Quantum Field Theory

Set 11: solutions

Exercise 1

We want to find the general form of an eigenstate of the annihilation operator:

$$a(\vec{q})|\psi>= \alpha(\vec{q})|\psi> \tag{1}$$

We can start from the ansatz:

$$|\psi>=\sum_{n=0}^{\infty} c_n \left(\int d^3k z(\vec{k})a^\dagger(\vec{k})\right)^n |0> \tag{2}$$

Using the commutation for the ladder operators $\left[a(\vec{q}), a^\dagger(\vec{k})\right] = \delta^3(\vec{k}-\vec{q})$ and since $a(\vec{q})|0>$, one can compute the left-hand side of Eq.(1):

$$a(\vec{q})|\psi>= z(\vec{q})\sum_{n=1}^{\infty} n\, c_n \left(\int d^3k z(\vec{k})a^\dagger(\vec{k})\right)^{n-1} |0>= z(\vec{q})\sum_{n=0}^{\infty}(n+1)\, c_{n+1} \left(\int d^3k z(\vec{k})a^\dagger(\vec{k})\right)^{n} |0> \tag{3}$$

One can see that $|\psi>$ satisfies Eq.(1) if we identify $z(\vec{q})$ as the eigenvalue and if we impose the following recursion relationship on the coefficients c_n:

$$c_{n+1} = \frac{1}{n+1} c_n \tag{4}$$

These are precisely the coefficients of the Taylor expansion of the exponential function (times a constant). Thus we can write:

$$|\psi>= c_0 \exp\left(\int d^3\vec{k} z(\vec{k})a^\dagger(\vec{k})\right)|0> \tag{5}$$

In the second point of the exercise we are required to fix the constant c_0 in order for the state to be normalized to 1.

$$<\psi|\psi>= |c_0|^2 <0|\exp\left(\int d^3\vec{k} z^*(\vec{k})a(\vec{k})\right)\exp\left(\int d^3\vec{k} z(\vec{k})a^\dagger(\vec{k})\right)|0> \tag{6}$$

For the sake of brevity, let's denote $A \equiv \int d^3\vec{k} z^*(\vec{k})a(\vec{k})$, $A^\dagger = \int d^3\vec{k} z(\vec{k})a^\dagger(\vec{k})$. One can easily show that $[A, A^\dagger] = \int d^3k |z(\vec{k})|^2$, i.e. the commutator is just a number (which commutes with everything). Then the Baker–Campbell–Hausdorff simply reads:

$$e^A e^{A^\dagger} = e^{A+A^\dagger} e^{\frac{1}{2}[A,A^\dagger]} \tag{7}$$

and, of course:

$$e^{A^\dagger} e^{A} = e^{A+A^\dagger} e^{\frac{1}{2}[A^\dagger,A]} \tag{8}$$

By applying the previous relations to (6) we arrive to:

$$< \psi|\psi >= |c_0|^2 e^{\int d^3k |z(\vec{k})|^2} < 0|e^{A^\dagger} e^{A}|0 >= |c_0|^2 e^{\int d^3k |z(\vec{k})|^2} \tag{9}$$

where we used $e^A|0 >= \left(1 + A + \frac{1}{2}A^2 + \dots\right)|0 >= |0 >$. Thus, we have to fix $c_0 = e^{-\frac{1}{2}\int d^3k|z(\vec{k})|^2}$.
In the last point of the exercise we are asked to calculate the expectation value of the real scalar field $\phi(x)$ on $|\psi >$.

$$\begin{aligned} < \psi|\phi(x)|\psi > &=< \psi| \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left[a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}\right]|\psi > \\ &=< \psi| \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left[\alpha(\vec{k})e^{-ikx} + \alpha^*(\vec{k})e^{ikx}\right]|\psi > \\ &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left[\alpha(\vec{k})e^{-ikx} + \alpha^*(\vec{k})e^{ikx}\right] \end{aligned} \tag{10}$$

(where as usual $kx \equiv \omega_{\vec{k}}t - \vec{k}\cdot\vec{x}$). This is just a real classical solution of the Klein-Gordon equation $\Box\phi + m^2\phi = 0$ (a superposition of plane waves). A coherent state represents indeed a quantum mechanical states which is the closest possible to a classical state, minimizing also the uncertainty $\Delta\phi\Delta\pi$ consistently with the quantum mechanical principles. Since it's not an eigenstate of energy, it oscillates unlike a state of the form $|\chi >\equiv a^\dagger(\vec{k}_1)\dots a^\dagger(\vec{k}_n)|0 >$, for which one could easily compute $< \chi|\phi(x)|\chi >= 0$.