Quantum Field Theory

Set 13: solutions

Exercise 1

For the first point, the transformed spinor bilinear is $\psi_R^\dagger \Lambda_R^\dagger \sigma^\mu \Lambda_R \psi_R$. Let us expand the quantity between the spinors infinitesimally:

$$\Lambda_R^\dagger \sigma^\mu \Lambda_R = e^{\frac{1}{2}(\vec{\eta}+i\vec{\theta})\vec{\sigma}} \sigma^\mu e^{\frac{1}{2}(\vec{\eta}-i\vec{\theta})\vec{\sigma}} \simeq (1+\frac{1}{2}(\vec{\eta}-i\vec{\theta})\vec{\sigma})\sigma^\mu(1+\frac{1}{2}(\vec{\eta}+i\vec{\theta})\vec{\sigma}) = \sigma^\mu + \frac{1}{2}\eta^i\{\sigma^i,\sigma^\mu\} + \frac{i}{2}\theta^i[\sigma^i,\sigma^\mu]$$

Let us distinguish the cases $\mu = 0$, $\mu = i$. Knowing the equations $[\sigma^i,\sigma^0] = 0$, $[\sigma^i,\sigma^j] = 2i\epsilon^{ijk}\sigma^k$, $\{\sigma^i,\sigma^0\} = 2\sigma^i$, $\{\sigma^i,\sigma^j\} = 2\delta^{ij}\sigma^0$, we find:

$$\Lambda_R^\dagger \sigma^0 \Lambda_R \simeq \sigma^0 + \eta^i\sigma^i$$
$$\Lambda_R^\dagger \sigma^i \Lambda_R \simeq \sigma^i + \eta^i\sigma^0 - \theta^j\epsilon^{jik}\sigma^k$$

We can guess that this represents the infinitesimal Lorentz transformation of a vector, with the boost part parametrized by η^i and the rotation part by θ^i. Indeed, the vector (spin 1) representation of the Lorentz transformation is given by $\Lambda(\eta,\theta)^\mu{}_\nu = e^{\omega^\mu{}_\nu} \simeq \delta^\mu{}_\nu + \omega^\mu{}_\nu$, with the constraint that $\omega^{\alpha\nu} = \eta^{\nu\alpha}\omega^\mu{}_\alpha$ is antisymmetric: $\omega^{\mu\nu} = -\omega^{\nu\mu}$, and the identification $\theta^i = \frac{1}{2}\epsilon^{ijk}\omega^{jk}$, $\eta^i = \omega^{i0}$.

$$\Lambda_R^\dagger \sigma^0 \Lambda_R \simeq \sigma^0 + \omega^{i0}\sigma^i = \sigma^0 + \omega^0{}_i\sigma^i$$
$$\Lambda_R^\dagger \sigma^i \Lambda_R \simeq \sigma^i + \omega^{i0}\sigma^0 - \frac{1}{2}\epsilon^{jlm}\omega^{lm}\epsilon^{jik}\sigma^k = \sigma^i + \omega^i{}_0\sigma^0 + \omega^i{}_k\sigma^k$$

where we used the obvious identities $\omega^{\mu i} = -\omega^\mu{}_i$, $\omega^{\mu 0} = \omega^\mu{}_0$. Thus, we have proved the infinitesimal form of $\Lambda_R^\dagger \sigma^\mu \Lambda_R = \Lambda^\mu{}_\nu \sigma^\nu$.

For the second point, the calculation is very simple:

$$\epsilon^{-1}\Lambda_L\epsilon = \epsilon^{-1}e^{-\frac{1}{2}(\vec{\eta}+i\vec{\theta})\vec{\sigma}}\epsilon = e^{-\frac{1}{2}(\vec{\eta}+i\vec{\theta})\epsilon^{-1}\vec{\sigma}\epsilon} = e^{-\frac{1}{2}(\vec{\eta}+i\vec{\theta})(-\vec{\sigma}^*)} = \left(e^{\frac{1}{2}(\vec{\eta}-i\vec{\theta})\vec{\sigma}}\right)^* = \Lambda_R{}^*$$

by using the properties of ϵ, this implies also $\epsilon^{-1}\Lambda_R\epsilon = \Lambda_L^*$, $\epsilon^{-1}\Lambda_R^\dagger\epsilon = \Lambda_L^T$.

For the third point, we can take the first result and do a similarity transformation through ϵ:

$$\epsilon^{-1}\Lambda_R^\dagger\sigma^\mu\Lambda_R\epsilon = \epsilon^{-1}\Lambda^\mu{}_\nu\sigma^\nu\epsilon$$

Since $\epsilon^{-1}\sigma^\mu\epsilon = \bar{\sigma}^{*\mu}$, the previous equation is equivalent to:

$$\Lambda_L^T\bar{\sigma}^{*\mu}\Lambda_L^* = \Lambda^\mu_\nu\bar{\sigma}^{*\nu}$$

The complex conjugate of this relation is $\Lambda_L^\dagger\bar{\sigma}^\mu\Lambda = \Lambda^\mu_\nu\bar{\sigma}^\nu$. This shows that the spinor bilinear $\psi_L^\dagger\bar{\sigma}^\mu\psi_L$ transforms as a vector.

Exercise 2

Given the results in the previous exercise, it is now easy to show that $\Lambda_D^{-1}\gamma^\mu\Lambda_D = \Lambda^\mu{}_\nu\gamma^\nu$. Indeed, given that $\Lambda_R^\dagger = \Lambda_L^{-1}$ and $\Lambda_L^\dagger = \Lambda_R^{-1}$, then

$$\Lambda_D^{-1} = \begin{pmatrix} \Lambda_L^{-1} & 0 \\ 0 & \Lambda_R^{-1} \end{pmatrix} = \begin{pmatrix} \Lambda_R^\dagger & 0 \\ 0 & \Lambda_L^\dagger \end{pmatrix},$$

and thus

$$\Lambda_D^{-1}\gamma^\mu\Lambda_D = \begin{pmatrix} \Lambda_R^\dagger & 0 \\ 0 & \Lambda_L^\dagger \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} = \begin{pmatrix} 0 & \Lambda_R^\dagger\sigma^\mu\Lambda_R \\ \Lambda_L^\dagger\bar{\sigma}^\mu\Lambda_L & 0 \end{pmatrix} = \Lambda^\mu{}_\nu\gamma^\nu.$$

To summarize:

$$\begin{aligned} \Lambda_R^\dagger\ \sigma^\mu\ \Lambda_R &= \Lambda^\mu{}_\nu\ \sigma^\mu, \\ \Lambda_L^\dagger\ \bar{\sigma}^\mu\ \Lambda_L &= \Lambda^\mu{}_\nu\ \bar{\sigma}^\mu, \\ \Lambda_D^{-1}\ \gamma^\mu\ \Lambda_D &= \Lambda^\mu{}_\nu\ \gamma^\nu. \end{aligned}$$

Recalling the definition of $\sigma^\mu = (1,\, \sigma^i)$ and $\bar{\sigma}^\mu = (1,\, -\sigma^i)$, one easily verifies:

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = \begin{cases} \mu = 0, \nu = 0 & \sigma^0\bar{\sigma}^0 + \sigma^0\bar{\sigma}^0 = \quad 2, \\ \mu = 0, \nu = i & \sigma^0\bar{\sigma}^i + \sigma^i\bar{\sigma}^0 = \quad -\sigma^i + \sigma^i = 0, \\ \mu = i, \nu = j & \sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i = \quad -\sigma^i\sigma^j - \sigma^j\sigma^i = \quad -\{\sigma^i, \sigma^j\} = -2\delta^{ij}. \end{cases}$$

The same holds for the identity $\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2\eta^{\mu\nu}$. Therefore, given the definition of the Dirac matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

one can deduce the anticommutation relation $\{\gamma^\mu,\, \gamma^\nu\} = 2\eta^{\mu\nu}$:

$$\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} + (\mu \leftrightarrow \nu) = \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu \end{pmatrix} + (\mu \leftrightarrow \nu) = 2\eta^{\mu\nu}1_4.$$