Exercise 1

We want to find the general form of an eigenstate of the annihilation operator:

\[ a(\vec{q})|\psi> = \alpha(\vec{q})|\psi> \]  \hspace{1cm} (1)

We can start from the ansatz:

\[ |\psi> = \sum_{n=0}^{\infty} c_n \left( \int d^3k z(\vec{k}) a^\dagger(\vec{k}) \right)^n |0> \]  \hspace{1cm} (2)

Using the commutation for the ladder operators \([a(\vec{q}), a^\dagger(\vec{k})] = \delta^3(\vec{k} - \vec{q})\) and since \(a(\vec{q})|0>\), one can compute the left-hand side of Eq.(1):

\[ a(\vec{q})|\psi> = z(\vec{q}) \sum_{n=1}^{\infty} n c_n \left( \int d^3k z(\vec{k}) a^\dagger(\vec{k}) \right)^{n-1} |0> \]  \hspace{1cm} (3)

One can see that \(|\psi>\) satisfies Eq.(1) if we identify \(z(\vec{q})\) as the eigenvalue and if we impose the following recursion relationship on the coefficients \(c_n\):

\[ c_{n+1} = \frac{1}{n+1} c_n \]  \hspace{1cm} (4)

These are precisely the coefficients of the Taylor expansion of the exponential function (times a constant). Thus we can write:

\[ |\psi> = c_0 \exp \left( \int d^3k z(\vec{k}) a^\dagger(\vec{k}) \right) |0> \]  \hspace{1cm} (5)

In the second point of the exercise we are required to fix the constant \(c_0\) in order for the state to be normalized to 1.

\[ <\psi|\psi> = |c_0|^2 <0|\exp \left( \int d^3k z^*(\vec{k}) a(\vec{k}) \right) \exp \left( \int d^3k z(\vec{k}) a^\dagger(\vec{k}) \right) |0> \]  \hspace{1cm} (6)

For the sake of brevity, let’s denote \(A \equiv \int d^3k z^*(\vec{k}) a(\vec{k}),\) \(A^\dagger = \int d^3k z(\vec{k}) a^\dagger(\vec{k})\). One can easily show that \([A, A^\dagger] = \int d^3k z(\vec{k})^2\), i.e. the commutator is just a number (which commutes with everything). Then the Baker–Campbell–Hausdorff simply reads:

\[ e^A e^{A^\dagger} = e^{A + A^\dagger} e^{\frac{1}{2}[A, A^\dagger]} \]  \hspace{1cm} (7)

and, of course:

\[ e^{A^\dagger} e^{A} = e^{A + A^\dagger} e^{\frac{1}{2}[A^\dagger, A]} \]  \hspace{1cm} (8)

By applying the previous relations to (6) we arrive to:
\[ < \psi | \psi > = |c_0|^2 e^{\int d^3 k |z(\vec{k})|^2} < 0 | e^{A^\dagger} e^A | 0 > = |c_0|^2 e^{\int d^3 k |z(\vec{k})|^2} \]

(9)

where we used \( e^{A^\dagger} |0 > = \left( 1 + A + \frac{1}{2} A^2 + \ldots \right) |0 > = |0 > \). Thus, we have to fix \( c_0 = e^{-\frac{1}{2} \int d^3 k |z(\vec{k})|^2} \).

In the last point of the exercise we are asked to calculate the expectation value of the real scalar field \( \phi(x) \) on \( |\psi > \).

\[ < \psi | \phi(x) | \psi > = < \psi | \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left[ a(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{ikx} \right] |\psi > 
\]
\[ = < \psi | \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left[ \alpha(\vec{k}) e^{-ikx} + \alpha^*(\vec{k}) e^{ikx} \right] |\psi > 
\]

(10)

\[ = \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left[ \alpha(\vec{k}) e^{-ikx} + \alpha^*(\vec{k}) e^{ikx} \right] \]

(where as usual \( kx \equiv \omega_k t - \vec{k} \cdot \vec{x} \)). This is just a real classical solution of the Klein-Gordon equation \( \Box \phi + m^2 \phi = 0 \) (a superposition of plane waves). A coherent state represents indeed a quantum mechanical states which is the closest possible to a classical state, minimizing also the uncertainty \( \Delta \phi \Delta \pi \) consistently with the quantum mechanical principles. Since it's not an eigenstate of energy, it oscillates unlike a state of the form \( |\chi > \equiv a^\dagger(\vec{k}_1) \ldots a^\dagger(\vec{k}_n) |0 > \), for which one could easily compute \( < \chi | \phi(x) | \chi > = 0 \).