Exercise 1

For the first point, the transformed spinor bilinear is $\psi_R^\dagger A_R^\dagger \sigma^\mu A_R \psi_R$. Let us expand the quantity between the spinors infinitesimally:

$$A_R^\dagger \sigma^\mu A_R = e^{\frac{1}{2}(\bar{q}+i\bar{\theta})\sigma^\mu} e^{\frac{1}{2}(i\bar{q}+\theta)\sigma^\mu} \simeq (1 + \frac{1}{2}(\bar{q} - i\bar{\theta})\sigma^\mu + \frac{1}{2}(\bar{q} + i\bar{\theta})\sigma^\mu) = \sigma^\mu + \frac{1}{2} \eta^i \{\sigma^i, \sigma^\mu\} + \frac{i}{2} \theta^i [\sigma^i, \sigma^\mu]$$

Let us distinguish the cases $\mu = 0, \mu = i$. Knowing the equations $[\sigma^i, \sigma^0] = 0$, $[\sigma^i, \sigma^j] = 2i \epsilon^{ijk} \sigma^k$, $\{\sigma^i, \sigma^0\} = 2 \sigma^i$, $\{\sigma^i, \sigma^j\} = 2 \delta^{ij} \sigma^0$, we find:

$$A_R^\dagger \sigma^0 A_R \simeq \sigma^0 + \eta^i \sigma^i$$
$$A_R^\dagger \sigma^i A_R \simeq \sigma^i + \eta^0 \sigma^0 - \eta^i \epsilon^{ijk} \sigma^k$$

We can guess that this represents the infinitesimal Lorentz transformation of a vector, with the boost part parametrized by $\eta^i$ and the rotation part by $\theta^i$. Indeed, the vector (spin 1) representation of the Lorentz transformation is given by $\Lambda(\eta, \theta)^\alpha_{\nu} = \epsilon^\mu_{\nu} \simeq \delta^\mu_{\nu} + \omega^\mu_{\nu}$, with the constraint that $\omega^{0\nu} = \eta^{\mu\nu} \omega^\mu_{\alpha}$ is antisymmetric: $\omega^{\mu0} = -\omega^{0\mu}$, and the identification $\theta^i = \frac{1}{2} \epsilon^{ijk} \omega_j^k$, $\eta^i = \omega^{0i}$.

$$A_R^\dagger \sigma^0 A_R \simeq \sigma^0 + \omega^0 \sigma^0 = \sigma^0$$
$$A_R^\dagger \sigma^i A_R \simeq \sigma^i + \omega^0 \sigma^0 - \frac{1}{2} \epsilon^{ijk} \omega_j^k \sigma^k = \sigma^i + \omega^0 \sigma^0 + \omega^i \sigma^k$$

where we used the obvious identities $\omega^{\mu\nu} = -\omega^{\nu\mu}$; $\omega^{0\nu} = \omega^{\nu0}$. Thus, we have proved the infinitesimal form of $A_R^\dagger A_R = \Lambda_{\mu}^\nu \sigma^\nu$.

For the second point, the calculation is very simple:

$$\epsilon^{-1} A_L \epsilon = \epsilon^{-1} e^{-\frac{1}{2}(\bar{q}+i\bar{\theta})\sigma^\mu} e^{-\frac{1}{2}(i\bar{q}+\theta)\sigma^\mu} = e^{-\frac{1}{2}(\bar{q} - i\bar{\theta})(-\bar{\sigma})} = \Lambda_R^*$$

by using the properties of $\epsilon$, this implies also $\epsilon^{-1} A_R \epsilon = \Lambda_L^\dagger, \epsilon^{-1} A_R^\dagger \epsilon = \Lambda_L^*$. Thus, we have proved the infinitesimal form of $\Lambda_R^\dagger A_R = \Lambda_{\mu}^\nu \sigma^\nu$.

For the third point, we can take the first result and do a similarity transformation through $\epsilon$:

$$\epsilon^{-1} A_R^\dagger \sigma^\mu A_R \epsilon = \epsilon^{-1} A_{\nu}^\dagger \sigma^\nu \epsilon$$

Since $\epsilon^{-1} \sigma^\mu \epsilon = \bar{\sigma}^\mu$, the previous equation is equivalent to:

$$A_R^\dagger \sigma^\mu A_R \epsilon = \Lambda_{\mu}^\nu \bar{\sigma}^\nu$$

The complex conjugate of this relation is $A_L^\dagger \sigma^\mu A_L^T = \Lambda_{\mu}^\nu \sigma^\nu$. This shows that the spinor bilinear $\psi_R^\dagger \bar{\sigma}^\mu \psi_L$ transforms as a vector.
Exercise 2

Given the results in the previous exercise, it is now easy to show that $\Lambda^{-1}_D \gamma^\mu \Lambda_D = \Lambda^\mu_\nu \gamma^\nu$. Indeed, given that $\Lambda_R^\dagger = \Lambda_L^{-1}$ and $\Lambda_L^\dagger = \Lambda_R^{-1}$, then

$$\Lambda_D^{-1} = \begin{pmatrix} \Lambda_R^{-1} & 0 \\ 0 & \Lambda_L^{-1} \end{pmatrix} = \begin{pmatrix} \Lambda_R^\dagger & 0 \\ 0 & \Lambda_L^\dagger \end{pmatrix},$$

and thus

$$\Lambda_D^{-1} \gamma^\mu \Lambda_D = \begin{pmatrix} \Lambda_R^\dagger & 0 \\ 0 & \Lambda_L^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} = \begin{pmatrix} 0 & \Lambda_R^\dagger \sigma^\mu \Lambda_R \\ \Lambda_L^\dagger \bar{\sigma}^\mu \Lambda_L & 0 \end{pmatrix} = \Lambda^\mu_\nu \gamma^\nu.$$ 

To summarize:

$$\Lambda_R^\dagger \sigma^\mu \Lambda_R = \Lambda^\mu_\nu \sigma^\mu,$$
$$\Lambda_L^\dagger \bar{\sigma}^\mu \Lambda_L = \Lambda^\mu_\nu \bar{\sigma}^\mu,$$
$$\Lambda_D^{-1} \gamma^\mu \Lambda_D = \Lambda^\mu_\nu \gamma^\nu.$$ 

Recalling the definition of $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$, one easily verifies:

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \begin{cases} \mu = 0, \nu = 0 & \sigma^0 \bar{\sigma}^0 + \sigma^0 \bar{\sigma}^0 = 2, \\ \mu = 0, \nu = i & \sigma^0 \bar{\sigma}^i + \sigma^i \bar{\sigma}^0 = -\sigma^i + \sigma^i = 0, \\ \mu = i, \nu = j & \sigma^i \bar{\sigma}^j + \sigma^j \bar{\sigma}^i = -\sigma^i \sigma^j - \sigma^j \sigma^i = -\{\sigma^i, \sigma^j\} = -2\delta^{ij}. \end{cases}$$

The same holds for the identity $\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2\eta^{\mu\nu}$. Therefore, given the definition of the Dirac matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

one can deduce the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$:

$$\left( \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \right) + (\mu \leftrightarrow \nu) = \left( \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu \\ \bar{\sigma}^\nu \sigma^\mu \end{pmatrix} \right) + (\mu \leftrightarrow \nu) = 2\eta^{\mu\nu} 1_4.$$