Quantum Field Theory

Set 9

Exercise 1: $SU(2)$ Noether's current

Consider a pair of complex scalar fields $\Phi = \binom{\phi_1}{\phi_2}$ transforming according to the representation $j = 1/2$ of a global $SU(2)$ symmetry (do not confuse this symmetry with the Lorentz transformations: they are completely uncorrelated):

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x^\mu, \\ \Phi(x) &\longrightarrow \Phi'(x') = \mathcal{U}\Phi(x), \\ \phi_a(x) &\longrightarrow \phi'_a(x') = \mathcal{U}_a{}^b \phi_b(x), \qquad a, b = 1, 2. \end{aligned}$$

where $\mathcal{U}$ is a $SU(2)$ matrix. Given the Lagrangian density

$$\mathcal{L} = \partial^\mu \Phi^\dagger \, \partial_\mu \Phi - m^2 \Phi^\dagger \Phi - \frac{\lambda}{2} \left(\Phi^\dagger \Phi\right)^2 ,$$

- Show that the Lagrangian density is invariant under $SU(2)$ transformations.
- Compute the Noether current for this symmetry.
- Add to the above Lagrangian the interaction with a real *triplet* $\vec{A} = (A_1, A_2, A_3)$ of $SU(2)$

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{1}{2} \partial^\mu A^T \, \partial_\mu A + A_i \Phi^\dagger \sigma^i \Phi. \tag{1}$$

  Is the new Lagrangian density invariant under $SU(2)$ transformations involving the field Φ and $\vec{A}$?

Exercise 2: Invariant measure

Consider the measure on thee dimensional momentum space $\frac{d^3k}{(2\pi)^3 2k^0}$, where $k_0 = \sqrt{|\vec{k}|^2 + m^2}$.

- Performing a boost in a particular direction, say $\hat{k}_3$, show the invariance of the measure under Lorentz transformations.
  (Hint: use the form of the boost transformations $k'_0 = \cosh(\eta) \; k_0 + \sinh(\eta) \; k_3$, $k'_3 = \cosh(\eta) \; k_3 + \sinh(\eta) \; k_0$).
- Show that the product $\delta^3(\vec{k}) d^3k$ is invariant under Lorentz transformation and deduce that the distribution $(2\pi)^3 2k^0 \delta^3(\vec{k})$ is invariant as well.

Exercise 3: Noether's current: a different approach

Given a Lagrangian density $\mathcal{L}(\phi_a, \partial_\mu \phi_b)$ consider the infinitesimal symmetry transformation defined by

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = f(x) \simeq x^\mu - \epsilon_i^\mu(x) \beta^i(x), \\ \phi_a(x) &\longrightarrow \phi'_a(x') = D[\phi](x) \simeq \phi_a(x') + \beta^i(x') \Delta_{ai}(x'), \end{aligned}$$

where the infinitesimal parameters of the transformation depend explicitly on the spacetime coordinates. The request that the Lagrangian in invariant under the transformation where β is a *constant* parameter implies the equation (see the proof of Noether's theorem)

$$\frac{\partial \mathcal{L}}{\partial \phi_a} \Delta_{ai}(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu \Delta_{ai}(x) - \partial_\mu \left(\mathcal{L} \epsilon_i^\mu\right) = 0$$

- Using the above equation show that the change in the action is given by

$$\int_{\Omega'} \mathcal{L}[\phi'_a](x')d^4x' - \int_{\Omega} \mathcal{L}[\phi_a](x)d^4x = \int_{\Omega} J_i^\mu \partial_\mu \beta^i(x)d^4x,$$

  where J_i^μ is the usual Noether's current defined for the *global* symmetry (the same symmetry but with constant parameters).

- Apply explicitly this procedure to find the Noether's current associated to the $U(1)$ symmetry in a complex scalar field theory.

Exercise 4: Decomposition in Fourier modes

Given the decomposition of a real scalar field in a finite cubic volume $V = L^3$

$$\phi(t, \vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^3} \frac{1}{\sqrt{V}} \phi_n(t)\, e^{i\frac{2\pi}{L}\vec{n}\cdot\vec{x}}$$

- Expand $\int d^3x (\vec{\nabla}\phi)^2(t, \vec{x})$ in Fourier modes $\phi_n(t)$.

Exercise 5: Ladder operators

Given the algebra of the ladder operators (non relativistic normalization)

$$\left[a(\vec{q}), a^\dagger(\vec{p})\right] = (2\pi)^3 \delta^3(\vec{q} - \vec{p}),$$

and the expression of the Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3}\, \omega(\vec{k})\, a^\dagger(\vec{k}) a(\vec{k}),$$

compute the commutation relations

$$[H, a(\vec{p})] = ? \qquad \left[H, a^\dagger(\vec{p})\right] = ?$$