagrangian in its manifestly hermitian form

$$\mathcal{L} = \frac{i}{2}\bar{\psi}(\partial_\mu - \overleftarrow{\partial}_\mu)\gamma^\mu\psi,$$

where the symbol $\overleftarrow{\partial}$ means that the derivative acts on what is on its left. Let's consider now the transformation properties of the Dirac field under the Lorentz group:

$$\psi_a(x) \longrightarrow \psi'_a(x') = (\Lambda_D)_a{}^b\psi_b(x).$$

From the definitions of ψ in terms of ψ_L and ψ_R, and of Λ_D in terms of Λ_R and Λ_L, one has

$$\Lambda_D = \exp\left[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right],$$

where

$$S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

The variation of the Dirac field at fixed coordinate is thus

$$\begin{aligned}\Delta_\psi(x) &\equiv \psi'(x) - \psi(x) \simeq \frac{\omega_{\mu\nu}}{2}\left[-iS^{\mu\nu} + (x^\mu\partial^\nu - x^\nu\partial^\mu)\right]\psi(x) = \Delta_\psi^{\mu\nu}\frac{\omega_{\mu\nu}}{2},\\ \Delta_{\bar\psi}(x) &\equiv \bar\psi'(x) - \bar\psi(x) \simeq \bar\psi(x)\left[iS^{\mu\nu} + (\overleftarrow{\partial}^\nu x^\mu - \overleftarrow{\partial}^\mu x^\nu)\right]\frac{\omega_{\mu\nu}}{2} = \Delta_{\bar\psi}^{\mu\nu}\frac{\omega_{\mu\nu}}{2}.\end{aligned}$$

The second of these relations has been derived from the first by hermitian-conjugating it, applying γ^0 on the right, and using $\gamma^0(S^{\mu\nu})^\dagger\gamma^0 = S^{\mu\nu}$ and $(\gamma^0)^2 = 1$. Of course, starting from the Lagrangian density and considering the variation of $\bar\psi$ one gets the same result.
Notice that the part containing derivatives is due to the variation of the *point* in which the field is evaluated (while keeping fixed the label x), so it is simply due to the fact that the field is a function of space-time, as it happened for the scalar field: this part is in fact common to all fields and when evaluated in its space components it gives rise to the orbital part of the angular momentum. On the contrary, the contribution $S^{\mu\nu}$ comes from Λ_D and thus depends on the representation of the Lorentz group the field belongs to: in particular it is zero for scalar fields, as we have seen in Set8. This is the spin contribution to total angular momentum.
To give a quantitative meaning to this statement one can compute the Noether's current associated to invariance under Lorentz transformations. Using the usual definitions one has

$$M^\rho_{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\rho\psi)}\Delta_{\psi,\mu\nu} + \Delta_{\bar\psi,\mu\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\rho\bar\psi)} - \epsilon^\rho_{\mu\nu}\mathcal{L} = x_\mu T^\rho_{\ \nu} - x_\nu T^\rho_{\ \mu} + \frac{1}{2}\bar\psi\{\gamma^\rho, S_{\mu\nu}\}\psi,$$

where $T_{\mu\nu}$ is the energy-momentum tensor, namely

$$\begin{aligned}T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi + \partial^\nu\bar\psi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)} - \eta^{\mu\nu}\mathcal{L}\\ &= \frac{i}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta})\bar\psi\gamma_\alpha(\partial_\beta - \overleftarrow{\partial}_\beta)\psi,\end{aligned}$$

and we have used $x^\mu - x'^\mu \equiv \epsilon^\mu_{\alpha\beta}\omega^{\alpha\beta}/2 = (x_\alpha\delta^\mu_\beta - x_\beta\delta^\mu_\alpha)\omega^{\alpha\beta}/2$.
The angular momentum is then

$$J^k \equiv \frac{1}{2}\epsilon^{ijk}\int d^3x\ M^0_{ij} = \frac{1}{2}\epsilon^{ijk}\int d^3x\left[x_iT^0_{\ j} - x_jT^0_{\ i} + \frac{1}{2}\bar\psi\{\gamma^0, S_{ij}\}\psi\right] \equiv \int d^3x\ \psi^\dagger(t,\vec{x})(L^k + \Sigma^k/2)\psi(t,\vec{x}),$$

where $L^k = [\vec{x}\wedge(-i\vec\nabla)]^k$ is the orbital part (obtained from the above definition of $T^{\mu\nu}$, integrating by parts), while

$$\Sigma^k = \epsilon^{ijk}S_{ij} = \begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}$$

is the spin operator. To write last two equations we have used the explicit form of S_{ij} and its hermiticity, which implies $\gamma^0 S_{ij}\gamma^0 = \gamma^0 S^\dagger_{ij}\gamma^0 = S_{ij}$.

Let's now consider the action of the spin on a generic one-particle state in position space, $\psi^\dagger_\alpha(t,\vec{x})|0\rangle \equiv |x,\alpha\rangle$. Defining

$$\vec{S} = \int d^3x\ \psi^\dagger(t,\vec{x})\frac{\vec\Sigma}{2}\psi(t,\vec{x}),$$

and recalling that the vacuum has zero spin, $\vec{S}|0\rangle = 0$, one has

$$\vec{S}^2\ |x,\alpha\rangle = [\vec{S}\cdot[\vec{S}, \psi^\dagger_\alpha(t,\vec{x})]]\ |0\rangle.$$

With the fermionic equal-time canonical (anti)commutation relations

$$\{\psi_\alpha(t,\vec{x}),\psi_\beta(t,\vec{y})\} = \{\psi^\dagger_\alpha(t,\vec{x}),\psi^\dagger_\beta(t,\vec{y})\} = 0,$$
$$\{\psi_\alpha(t,\vec{x}),\psi^\dagger_\beta(t,\vec{y})\} = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}),$$

one gets

$$\begin{aligned}[\vec{S},\psi^\dagger_\alpha(t,\vec{x})] &= \int d^3y\ \frac{\vec{\Sigma}^{\gamma\beta}}{2}[\psi^\dagger_\gamma(t,\vec{y})\psi_\beta(t,\vec{y}),\psi^\dagger_\alpha(t,\vec{x})] = \int d^3y\ \frac{\vec{\Sigma}^{\gamma\beta}}{2}[-\psi^\dagger_\gamma\psi^\dagger_\alpha\psi_\beta + \psi^\dagger_\gamma(t,\vec{y})\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}) - \psi^\dagger_\alpha\psi^\dagger_\gamma\psi_\beta] \\ &= \psi^\dagger_\gamma(t,\vec{x})\frac{\vec{\Sigma}^{\gamma\alpha}}{2},\end{aligned}$$

then

$$[\vec{S}\cdot[\vec{S},\psi^\dagger_\alpha(t,\vec{x})]] = \frac{1}{4}\psi^\dagger_\beta(t,\vec{x})\vec{\Sigma}^{\beta\gamma}\cdot\vec{\Sigma}^{\gamma\alpha} = \frac{\psi^\dagger_\beta(t,\vec{x})}{4}\sum_k \begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}^{\beta\alpha} = \frac{3}{4}\psi^\dagger_\alpha(t,\vec{x}).$$

Thus

$$\vec{S}^2\ |x,\alpha\rangle \equiv s(s+1)\ |x,\alpha\rangle = \frac{3}{4}\ |x,\alpha\rangle \quad\Longrightarrow\quad s=\frac{1}{2}.$$

The the operator $\psi^\dagger$ creates particles characterized by spin one half.