Solution 11

a) Knowing that

\[ V \sim L^{D_f} \]

we choose \( L \) to be the number of divisions of the edge of the initial triangle. \( V \) is then the number of \textit{colored} triangles.

After \( n \) iterations: \( L = 2^n \) and \( V = 3^n \).

So:

\[ 3^n = 2^{nD_f} \Leftrightarrow D_f = \log_2 3 \]

b) The terms with \( s_2, s_4, s_6 \) in the partition function look like:

\[ Z = \sum_{s_2, s_4, s_6} \exp(Ks_1s_2 + Ks_2s_3 + Ks_3s_4 + Ks_4s_5 + Ks_5s_6 + Ks_6s_1 + Ks_2s_4 + Ks_4s_6 + Ks_6s_2) \]

We perform the sum over \( s_2 \), than over \( s_4 \) and finally over \( s_6 \). Things are simplified by the fact that every time one term appears with an odd power (i.e. \( s_2 \)) this will disappear under the sum over \( s_2 = \pm 1 \).

It is worth to make the substitution \( e^{Ks_is_j} = \cosh(K)(1 + s_is_jv) \) where \( v = \tanh(K) \). The final result is:

\[ Z = 8\cosh^9(K) \left( 1 + 4v^3 + 3v^4 + 3v^5 + 4v^6 + v^9 + (s_1s_3 + s_3s_5 + s_5s_1)(v^2 + 3v^3 + 4v^4 + 4v^5 + 3v^6 + v^7) \right) \]
c) The renormalisation transformation is obtained comparing (1) with:

\[ C \cosh^3(K') (1 + v's_1s_3)(1 + v's_3s_5)(1 + v's_5s_1) = C \cosh^3(K') (1 + v'^3 + (s_1s_3 + s_1s_5 + s_5s_3)(v' + v'^2)) \]

Evaluating the different values of the spins, the term \( s_1s_3 + s_1s_5 + s_5s_3 \) can only assume the values \(-1\) or \(3\), which give two equations:

\[
\begin{align*}
C \cosh^3(K')(1 + 3v' + 3v'^2 + v'^3) &= 8 \cosh^9(K) \left(1 + 3v^2 + 13v^3 + 15v^4 + 15v^5 + 13v^6 + 3v^7 + v^9\right) \\
C \cosh^3(K')(1 - v' - v'^2 + v'^3) &= 8 \cosh^9(K) \left(1 - v^2 + v^3 - v^4 - v^5 - v^6 - v^7 + v^9\right)
\end{align*}
\]

d) \( v = v' = 0 \) implies \( C = 8 \).

\( v = v' = 1 \) implies \( C = 64 \cosh^6(K) = 64 \cosh^6(\arctanh(v)) \). This means that \( C \) will take an infinite value as \( K \) does, which is not a problem (not as the problem we encountered while solving the Ising model in one dimension).

In conclusion \( v = 0 \) and \( v = 1 \) are two fixed point of the transformation.
e) The ratio between the two equations of renormalization give:

\[
\frac{1 + 3v' + 3v'^2 + v'^3}{1 - v' - v'^2 + v'^3} = \frac{1 + 3v^2 + 13v^3 + 15v^4 + 15v^5 + 13v^6 + 3v^7 + v^9}{1 - v^2 + v^3 - v^4 - v^5 + v^6 - v^7 + v^9}
\]

We pose:

\[
h(v') = \frac{1 + 3v' + 3v'^2 + v'^3}{1 - v' - v'^2 + v'^3}
\]

\[
f(v) = \frac{1 + 3v^2 + 13v^3 + 15v^4 + 15v^5 + 13v^6 + 3v^7 + v^9}{1 - v^2 + v^3 - v^4 - v^5 + v^6 - v^7 + v^9}
\]

The function \( h(v') \) is monotonous and continuous over the interval \([0, 1]\), so it is invertible. (I used Mathematica to see that...). It follows that: \( (h^{-1}(f(v)))' = \frac{1}{h'(f(v))} f'(v) \). Also the development of \( f(v) \) around \( v = 0 \) has no term of first order, which means \( f'(v = 0) = 0 \Rightarrow (h^{-1}(f(0)))' = 0 \) and the fixed point \( v = 0 \) is stable.

f) We know that the system has three fixed points (refer to Figure 1) with \( |h^{-1}(f(v = 0))'| = 0 < 1 \).

As the function \( h^{-1}(f(v)) \) is continuous over \([0, 1]\), we conclude that \( (h^{-1}(f(v^*)))' \geq 1 \) and \( |h^{-1}(f(v^*))'| \neq 1 \) (because otherwise it would never cross the line \( v \)). So \( (h^{-1}(f(v^*)))' > 1 \) and this shows that the point \( v^* \) is unstable. By consequence the point \( v = 1 \) is stable because there can not be two unstable points without a stable point in between.

The existence of the unstable point means that if \( v < v^* \) the system behaves as \( v = 0 (T = \infty) \), while if \( v > v^* \) the system behaves as \( v = 1 (T = 0) \). This implies the existence of a phase transition in \( v^* \). Numerically we obtain \( v^* = 0.49386 \).
Figure 1: Qualitative graph of the behavior of $h^{-1}(f(v))$, which is $\nu'(\nu)$ ($\nu'$ being the renormalized $\nu$)