Solution 11

a) Knowing that

$$V \sim L^{D_f}$$

we choose L to be the number of divisions of the edge of the initial triangle. V is then the number of *colored* triangles.

After n iterations: $L = 2^n$ and $V = 3^n$.

So:

$$3^n = 2^{nD_f} \Leftrightarrow D_f = \log_2 3$$

b) The terms with s_2, s_4, s_6 in the partition function look like:

$$Z = \sum_{s_2,s_4,s_6} \exp(Ks_1s_2 + Ks_2s_3 + Ks_3s_4 + Ks_4s_5 + Ks_5s_6 + Ks_6s_1 + Ks_2s_4 + Ks_4s_6 + Ks_6s_2)$$

We perform the sum over s_2, than over s_4 and finally over s_6. Things are simplified by the fact that every time one term appears with an odd power (i.e. s_2) this will disappear under the sum over $s_2 = \pm 1$.

It is worth to make the substitution $e^{Ks_is_j} = \cosh(K)(1 + s_is_jv)$ where $v = \tanh(K)$. The final result is:

$$Z = 8\cosh^9(K)\left(1 + 4v^3 + 3v^4 + 3v^5 + 4v^6 + v^9 + (s_1s_3 + s_3s_5 + s_5s_1)(v^2 + 3v^3 + 4v^4 + 4v^5 + 3v^6 + v^7)\right) \quad (1)$$

c) The renormalisation transformation is obtained comparing (1) with:

$$C\cosh^3(K')(1+v's_1s_3)(1+v's_3s_5)(1+v's_5s_1) = C\cosh^3(K')\left(1+v'^3+(s_1s_3+s_1s_5+s_5s_3)(v'+v'^2)\right)$$

Evaluating the different values of the spins, the term $s_1s_3+s_1s_5+s_5s_3$ can only assume the values -1 or 3, which give two equations:

$$\begin{cases} C\cosh^3(K')(1+3v'+3v'^2+v'^3) &= 8\cosh^9(K)\left(1+3v^2+13v^3+15v^4+15v^5+13v^6+3v^7+v^9\right) \\ C\cosh^3(K')(1-v'-v'^2+v'^3) &= 8\cosh^9(K)\left(1-v^2+v^3-v^4-v^5+v^6-v^7+v^9\right) \end{cases}$$

d) $v=v'=0$ implies $C=8$.
$v=v'=1$ implies $C=64\cosh^6(K)=64\cosh^6(\text{arctanh}(v))$. This means that C will take an infinite value as K does, which is not a problem (not as the *problem* we encountered while solving the Ising model in one dimension).
In conclusion $v=0$ and $v=1$ are two fixed point of the transformation.

e) The ratio between the two equations of renormalization give:

$$\frac{1+3v'+3v'^2+v'^3}{1-v'-v'^2+v'^3} = \frac{1+3v^2+13v^3+15v^4+15v^5+13v^6+3v^7+v^9}{1-v^2+v^3-v^4-v^5+v^6-v^7+v^9}$$

We pose:

$$h(v') = \frac{1+3v'+3v'^2+v'^3}{1-v'-v'^2+v'^3}$$
$$f(v) = \frac{1+3v^2+13v^3+15v^4+15v^5+13v^6+3v^7+v^9}{1-v^2+v^3-v^4-v^5+v^6-v^7+v^9}$$

The function $h(v')$ is monotonous and continuous over the interval $[0,1)$, so it is invertible. (I used Mathematica to see that...). It follows that: $(h^{-1}(f(v)))' = \frac{1}{h'(f(v))} f'(v)$. Also the development of $f(v)$ around $v=0$ has no term of first order, which means $f'(v=0)=0 \Rightarrow (h^{-1}(f(0)))' = 0$ and the fixed point $v=0$ is stable.

f) We know that the system has three fixed points (refer to Figure 1) with $|h^{-1}(f(v=0))'| = 0 < 1$.
As the function $h^{-1}(f(v))$ is continuous over $[0,1)$, we conclude that $(h^{-1}(f(v^*)))' \geq 1$ and $|h^{-1}(f(v^*))'| \neq 1$ (because otherwise it would never cross the line v). So $(h^{-1}(f(v^*)))' > 1$ and this shows that the point v^* is unstable. By consequence the point $v=1$ is stable because there can not be two unstable points without a stable point in between.

The existence of the unstable point means that if $v < v^*$ the system behaves as $v=0$ ($T=\infty$), while if $v > v^*$ the system behaves as $v=1$ ($T=0$). This implies the existence of a phase transition in v^*. Numerically we obtain $v^* = 0.49386$.

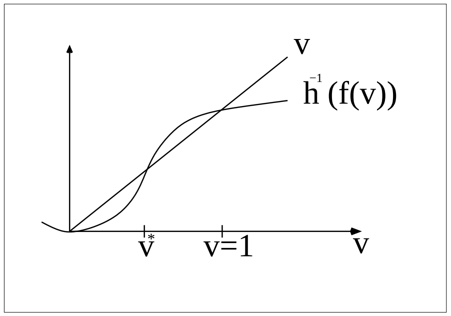

Figure 1: Qualitative graph of the behavior of $h^{-1}(f(v))$, which is $v'(v)$ (v' being the renormalized v)