Quantum Field Theory

Set 10: solutions

Exercise 1

Let us consider the expansion of a scalar field in terms of the ladder operators:

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2k_0} \left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right] e^{i\vec{k}\cdot\vec{x}}.$$

We want to show that this satisfies the Klein-Gordon equation:

$$(\Box + m^2)\phi(\vec{x},t) = 0.$$

Thus:

$$(\Box + m^2)\phi(\vec{x},t) = (\partial_t^2 - \partial_i^2 + m^2) \int \frac{d^3k}{(2\pi)^3 2k_0} \left[a(\vec{k}) e^{-ik_0t + i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k}) e^{ik_0t - i\vec{k}\cdot\vec{x}}\right]$$
$$= \int \frac{d^3k}{(2\pi)^3 2k_0} \left[(m^2 - k_0^2 + |\vec{k}|^2)\ a(k,t) e^{-ik_0t + i\vec{k}\cdot\vec{x}} + (m^2 - k_0^2 + |\vec{k}|^2)\ a^\dagger(k,t) e^{ik_0 - i\vec{k}\cdot\vec{x}}\right] = 0,$$

where we have used the mass shell condition $k_0^2 = |\vec{k}|^2 + m^2$.

Exercise 2

Given a real free massive scalar field ϕ one can obtain the energy momentum tensor using Noether's prescription as usual:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}.$$

In order to compute the Noether's charge one needs

$$\begin{aligned} T_{00} &= \dot{\phi}^2 - \mathcal{L} = \mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right), \\ T_{0i} &= \pi\partial_i\phi. \end{aligned} \tag{1}$$

The decomposition of the fields ϕ, π in terms of the operator $a(\vec{k})$ and $a^\dagger(\vec{k})$ reads:

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3 2k_0} \left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right] e^{i\vec{k}\cdot\vec{x}}, \\ \pi(x) &= \int \frac{d^3k}{(2\pi)^3 2k_0} (-ik_0) \left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right] e^{i\vec{k}\cdot\vec{x}}, \end{aligned}$$

where we have used the notation $x \equiv (t,\vec{x})$. Therefore:

$$\begin{aligned} P_0 = \int d^3x\ T_{00} \quad &= \frac{1}{2}\int d^3x \int \frac{d^3k}{(2\pi)^3 2k_0} \frac{d^3q}{(2\pi)^3 2q_0} \Big\{-k_0q_0 \left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right] \left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &+ \left(-\vec{k}\cdot\vec{q} + m^2\right) \left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right] \left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Big\} e^{i(\vec{k}+\vec{q})\cdot\vec{x}}. \end{aligned}$$

Using the relation

$$\int d^3x\, e^{i(\vec{k}+\vec{q})\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{k}+\vec{q}),$$

one can integrate over d^3k and set $\vec{k}=-\vec{q}$. In addition, $k_0=\sqrt{m^2+|\vec{k}|^2}=\sqrt{m^2+|\vec{q}|^2}=q_0$. Thus:

$$\begin{aligned} P_0 = \frac{1}{4}\int \frac{d^3q}{(2\pi)^3 2q_0} \Big\{ & -q_0\left[a(-\vec{q},t)-a^\dagger(\vec{q},t)\right]\left[a(\vec{q},t)-a^\dagger(-\vec{q},t)\right] \\ & + \left(\frac{|\vec{q}|^2+m^2}{q_0}\right)\left[a(-\vec{q},t)+a^\dagger(\vec{q},t)\right]\left[a(\vec{q},t)+a^\dagger(-\vec{q},t)\right]\Big\} \\ = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2q_0} & q_0\left\{a(\vec{q},t)a^\dagger(\vec{q},t)+a^\dagger(\vec{q},t)a(\vec{q},t)\right\}, \end{aligned}$$

where in the last step we have used the fact that the measure and the extremes are invariant under $\vec{q}\longrightarrow-\vec{q}$, so $a(-\vec{q},t)a^\dagger(-\vec{q},t)$ can be replaced by $a(\vec{q},t)a^\dagger(\vec{q},t)$. Finally one can commute the operators to achieve the normal ordered expression plus an irrelevant infinite constant:

$$P_0 = \int \frac{d^3q}{(2\pi)^3 2q_0}\; q_0\; a^\dagger(\vec{q},t)\; a(\vec{q},t)\; + \text{const}.$$

Similarly:

$$P_i = \int d^3x\; T_{0i} = \int d^3x \int \frac{d^3k}{(2\pi)^3 2k_0}\frac{d^3q}{(2\pi)^3 2q_0}\left\{-k_0 q_i\left[a(\vec{k},t)-a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t)+a^\dagger(-\vec{q},t)\right]\right\}e^{i(\vec{k}+\vec{q})\cdot\vec{x}}.$$

Note that the minus sign in front of k_0q_i is due to the fact that $\partial_i e^{i\vec{q}\cdot\vec{x}}=\partial_i e^{iq^ix^i}=iq^ie^{iq^ix^i}=-iq_ie^{i\vec{q}\cdot\vec{x}}$. Again, integrating first over d^3x to generates the delta function on the momentum space and then integrating over one of the momenta one gets:

$$\begin{aligned} P_i &= \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2q_0}\left\{-q_i\left[a(-\vec{q},t)-a^\dagger(\vec{q},t)\right]\left[a(\vec{q},t)+a^\dagger(-\vec{q},t)\right]\right\} \\ &= \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2q_0} q_i\left\{a^\dagger(\vec{q},t)a(\vec{q},t)-a(-\vec{q},t)a^\dagger(-\vec{q},t)\right\} + \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2q_0} q_i\left\{a^\dagger(\vec{q},t)a^\dagger(-\vec{q},t)-a(-\vec{q},t)a(\vec{q},t)\right\} \\ &= \int \frac{d^3q}{(2\pi)^3 2q_0}\; q_i\; a^\dagger(\vec{q},t)\; a(\vec{q},t) + \text{const}. \end{aligned}$$

The second term in second line vanishes since the integrand is odd under $q\to-q$: indeed $\vec{q}\; a(-\vec{q},t)\; a(\vec{q},t)\to -\vec{q}\, a(\vec{q},t)\; a(-\vec{q},t) = -\vec{q}\, a(-\vec{q},t)\; a(\vec{q},t)$, because $a(\vec{q},t)$ commutes with itself for any q. Finally in the last equality of last equation we have used again $aa^\dagger = a^\dagger a$ + const.

Let us consider the Noether's current associated to rotations and boosts; recalling the transformation properties $\phi'(x)\simeq\phi(x)+\frac{1}{2}(x_\rho\partial_\sigma-x_\sigma\partial_\rho)\phi(x)\omega^{\rho\sigma}$, one can define $\Delta_\phi=(x_\rho\partial_\sigma-x_\sigma\partial_\rho)\phi(x)\omega^{\rho\sigma}$ and therefore

$$M_{\mu\rho\sigma} = \partial_\mu\phi(x_\rho\partial_\sigma\phi - x_\sigma\partial_\rho\phi) - (x_\rho\eta_{\mu\sigma}-x_\sigma\eta_{\mu\rho})\mathcal{L} = x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho}.$$

Notice that the Noether's current is defined up to constant rescaling of the transformation parameter: in this case we have considered $\omega^{\rho\sigma}/2$ as parameters. However, if the theory contains objects transforming according some other representation of the Lorentz group, the definition of what the parameters are has to be consistent. In the present case the Noether's charge reads:

$$\int d^3x\; M_{0\rho\sigma} = \int d^3x\left\{x_\rho T_{0\sigma}-x_\sigma T_{0\rho}\right\}.$$

In particular the generator of boosts can be extracted taking the timelike component of previous expression:

$$K_i = \int d^3x\; M_{00i} = \int d^3x\left\{x_0T_{0i}-x_iT_{00}\right\} = t\,P_i - \int d^3x\,\mathcal{H}\,x_i.$$

The first term is t times the generator of translation and has been already computed, while the second one involves the Hamiltonian density. Let us compute this quantity:

$$\begin{aligned} \int d^3x\,\mathcal{H}\,x_i = \frac{1}{2}\int d^3x\int \frac{d^3q}{(2\pi)^3 2q_0}\frac{d^3k}{(2\pi)^3 2k_0} \Big\{ & -k_0q_0\left[a(\vec{k},t)-a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t)-a^\dagger(-\vec{q},t)\right] \\ & + \left(-\vec{k}\cdot\vec{q}+m^2\right)\left[a(\vec{k},t)+a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t)+a^\dagger(-\vec{q},t)\right]\Big\}\, x_i e^{i(\vec{k}+\vec{q})\cdot\vec{x}}. \end{aligned}$$

Let us use the following relation:

$$\int d^3x\, x_i\, e^{i(\vec{k}+\vec{q})\cdot\vec{x}} = \int d^3x\, i\frac{\partial}{\partial k^i} e^{i(\vec{k}+\vec{q})\cdot\vec{x}} = i(2\pi)^3 \frac{\partial}{\partial k^i}\delta^3(\vec{k}+\vec{q}).$$

We can then integrate by parts the derivative with respect to k^i:

$$\begin{aligned}\int d^3x\, \mathcal{H}\, x_i &= \frac{i}{4}\int \frac{d^3q}{(2\pi)^3 2q_0} d^3k \frac{\partial}{\partial k^i}\Bigg\{ q_0 \left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &\quad - \left(\frac{-\vec{k}\cdot\vec{q}+m^2}{k_0}\right)\left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Bigg\}\delta^3(\vec{k}+\vec{q}).\end{aligned}$$

The only subtle point arises in the derivation of the fraction in parentheses:

$$\frac{\partial}{\partial k^i}\left(\frac{-\vec{k}\cdot\vec{q}+m^2}{k^0}\right) = -\frac{q^i}{k_0} + (-\vec{k}\cdot\vec{q}+m^2)\left(-\frac{k^i}{k_0^3}\right),$$

and since the integral contains $\delta^3(\vec{k}+\vec{q})$, after the integration on d^3k it will be $\vec{k}=\vec{q}$, $q_0=k_0$, and this term will vanish. Therefore we neglect it from now on. Hence

$$\begin{aligned}\int d^3x\, \mathcal{H}\, x_i &= \frac{i}{4}\int \frac{d^3q}{(2\pi)^3 2q_0} d^3k \Bigg\{ q_0 \left(\frac{\partial}{\partial k^i}\left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\right)\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &\quad - \left(\frac{-\vec{k}\cdot\vec{q}+m^2}{k^0}\right)\left(\frac{\partial}{\partial k^i}\left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right]\right)\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Bigg\}\delta^3(\vec{k}+\vec{q}),\end{aligned}$$

and finally, integrating over d^3k (and then setting $\vec{k}=\vec{q}$):

$$\begin{aligned}\int d^3x\, \mathcal{H}\, x_i &= -\frac{i}{4}\int \frac{d^3q}{(2\pi)^3 2q_0}\, q_0 \Bigg\{\left(\frac{\partial}{\partial q^i}\left[a(-\vec{q},t) - a^\dagger(\vec{q},t)\right]\right)\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &\quad - \left(\frac{\partial}{\partial q^i}\left[a(-\vec{q},t) + a^\dagger(\vec{q},t)\right]\right)\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Bigg\} \\ &= \frac{i}{2}\int \frac{d^3q}{(2\pi)^3 2q_0}\, q_0 \left\{\frac{\partial}{\partial q^i} a(-\vec{q},t)\, a^\dagger(-\vec{q},t) + \frac{\partial}{\partial q^i} a^\dagger(\vec{q},t)\, a(\vec{q},t)\right\} \\ &= -i\int \frac{d^3q}{(2\pi)^3 2q_0}\, q_0 \left(a^\dagger(\vec{q},t)\, \frac{\partial}{\partial q^i} a(\vec{q},t)\right),\end{aligned}$$

where in the last step we have integrated the second term by parts and commuted the operators $a^\dagger$ and ∂a in the first. This is possible for the following reason: defining $C_i(q) \equiv [a^\dagger(\vec{q},t), \frac{\partial}{\partial q^i} a(\vec{q},t)]$, it is easy to show that $[C_i(q), a(\vec{p},t)] = [C_i(q), a^\dagger(\vec{p},t)] = 0$, so $C_i(q)$ is a $\mathbb{C}$-number, that can be neglected for the purposes of this exercise, as it has been done previously as well. Finally one can compute the generator of boosts:

$$K^i = t\, P^i - \int d^3x\, \mathcal{H}\, x^i = \int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q},t)\left(tq^i - iq_0\frac{\partial}{\partial q^i}\right) a(\vec{q},t).$$

In the same way one can compute the generators of rotations:

$$J^{ij} = \int d^3x\, M_0{}^{ij} = \int d^3x \left\{x^i T_0{}^j - x^j T_0{}^i\right\}.$$

Proceeding as before one has

$$\int d^3x\, x^i T_0{}^j = \int d^3x \int \frac{d^3q}{(2\pi)^3 2q_0}\frac{d^3k}{(2\pi)^3 2k_0}\left\{-k_0 q^j \left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\right\} x^i e^{i(\vec{k}+\vec{q})\cdot\vec{x}},$$

and integrating by parts the derivative with respect to k^i (this time it's straightforward since k_0 simplifies):

$$\int d^3x\, x^i T_0{}^j = -\frac{i}{2}\int \frac{d^3q}{(2\pi)^3 2q_0} d^3k \left\{q^j \frac{\partial}{\partial k^i}\left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\right\}\delta^3(\vec{k}+\vec{q}).$$

Finally integrating over d^3k gives:

$$\int d^3x\, x^i {T_0}^j = \frac{i}{2}\int \frac{d^3q}{(2\pi)^3 2q_0} q^j \left\{\frac{\partial}{\partial q^i}\left[a(-\vec{q},t) - a^\dagger(\vec{q},t)\right]\right\}\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right].$$

In the end we have:

$$\begin{aligned}
J^{ij} &= \int d^3x\ {M_0}^{ij} = \int d^3x \left\{x^i {T_0}^j - x^j {T_0}^i\right\} \\
&= \frac{i}{2}\int \frac{d^3q}{(2\pi)^3 2q_0}\left\{\left[q^j \frac{\partial}{\partial q^i} a(-\vec{q},t)\right] a(\vec{q},t) - \left[q^j \frac{\partial}{\partial q^i} a^\dagger(\vec{q},t)\right] a^\dagger(-\vec{q},t)\right. \\
&\quad + \left.\left[q^j \frac{\partial}{\partial q^i} a(-\vec{q},t)\right] a^\dagger(-\vec{q},t) - \left[q^j \frac{\partial}{\partial q^i} a^\dagger(\vec{q},t)\right] a(\vec{q},t)\right\} - (i \leftrightarrow j).
\end{aligned}$$

The antisymmetrization causes the first line to vanish, while the two terms in the second are identical (up to some infinite constant). Indeed integrating by parts one can show that the former terms are symmetric in i, j:

$$\begin{aligned}
&\int \frac{d^3q}{(2\pi)^3 2q_0}\left(q^j \frac{\partial}{\partial q^i} a(-\vec{q},t)\right) a(\vec{q},t) - (i \leftrightarrow j) = -\left(\int \frac{d^3q}{(2\pi)^3}\, a(-\vec{q},t) \frac{\partial}{\partial q^i}\left[\frac{q^j}{2q_0} a(\vec{q},t)\right] - (i \leftrightarrow j)\right) \\
&= -\left(\int \frac{d^3q}{(2\pi)^3 2q_0}\, a(-\vec{q},t) q^j \frac{\partial}{\partial q^i} a(\vec{q},t) + \int \frac{d^3q}{(2\pi)^3}\, a(-\vec{q},t) a(\vec{q},t) \frac{\partial}{\partial q^i}\left(\frac{q^j}{2q_0}\right) - (i \leftrightarrow j)\right) \\
&= -\left(\int \frac{d^3q}{(2\pi)^3 2q_0}\, a(-\vec{q},t) q^j \frac{\partial}{\partial q^i} a(\vec{q},t) - (i \leftrightarrow j)\right) - \underbrace{\left(\int \frac{d^3q}{(2\pi)^3}\, a(-\vec{q},t) a(\vec{q},t)\left(\frac{\delta^{ij}}{2q_0} - \frac{q^i q^j}{2q_0^3}\right) - (i \leftrightarrow j)\right)}_{\text{symmetric} \Longrightarrow = 0} \\
&= -\left(\int \frac{d^3q}{(2\pi)^3 2q_0}\left(q^j \frac{\partial}{\partial q^i} a(-\vec{q},t)\right) a(\vec{q},t) - (i \leftrightarrow j)\right) = 0,
\end{aligned}$$

since we have shown that this term is equal to minus itself. In the last line we have commuted the operators and changed sign to $\vec{q}$. At the very end the generators J^{ij} read:

$$J^{ij} = i\int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q},t)\left(q^j \frac{\partial}{\partial q^i} - q^i \frac{\partial}{\partial q^j}\right) a(\vec{q},t).$$

Finally one can show that these charges don't depend on time, even if $a(\vec{q},t) = a(\vec{q})e^{-iq_0 t}$ does. Clearly in the product $a^\dagger(\vec{q},t)a(\vec{q},t)$ the factors cancels. Therefore:

$$\begin{aligned}
P^\mu &= \int \frac{d^3q}{(2\pi)^3 2q_0} q^\mu\, a^\dagger(\vec{q},t) a(\vec{q},t) = \int \frac{d^3q}{(2\pi)^3 2q_0} q^\mu\, a^\dagger(\vec{q}) a(\vec{q}), \\
K^i &= \int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q},t)\left(tq^i - iq_0 \frac{\partial}{\partial q^i}\right) a(\vec{q},t) \\
&= -\int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q})\left(iq_0 \frac{\partial}{\partial q^i}\right) a(\vec{q}) + \int \frac{d^3q}{(2\pi)^3 2q_0}\, a^\dagger(\vec{q}) a(\vec{q}) \underbrace{\left(tq^i - iq_0(-it)\frac{\partial}{\partial q^i} q_0\right)}_{=0}, \\
J^{ij} &= -i\int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q})\left(q^i \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial q^i}\right) a(\vec{q}) - \int \frac{d^3q}{(2\pi)^3 2q_0}\, a^\dagger(\vec{q}) a(\vec{q}) \underbrace{\left(q^i \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial q^i}\right)(iq_0 t)}_{\propto\, q^i q^j - q^j q^i = 0}.
\end{aligned}$$

Additional note

Instead of using the representation of the canonical fields $\phi(\vec{x},t)$, $\pi(\vec{x},t)$ in terms of the ladder operators $a(\vec{k})$, $a^\dagger(\vec{k})$, one could directly work with ϕ and π. Indeed, the following equation holds:

$$\frac{dQ_i}{dt} = \frac{\partial Q_i}{\partial t} + i[H, Q_i]$$

since H is the generator of time translations. Thus, just by using the commutation relations $[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$, one could check that the right-hand side of the previous equation vanishes. Notice that only the boost generators K_i have an explicit time-dependence, $\frac{\partial K_i}{\partial t} \neq 0$. For the other ones only the relation $[H, Q_i(t)] = 0$ must be checked (which means that the Hamiltonian is invariant under the transformations generated by Q_i, as expected).

Exercise 3

Given the canonical commutation relation at equal time:

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}),$$

we want to show that the Noether charges are the generators of the infinitesimal transformation in the following sense: if a transformation acts on coordinates and fields as $x'^\mu = x^\mu - \epsilon_i^\mu(x)\alpha^i$, $\phi'(x) = \phi(x) + \Delta_i(x)\alpha^i$ then:

$$[Q_i, \phi(x)] = i\Delta_i(x).$$

In Solution8 we have shown the analogous of this expression for classical field theory, where the commutators are replaced by Poisson brackets. One could start from the Noether's charge

$$Q_i = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \Delta_i - \epsilon_i^0 \mathcal{L} \right),$$

and derive the result following the same steps, since the canonical commutation relation have the same form as the Poisson brackets. Indeed, since the charges do not depend on time we can choose t in order to use equal time commutation rules. For example:

$$\begin{aligned}
[J^{ij}, \phi(\vec{x},t)] &= \int d^3y \; [y^i\pi(\vec{y},t)\partial^j\phi(\vec{y},t) - y^j\pi(\vec{y},t)\partial^i\phi(\vec{y},t), \phi(\vec{x},t)] \\
&= \int d^3y \left\{ y^i[\pi(\vec{y},t),\phi(\vec{x},t)]\partial^j\phi(\vec{y},t) - (i \leftrightarrow j) \right\} \\
&= -i \int d^3y \; \delta^3(\vec{x}-\vec{y}) \left\{ y^i\partial^j\phi(\vec{y},t) - (i \leftrightarrow j) \right\} \\
&= -i \left(x^i\partial^j - x^j\partial^i \right) \phi(\vec{x},t).
\end{aligned}$$

Similarly:

$$\begin{aligned}
[P^i, \phi(\vec{x},t)] &= \int d^3y \; [\pi(\vec{y},t)\partial^i\phi(\vec{y},t), \phi(\vec{x},t)] \\
&= \int d^3y [\pi(\vec{y},t), \phi(\vec{x},t)]\partial^i\phi(\vec{y},t) \\
&= -i \int d^3y \, \delta^3(\vec{x}-\vec{y})\partial^i\phi(\vec{y},t) = -i\partial^i\phi(\vec{x},t).
\end{aligned}$$

One can check the consistency of the result using the Jacobi identity:

$$\begin{aligned}
[[J^{ij}, P^k], \phi(\vec{x},t)] &= -[P^k, [J^{ij}, \phi(\vec{x},t)]] + [J^{ij}, [P^k, \phi(\vec{x},t)]] \\
&= \left(x^i\partial^j - x^j\partial^i \right) \partial^k\phi(\vec{x},t) - \partial^k \left(x^i\partial^j - x^j\partial^i \right) \phi(\vec{x},t) \\
&= (\delta^{jk}\delta^{in} - \delta^{ik}\delta^{jn})\partial^n\phi(\vec{x},t) \\
&= -i(\delta^{ik}\delta^{jn} - \delta^{jk}\delta^{in})[P^n, \phi(\vec{x},t)],
\end{aligned}$$

from which one can deduce $[J^{ij}, P^k] = -i(\delta^{ik}\delta^{jn} - \delta^{jk}\delta^{in})P^n$, as expected from the Poincaré algebra.